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ROTATION NUMBERS FOR AUTOMORPHISMS OF C^* ALGEBRAS

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Poincaré's notion of rotation number for a homeomorphism of the circle is generalized to a large class of automorphisms of C^* algebras. This is accomplished by the introduction of a C^* algebraic notion of determinant. A formula is obtained for the range of a trace on the K_0 group of a cross product by Z in terms of the rotation number of the automorphism involved.

Introduction. In [16] Poincaré introduced the notion of rotation number for homeomorphisms of the circle. The idea is to associate to any orientation-preserving homeomorphism of the circle a complex number of absolute value one which, in some sense, represents the average amount by which each individual point in the circle is "rotated" by the given homeomorphism. If R_θ denotes the rotation by the angle θ on the circle, that is, the transformation $z \to e^{i\theta}z$, we may compute its rotation number which, not surprisingly, turns out to be equal to $e^{i\theta}$.

Suppose we replace the circle by the 2-torus T^2 (viewed as the cartesian product of two circles) and let $R_{\eta,\theta}$ be the homeomorphism of T^2 which rotates the first and second circle coordinates by different angles η and θ. It seems plausible to assert that $R_{\eta,\theta}$ admits two rotation numbers, namely $e^{i\eta}$ and $e^{i\theta}$.

The reader could certainly think of other examples where rotation numbers may be heuristically defined. The discussion above suggests that there may be a general notion of rotation number for homeomorphisms of spaces other than the circle.

Compact topological spaces are in one-to-one correspondence with commutative C^* algebras containing a unit by **[5]**, so we may generalize further and ask whether it is possible to define a suitable notion of rotation number for automorphisms of any C^* algebra. This is precisely what is studied in this work.

As the example of the 2-torus suggests, there may be more than one rotation number involved. To account for this, our rotation number is defined to be a function into the circle instead of a single number, so we shall refer to it as the rotation number map.

Before giving the precise definition of the rotation number map we need to extend the notion of determinant to the context of C^* algebras and this turns out to be a very interesting problem in itself. The idea is to associate to a trace τ on a unital C^* algebra A a homomorphism, det, from the group of unitary matrices over A into the circle group which satisfies the familiar property $\det(e^{ih}) = e^{i\tau(h)}$ for all self adjoint matrices h over A. We find that this is not alwlays possible unless the pair (A, τ) satisfies a certain K-theoretical property which we call integrality. Roughly, this is an extