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EXTENSIONS OF REPRESENTATIONS OF LIE ALGEBRAS

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Let ϕ: $L_1 \to L_2$ be a morphism of finite-dimensional Lie algebras over a field of characteristic zero. Our problem is this: given a finite-dimensional L_1-module, V say, when does V embed as a sub L_1-module of some finite-dimensional L_2-module? The problem clearly reduces to the case in which ϕ is injective. We provide here (Thm. 3.6) a solution in two separate cases: (i) under the assumption that ϕ maps the radical of L_1 into the radical of L_2, or (ii) under the assumption that L_1 is its own commutator ideal.

0. Introduction. A theorem of Bialynicki-Birula, Hochschild, and Mostow ([**1**, Thm. 1]) gives conditions for a finite-dimensional module for a subgroup of an algebraic group to embed as a submodule into a finite-dimensional module for the whole group. It is with a modification of this result that we obtain criteria for modules of Lie algebras.

Throughout this paper, k will denote a field of characteristic zero, and K will be an algebraic closure of k. For a Lie algebra L over k, $U(L)$ will denote the universal enveloping algebra of L; $H(L)$ will denote the Hopf algebra of representative functions associated with L. All of our Lie algebras, modules, and representations are taken to be *finite-dimensional* unless otherwise specified. We will regard a module for a Lie algebra L as also a left $U(L)$-module or as a right $H(L)$-comodule, and vice versa.

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1. Reduction of the problem to representative functions.

DEFINITION. Let ϕ: $H_1 \to H_2$ be a morphism of coalgebras over k. ϕ induces an H_2-comodule structure on any H_1-comodule ψ: $V \to V \otimes H_1$ by following up ψ with $(i \otimes \phi)$, where i is the identity map. We say that an H_2-comodule ξ: $U \to U \otimes H_2$ is *extendable* to H_1 if there is an H_1-comodule ψ: $V \to V \otimes H_1$ and a linear injection j: $U \hookrightarrow V$ such that $(j \otimes i) \circ \xi = (i \otimes \phi) \circ \psi \circ j$.

Clearly, a necessary condition for ξ to be extendable is that $\xi(U)$ should be contained in $U \otimes \phi(H_1)$. We say that ϕ is *manageable* if, for all H_2-comodules U, the above condition is also sufficient.

Note. [**1**, §6] contains examples of morphisms ϕ that are not manageable.

We remark that, if H is a bialgebra, then the multiplication map on H enables us to construct an H-comodule structure on the tensor product of two comodules. If H is commutative, then the H-comodule structure on the tensor algebra $\otimes V$ of an H-comodule V factors to give an H-comodule structure on each homogeneous component of the exterior algebra $\wedge V$ built on V. Finally, we note that, if H is a Hopf algebra, then the antipode map on H enables us to construct an H-comodule structure on the linear dual V° of an H-comodule V.

The following is a generalization of [**1**, Thm. 1].

LEMMA 1.1. *Let $\phi\colon H_1 \to H_2$ be a morphism of commutative Hopf algebras over k. Then ϕ is manageable if and only if, for every one-dimensional H_2-comodule that is extendable to H_1, the dual comodule V° is also extendable to H_1.*

Proof. The necessity of the condition is clear. Now, suppose that the condition on one-dimensional H_2-comodules is satisfied. Let $\xi\colon U \to U \otimes H_2$ be an H_2-comodule, and let us assume that $\xi(U)$ is contained in $U \otimes \phi(H_1)$. Then, U is isomorphic with a subcomodule of the direct sum of finitely many copies of $\phi(H_1)$, where the (locally finite-dimensional) H_2-comodule structure on $\phi(H_1)$ is given by the restriction of the comultiplication of H_2.

If we take inverse images under the map that sends $H_1 \oplus \cdots \oplus H_1$ to $\phi(H_1) \oplus \cdots \oplus \phi(H_1)$, we can choose a finite-dimensional sub H_1-comodule Z of $H_1 \oplus \cdots \oplus H_1$ and a sub H_2-comodule X of Z that maps onto U (in $\phi(H_1) \oplus \cdots \oplus \phi(H_1)$) with kernel Y, say. Let n be the dimension of Y. Now, $U \otimes \wedge^n Y$ is an H_2-comodule and, with the identification of U with X/Y, the multiplication of the algebra $\wedge Z$ yields an isomorphism from $U \otimes \wedge^n Y$ to $X(\wedge^n Y)$. Now, observe that $\wedge^n Y$ is a one-dimensional H_2-comodule that is extendable to H_1; by assumption, then, there is an H_1-comodule V that contains $(\wedge^n Y)^{\circ}$ as a sub H_2-comodule. It is thus clear that there is an embedding of U ($\cong U \otimes \wedge^n Y \otimes (\wedge^n Y)^{\circ}$) into the H_1-comodule $\wedge^{n+1} Z \otimes V$.

COROLLARY 1. *If the only one-dimensional H_2-comodule is the trivial comodule, then every morphism of Hopf algebras ϕ: $H_1 \to H_2$ is manageable.*

COROLLARY 2. *If H_1 is a pointed Hopf algebra, that is, if the simple H_1-comodules are one-dimensional, then every ϕ: $H_1 \to H_2$ is manageable.*

Proof. If a one-dimension H_2-comodule U embeds in some H_1-comodule, then it embeds in a simple H_1-comodule, say V. Since V is one-dimensional, the embedding of U into V is an H_2-comodule isomorphism which yields an embedding of U° as a sub H_2-comodule of the H_1-comodule V°.

At this stage, it is useful to simplify the problem by working over an algebraically closed field. This involves no loss of generality.

THEOREM 1.2. *Let ϕ: $L_1 \to L_2$ be a morphism of Lie algebras over K, and let ϕ^*: $H(L_2) \to H(L_1)$ be the corresponding map of representative functions. Suppose that either (a) ϕ sends the radical R_1 of L_1 into the radical R_2 of L_2, or* (b) *$L_1 = [L_1, L_1]$. Then the map ϕ^* is manageable.*

Proof. Condition (b) implies that the one-dimensional L_1-modules are trivial, i.e. that the one-dimensional $H(L_1)$-comodule are trivial. Therefore the manageability of ϕ^* in this case follows from Corollary 1 above.

Suppose now that we are in case (a). Let U be a one-dimensional L_1-module that embeds in an L_2-module V. We can assume that V is simple (consider a composition series for V). Then, in particular, $[L_2, R_2]$ annihilates V. By a well-known result, V will be semisimple as an R_2-module and consequently, as such, is a direct sum $\oplus V_i$ of one-dimensional sub R_2-modules. It is clear that we can embed U as a sub R_1-module into one of these R_2-modules V_i. Let S_1 be a maximal semisimple subalgebra of L_1 and let S_2 be a maximal semisimple subalgebra of L_2 that contains $\phi(S_1)$. Since U is a one-dimensional L_1-module, it is annihilated by S_1. Since $[L_1, L_2]$ annihilates V_i, we can extend the R_2-module structure on V_i to an L_2-module structure by making S_2 act trivially. Then the embedding of U into V_i is an embedding of L_1-modules. As we have seen in the proof of Corollary 2 above, it follows that the dual of U can be embedded in the dual of V_i. In view of Lemma 1.1, this completes the proof.

2. An analysis of the Hopf algebra of representative functions.

NOTATION. For any Lie algebra L, we denote by $Q(L)$ the (multiplicative) group of group-like elements of $H(L)$, and by $P(L)$ the (additive) group of primitive elements of $H(L)$. Note that there is an isomorphism of groups from $P(L)$ to $Q(L)$ given by the exponential map.

DEFINITION. ([**3**], [**5**], [**6**]). Let L be a Lie algebra over K. A subalgebra J of $H(L)$ is called a *basic subalgebra* if the multiplication map yields an algebra isomorphism from $J \otimes K[Q(L)]$ to $H(L)$. Let R denote the radical of L. A basic subalgebra J is called a *normal basic subalgebra* if the semisimple part J_s of J (i.e. the subalgebra consisting of representative functions belonging to semisimple representations of L) is exactly the left R-annihilated part $H(L)^R$ of $H(L)$ and if J is a left $H(L)$-comodule.

The main results on basic subalgebras that we need are that, for any Lie algebra L, a normal basic subalgebra of $H(L)$ always exists ([**6**, p. 610]), that any two normal basic subalgebras of $H(L)$ are conjugate via an automorphism of $H(L)$ of the form $\text{Exp}(t(x))$ where t is the left-translation map and x is in $[L, R]$ ([**3**, Thm. 4.1]), that every normal basic subalgebra contains the group $P(L)$ of primitive elements and is finitely generated as an algebra ([**6**, Thm. 4]).

The existence and conjugacy of normal basic subalgebras implies the existence of a unique small sub Hopf algebra $B(L)$ of $H(L)$ such that $B(L)$ contains some (and hence every) normal basic subalgebra of $H(L)$. We call $B(L)$ the *basic sub Hopf algebra* of $H(L)$.

In the rest of this section, L is a Lie algebra over K, R is the radical of L, $N = [L, R]$ (this coincides with the intersection of the kernels of all semisimple representations of$L$), t is the left translation map on $H(L)$ and t_r the right-translation map.

LEMMA 2.1. *Let H be a sub Hopf algebra of $H(L)$ and suppose that H contains a normal basic subalgebra J. Then, the intersection of H with $Q(L)$ is a set of free generators for H as a J-module.*

Proof. As is easy to see, it is sufficient to show that $H \cap Q(L)$ generates H as a J-module.

Since J is basic, every element of $H(L)$ can be written as a sum $\sum j_i q_i$, where the j_i's are in J and the q_i's in $Q(L)$. Now, $Q(L)$ is clearly contained in the right N-annihilated part ${}^N H(L)$ of $H(L)$, and repeated right-translation by elements of N will annihilate any element of $H(L)$. If the result of the lemma does not hold, then we can find an element h of H which has an expression as $\sum j_i q_i$, where not all the q_i's are in H, and among such elements $h = \sum j_i q_i$, we can pick one that is of minimal length and such that all of the j_i's lie in the right-N-annihilated part ${}^N J$ of J. The reason for making such a choice of h is that by [**3**, Lemma 4.3], ${}^N J = P(L)H(L)^R$, which is stable under both left and right translations.

Let x be an element of $U(L)$. Since H is two-sidedly stable, $t_r(x)h$ is also an element of H. Let δ denote the comultiplication map, and let $\delta(x) = \sum_\alpha x'_\alpha \otimes x''_\alpha$. Then,

$$t_r(x)h = \sum_{\alpha,i} t_r(x'_\alpha)(j_i)q_i(x''_\alpha)q_i.$$

Thus, multiplying by j_1, we see that H contains the element

$$\sum_{\alpha,i} j_1 t_r(x'_\alpha)(j_i)q_i(x''_\alpha)q_i.$$

H also contains the following, which is a J-multiple of h

$$\sum_{\alpha,i} j_i t_r(x'_\alpha)(j_1)q_1(x''_\alpha)q_i.$$

Subtracting the second of these from the first, and using the minimality of the length of the expression for h, we get

$$\sum_\alpha j_1 t_r(x'_\alpha)(j_i)q_i(x''_\alpha) = \sum_\alpha j_i t_r(x'_\alpha)(j_1)q_1(x''_\alpha).$$

If we evaluate each side of the above equation at an element y of $U(L)$, then, denoting $\delta(y)$ by $\sum_\beta y'_\beta \otimes y''_\beta$, we get

$$\sum_{\beta,\alpha} j_1(y'_\beta)j_i(x'_\alpha y''_\beta)q_i(x''_\alpha) = \sum_{\beta,\alpha} j_i(y'_\beta)j_1(x'_\alpha y''_\beta)q_1(x''_\alpha).$$

The above can be re-written.

$$\sum_\beta \{j_1(y'_\beta)t(y''_\beta)(j_i)q_i\}(x) = \sum_\beta \{j_i(y'_\beta)t(y''_\beta)(j_1)q_1\}(x).$$

Moreover, the $t(y)(j)$'s are in J since we have chosen the j's to be in a two-sidedly stable subspace of J. Owing to the freeness of the q's over J, it follows that, for all y in $U(L)$ and all $i > 1$,

$$\sum_\beta j_1(y'_\beta)t(y''_\beta)(j_i) = 0.$$

Applying this to the element 1 of $U(L)$, we obtain the equation $(j_1 j_i)(y) = 0$, for all y in $U(L)$. This means that $j_1 j_i$ must be the zero function, so that the chosen element h must be just $j_1 q_1$.

Now, we see that $(Hq_1 \cap H)$ is a non-zero left *Hopf module* for H, so that, by [**8**, Thm. 4.1.1], there is a non-zero element g of $(Hq_1 \cap H)$ such that $\delta(g) = 1 \otimes g$. This is possible only if q_1^{-1} is in H. Since H is closed under the antipode map, this means that q_1 is in H, which establishes the lemma.

LEMMA 2.2. *Let H be any sub Hopf algebra of $H(L)$ that separates the elements of L. Then, H contains the representative functions of the adjoint representation of L.*

Proof. The representative functions of the adjoint representation of L lie in the space of representative functions of the adjoint representation of the Lie algebra $L(H)$ of H, which clearly are contained in H.

LEMMA 2.3. *Let ρ^R be the restriction map $H(L) \to H(R)$. Then, ρ^R is injective on $Q(L)$, and there is a normal basic subalgebra J of $H(L)$ such that $\rho^R(J)$ is a normal basic subalgebra of $H(R)$.*

Proof. The first result is clear; the second follows from the constructions in [**5**] and [**6**].

For any L-module V, we denote by V' the *semisimple L-module associated with V*, i.e. the direct sum of the simple factor modules in a composition series for V. The following result is well known, but, in the absence of a convenient reference, we give a proof here.

LEMMA 2.4. *For any L-module V, the space* $\mathrm{Rep}(V)_s$ *of semisimple representative functions of V is identical with the space* $\mathrm{Rep}(V')$ *of representative functions of the associated semisimple L-module.*

Proof. For any L-modules U and W, we say that U is *subordinate* to W if U is isomorphic to a module obtained from W by a finite sequence of steps each of which is either the selection of a submodule, or the selection of a homomorphic image, or the direct sum of such modules. It is then straightforward to see that, if U is subordinate to W then $\mathrm{Rep}(U)$ is contained in $\mathrm{Rep}(W)$. Further, if U is semisimple and subordinate to W then U' is subordinate to W'.

The space $\mathrm{Rep}(V)$ is a direct sum of copies of homomorphic images of V, so is subordinate to V. Thus $\mathrm{Rep}(V)_s$ is subordinate to V, and, thus, to V'. But $\mathrm{Rep}(V)_s$ is a coalgebra, so is its own space of representative functions. This shows that $\mathrm{Rep}(V)_s$ is contained in $\mathrm{Rep}(V')$; the inclusion in the other direction is clear.

Let P be a solvable Lie algebra over K and V a semisimple P-module. Then the space of representative functions of V is spanned by elements of $Q(P)$ which we call the *component functions* of the representation. We denote by $A(P)$ the subgroup of $Q(P)$ that is generated by the component functions of the semisimple representation associated with the adjoint representation of P. By [7, Lemma 2.1], any P-module U is a direct sum of sub P-modules U_μ where the μ's are equivalence classes of component functions of U' modulo the group $A(P)$, and where the component functions of the P-module U_μ' lie in the class μ.

DEFINITION. Let L, R be as before. An L-module V is called an *essential L-module* if the component functions of V' as an R-module lie in $A(R)$. An element of $H(L)$ is called an *essential representative function* if it belongs to an essential L-module.

THEOREM 2.5. *The basic sub Hopf algebra $B(L)$ of $H(L)$ coincides with the Hopf algebra of all essential representative functions of L.*

Proof. Let $A(R)\#$ denote the inverse image (under the restriction map) in $Q(L)$ of the group $A(R)$. Let J be a normal basic subalgebra of $H(L)$ such that its restriction image $\rho^R(J)$ in $H(R)$ is a normal basic subalgebra (Lemma 2.3). We show first that $J[A(R)\#]$ in a Hopf algebra.

Clearly $J[A(R)\#]$ is stable under right-translations. To prove stability under left-translations is equivalent to proving the right-stability of $\eta(J)[A(R)\#]$, where η is the antipode map of $H(L)$. We claim first that every left sub R-module of $\eta(J)$ is an essential R-module. By the result quoted above from [7], it suffices to show that the component function of any one-dimensional sub R-moldule of $\eta(J)$ lies in $A(R)$. Let $u \in \eta(J)$ span a one-dimensional sub R-module with component function h, say. h is the restriction of an element of $H(L)$ and is semisimple; it is easy to see that h must be the restriction of a semisimple element of $H(L)$. Thus, $[L, R]$ annihilates h, and there is thus an element, g say, of $Q(L)$ that restricts to h. Then, $g^{-1}u$ is in $H(L)^R$ which is contained in $\eta(J)$. Since u is chosen to be in $\eta(J)$ this implies that $g = 1$. Now, for an element q of $Q(L)$, let π_q denote the projection from $H(L)$ to $\eta(J)q$. If $x \in L$, then

$\pi_q \circ t_r(x)$ is a left L-module endomorphism of $H(L)$, and so, maps essential R-modules to essential R-modules. It is clear then that $\pi_q \circ t_r(x)(\eta(J)) = (0)$ unless $q \in A(R)\#$. Thus, $J[A(R)\#]$ is stable under both left and right translations.

To complete the proof that $J[A(R)\#]$ is a Hopf algebra, we need to prove stability under the antipode η. Since we already have two-sided stability, it suffices to show that, whenever $J[A(R)\#]$ contains the representative functions of an L-module V, it also contains the representative functions of the dual module V^0. If V is an n-dimensional module, then the 'interior product' yields an isomorphism of L-modules $V^0 \otimes \wedge^n V \cong \wedge^{n-1} V$. The space $\wedge^n V$ is one-dimensional, so that its representative functions are the K-multiples of a group-like element q of $H(L)$. If the representative functions of V lie in $J[A(R)\#]$, then q^{-1} is in $A(R)\#$, and the representative functions for V^0 also lie in $J[A(R)\#]$.

Let f be an essential representative function, and let $T(f)$ be the L-module of left-translates of f. Let $\operatorname{Rep}(T(f))$ be the space of representative functions for $T(f)$; $\operatorname{Rep}(T(f))$ is a finite-dimensional sub coalgebra of $H(L)$. Clearly, there are finite dimensional sub coalgebras $Y_1, \dots, Y_n$ of $J[A(R)\#]$ and elements $q_1, \dots, q_n$ of $Q(L)$ that are distinct modulo $A(R)\#$ such that $\operatorname{Rep}(T(f))$ is contained in $\sum Y_i q_i$. Let π_i be the projection of $\operatorname{Rep}(T(f))$ onto $Y_i q_i$. Each π_i commutes with left (or right) translations so that the image of π_i is contained in $\operatorname{Rep}(T(f))$. Thus, we can assume that $\operatorname{Rep}(T(f))$ is the direct sum of the $Y_i q_i$'s and that none of the Y_i's are (0). The semisimple elements of $\operatorname{Rep}(T(f))$ are thus exactly $\Sigma(Y_i)_s q_i$ (see [**3**, Lemma 3.3]). Moreover, a non-zero coalgebra has a non-zero simple coalgebra so that none of the $(Y_i)_s$'s are (0). It is also clear that the restriction map ρ^R does not annihilate any of the $(Y_i)_s$'s (indeed, since $(Y_i)_s$ is stable under translations, evaluation at the element 1 of $U(L)$ is not the zero map).

Now, $J_s = H(L)^R$, so by [**3**, Lemma 3.3], $\rho^R(J[A(R)\#]) = K[A(R)]$. Since the element f above is essential, we must have that $\rho^R(\operatorname{Rep}(T(f))_s) \subset K[A(R)]$. From Lemma 2.1, and from the above remarks on the $(Y_i)_s$'s, we see that each q_i must be in $A(R)\#$, i.e. that $\operatorname{Rep}(T(f))$ must be contained in $J[A(R)\#]$. Thus, $J[A(R)\#]$ is the Hopf algebra of all essential representative functions of L. It remains only to show that $J[A(R)\#] = B(L)$, i.e. that there is no proper sub Hopf algebra of $J[A(R)\#]$ that contains J. Since J contains $P(L)$, it separates the elements of L (indeed, a zero of J is thus a zero of $Q(L)$, the exponential image of $P(L)$, and, thus, of all of $H(L)$). By Lemma 2.2, then, $\rho^R(B(L))$ must contain $A(R)$, so that, by Lemma 2.1, $B(L)$ must contain $A(R)\#$. This completes the proof.

THEOREM 2.6. *Let L be a Lie algebra over K, R the radical of L, and let V be an L-module. Let $\{q_1\#, \ldots, q_n\#\}$ be the subset of $Q(L)$ whose restriction image in $H(R)$ is the set of component functions of V' as an R-module. Then the space of representative functions of V is contained in $\Sigma_1^n B(L) q_i \#$.*

Proof. The proof is almost identical with the second part of the proof of Theorem 2.5. In fact, there are finite-dimensional sub coalgebras $Y_1, \ldots, Y_m$ of $B(L)$, and elements $p_1, \ldots, p_m$ of $Q(L)$ that are distinct modulo $A(R)\#$ and such that the representative functions of V are contained in $\Sigma Y_i p_i$. As in the proof of Theorem 2.5, we find that each p_i must be equivalent modulo $A(R)\#$ to one of the $q_j\#$'s. In view of Theorem 2.5, this completes the proof.

DEFINITION. A sub Lie algebra L_1 of a Lie algebra L_2 is called an *essential subalgebra* if the radical of L_1 is contained in the radical of L_2 and if every essential representation of L_2 restricts to an essential representation of L_1.

LEMMA 2.7. *A sub Lie algebra L_1 of a Lie algebra L_2 is an essential subalgebra if and only if the radical of L_1 is contained in that of L_2 and the adjoint representation of L_2 restricts to an essential representation of L_1.*

(The proof is straightforward.)

Note. It is an easy consequence of Lemma 2.7 that an ideal in a Lie algebra is an essential subalgebra.

3. Behavior of $H(L)$ with respect to restriction; the extension results.

THEOREM 3.1. *Let R_1, R_2 be solvable Lie algebras over K, let ϕ: $R_1 \to R_2$ be an injective morphism and let ϕ^* be the induced morphism of Hopf algebras $H(R_2) \to H(R_1)$. Then the basic sub Hopf algebra $B(R_1)$ of $H(R_1)$ is contained in $\phi^*(B(R_2))$.*

Proof. We prove this in two steps, according to the following two lemmas.

LEMMA 3.2. *Let ϕ: $L_1 \to L_2$ be an injection of Lie algebras such that $\phi(L_1)$ is an ideal of L_2. Then $B(L_1) = \phi^*(B(L_2))$.*

Proof. It is a consequence of the note at the end of §2 that $\phi^*(B(L_2))$ is contained in $B(L_1)$. By a theorem of Zassenhaus (see, for example, [**2**, Chap. I, §7]), every L_1-module on which $[L_2, L_2] \cap \mathrm{Rad}(L_1)$ acts nilpotently can be embedded in an L_2-module. In particular, this covers the case of an essential L_1-module. Thus, $B(L_1)$ is contained in $\phi^*(H(L_2))$. It is then easy from Theorem 2.5 to show that $B(L_1)$ must actually be contained in $\phi^*(B(L_2))$.

LEMMA 3.3. *Let* ϕ, R_1, R_2 *be as in the statement of Theorem* 3.1, *and let* ψ: $R_2 \to \mathrm{End}(V)$ *be a faithful representation of* R_2. *Let* R_1^+, R_2^+ *be the smallest algebraic subalgebras of* $\mathrm{End}(V)$ *to contain* $\psi(\phi(R_1)), \psi(R_2)$ *respectively, and let* ρ *be the restriction map from* $H(R_2^+)$ *to* $H(R_1^+)$. *Then,* $B(R_1^+)$ *is contained in* $\rho(B(R_2^+))$.

Proof. The idea of this lemma is that R_1^+ is sufficiently nicely embedded in R_2^+ to enable us to construct a normal basic subalgebra J_2 of $H(R_2^+)$ such that $\rho(J_2)$ is a normal basic subalgebra of $H(R_1^+)$. Specifically, each R_i^+ can be written as a semidirect sum of a nilpotent ideal X_i (that contains the commutator ideal) and an abelian subalgebra Y_i in such a way that $X_1 \subset X_2$ and $Y_1 \subset Y_2$. In [**6**, pp. 610–611], a normal basic subalgebra is constructed starting with an ordered basis of the Lie algebra. If we use the semidirect sum decompositions above for each R_i^+ in choosing the basis of R_i^+, we can construct normal basic subalgebras J_i of $H(R_i^+)$ such that $J_1 = \rho(J_2)$. The result follows immediately.

Proof of Theorem 3.1. We note that, in the notation of Lemma 3.3, $\psi \circ \phi$ is an injection of R_1 as an ideal of R_1^+, while ψ is an injection of R_2 as an ideal of R_2^+. By applying the result of Lemma 3.2 to both of these injections, we obtain Theorem 3.1 from Lemma 3.3.

Let ϕ: $S_1 \to S_2$ be an injection of semisimple Lie algebras over K, and let ϕ^*: $H(S_2) \to H(S_1)$ be the induced morphism of Hopf algebras. Clearly, $H(S)$ coincides with $B(S)$, and the group $G(S)$ of algebra homomorphisms from $H(S)$ to K is an affine algebraic group. By [**4**, Chap. XVIII], the Lie algebra of $G(S)$ is S. We see, then, that the injection ϕ induces a morphism of algebraic groups Φ: $G(S_1) \to G(S_2)$ whose kernel, T say, is a finite central subgroup of $G(S_1)$. Now, there are Cartan subalgebras C_1 of S_1 and C_2 of S_2 such that $\phi(C_1)$ is contained in C_2. Let Λ_i be the set of (integral) weights of S_i with respect to C_i (for $i = 1, 2$). Let ϕ^Λ be the restriction map from Λ_2 to Λ_1.

THEOREM 3.4. *In the above notation, if V is an S_1-module, then the space of representative functions of V is contained in $\phi^*(H(S_2))$ iff the weights of V are in $\phi^\Lambda(\Lambda_2)$.*

Proof. Let T_1, T_2 be the maximal toroids of $G(S_1), G(S_2)$ whose Lie algebras are C_1, C_2 respectively. Since the kernel T of the map Φ: $G(S_1) \to G(S_2)$ is finite and central, it is in T_1 and is, therefore, the kernel of the restriction θ: $T_1 \to T_2$ of Φ. For $i = 1, 2$, let $\chi(T_i)$ be the group of those polynomial characters of T_i that occur in restrictions to T_i of polynomial representations of $G(S_i)$. Then, θ induces a map θ^χ that sends $\chi(T_2)$ into $\chi(T_1)$. By means of the connection between finite-dimensional S_i-modules and $G(S_i)$-modules, there is an isomorphism of groups $\chi(T_i) \cong \Lambda_i$ that is compatible with the restriction maps θ^χ and ϕ^Λ.

Now, if a weight λ of V is in $\phi^\Lambda(\Lambda_2)$, then the corresponding character must be in $\theta^\chi(\chi(T_2))$ and vice versa. Since T is the kernel of the map θ, this means that T must act trivially on the λ-weight space of V. Since V is a sum of weight spaces, T will act trivially on V iff all the weights of V are in $\phi^\Lambda(\Lambda_2)$. It is clear from the theory of factor groups that $\phi^*(H(S_2))$ is the T-fixed part of $H(S_1)$, and, thus, that T acts trivially on V iff the representative functions of V are in $\phi^*(H(S_2))$. This completes the proof.

THEOREM 3.5. *Let ϕ: $L_1 \to L_2$ be an injection of Lie algebras over K and let S_1, S_2 be maximal semisimple subalgebras of L_1, L_2 respectively such that $\phi(S_1) \subset S_2$. Suppose that $L_1 = [L_1, L_1]$. Then, the representative functions for an L_1-module V lie in $\phi^*(H(L_2))$ iff the representative functions of V qua S_1-module lie in the restriction image $\phi_S^*(H(S_2))$ of $H(S_2)$ in $H(S_1)$.*

Proof. By [4, Chap. XVIII], $L_1 = [L_1, L_1]$ iff $H(L_1)$ is finitely generated as an algebra. Moreover, in such a case, the Lie algebra of $H(L_1)$ is L_1. Let $G(L_1)$, $G(L_2)$ be the pro-affine algebraic groups corresponding to the Hopf algebras $H(L_1)$, $H(L_2)$ respectively, and let Φ be the induced morphism $G(L_1) \to G(L_2)$. As in the proof of Theorem 3.4, the kernel, T say, of Φ is a finite central subgroup, and is thus contained in every maximal linearly reductive subgroup of $G(L_1)$.

Let $G(L_1) = G_u \cdot P$ be a decomposition of $G(L_1)$ as a semidirect product of its unipotent radical G_u and a maximal linearly reductive subgroup P. Since every (finite-dimensional) L_1-module is a $G(L_1)$-module, the Lie algebra of G_u is the intersection of the kernels of all semisimple L_1-modules. In the case where $L_1 = [L_1, L_1]$, this is the radical of L_1. Consequently, the (linearly reductive) subgroups corresponding to maximal semisimple subalgebras of L_1 are maximal linearly reductive subgroups. By the conjugacy of such subgroups, we see that we may suppose that the Lie algebra of P is S_1.

The injection ϕ_S: $S_1 \to S_2$ induces a morphism of algebraic groups Φ_S: $G(S_1) \to G(S_2)$, where the $G(S)$'s are as in Theorem 3.4. Let T_S be the kernel of Φ_S. Now, the injection of S_1 into L_1 induces an *injection* of $G(S_1)$ into $G(L_1)$, and similarly for S_2 (as follows from the fact that every S_i-module can be regarded as an L_i-module). The image of $G(S_1)$ in $G(L_1)$ is clearly P, and the map $G(S_1) \to P$ is an isomorphism. Now, we need only note that T is the kernel of the map (the restriction of Φ) from P to $G(L_2)$, while T_S is the kernel of the map from $G(S_1)$ to $G(S_2) \subset G(L_2)$. Therefore the isomorphism from $G(S_1)$ to P maps T_S onto T. Since $\phi^*(H(L_2))$ is the T-fixed part of $H(L_1)$, and $\phi_S^*(H(S_2))$ the T_S-fixed part of $H(S_1)$, the result of the theorem now follows.

We are now in a position to prove the extension theorem for representations of Lie algebras that was mentioned at the beginning.

Let L_1 be a subalgebra of a Lie algebra L_2 over K, and let R_1 and R_2 be the radicals of L_1 and L_2. Let S_1 and $_2$ be maximal semisimple subalgebras of L_1 and L_2 such that S_1 is contained in S_2. Let C_1 and C_2 be Cartan subalgebras of S_1 and S_2 such that C_1 is contained in C_2. Let V be a finite-dimensional L_1-module.

THEOREM 3.6. *In the above notation, assume that either* (a) R_1 *is contained in* R_2 *or* (b) $L_1 = [L_1, L_1]$. *Then* V *can be embedded as a sub* L_1*-module in a finite dimensional* L_2*-module iff both* (i) $[L_2, L_2] \cap R_1$ *acts nilpotently on* V *and* (ii) *the weights for* V *as a* C_1*-module are restrictions of integral weights for* C_2.

Proof. Condition (ii) is evidently necessary in all cases. In case (a), $[L_1, L_2] \cap R_1$ is contained in $[L_2, R_2]$ which acts nilpotently on any L_2-module, while, in case (b), all of R_1 necessarily acts nilpotently on an L_1-module. Thus, in both cases, conditions (i) and (ii) are necessary.

The sufficiency in case (b) is a consequence of Theorems 1.3, 3.4, and 3.5. We may restrict ourselves, then, to case (a).

The Levi decompositions (suppressing the indices 1 and 2) $L = R + S$ induce isomorphisms of algebras from $H(L)^R \otimes {}^SH(L)$ to $H(L)$, where $H(L)^R$ denotes the subspaces of $H(L)$ that is annihilated by left-translations by elements of R, and ${}^SH(L)$ the subspace annihilated by right-translations by elements of S (see [**4**, XVIII.4]). We need to make the isomorphisms explicit. Let ρ^R: $H(L) \to H(R)$ and ρ^S: $H(L) \to H(S)$ be the restrictions; we note that ρ^S is surjective. The restriction maps are pre-inverted by algebra isomorphisms j^R: $\rho^R(H(L)) \to {}^S H(L)$ and j^S: $H(S) \to H(L)^R$. If δ is the comultiplication and μ the multiplication on $H(L)$, then $\mu \circ \{(j^S \circ \rho^S) \otimes (j^R \circ \rho^R)\} \circ \delta$ is the identity map on $H(L)$.

Let ϕ denote the injection of L_1 into L_2, ϕ_R that of R_1 into R_2 and ϕ_S that of S_1 into S_2. By Theorem 2.6 the space of representative functions of V qua R_1-module is contained in $\sum B(R_1)q_i$, where the q_i's are the component functions of the associated semisimple R-module V'. By Theorem 3.1, $B(R_1)$ is contained in $\phi_R^*(B(R_2))$, while, by Lemma 3.2, $B(R_2) = \rho^R(B(L_2))$. If condition (i) holds, then the restriction to R_1 of each component function q_i is a Lie algebra homomorphism $R_1 \to K$ that annihilates $[L_2, L_2] \cap R_1$ and, thus, extends to a Lie algebra homomorphism $L_2 \to K$. This implies that each q_i is in $\phi_R^*(\rho^R(H(L_2)))$, whence all of the representative functions of V as an R_1-module lie in $\phi_R^*(\rho^R(H(L_2)))$. By Theorem 3.4, condition (ii) implies that the representative functions of V as an S_1-module lie in $\phi_S^*(H(S_2))$.

To complete the proof, we remark that the algebra homomorphism j^R: $\rho^R(H(L_1)) \to {}^S(H(L_1))$ maps $\phi_R^*(\rho^R(H(L_2)))$ into $\phi^*({}^S H(L_2))$; similarly, j^S maps $\phi_S^*(H(S_2))$ into $\phi^*(H(L_2)^R)$. We now apply the map $\mu \circ \{(j^S \circ \rho^S) \otimes (j^R \circ \rho^R)\} \circ \delta$ to the space of representative functions of V. Since the space of representative functions is a sub-coalgebra, δ sends it into its tensor square. Now ρ^R maps the space of representative functions of V into the space of representative functions of V qua R_1-module; similarly for ρ^S. It is clear from the above, then, that $\mu \circ \{(j^S \circ \rho^S) \otimes (j^R \circ \rho^R)\} \circ \delta$ maps the space of representative functions of V into $\phi^*(H(L_2))$. This map is, however, the identity, so, by Theorem 1.2, the proof is complete.

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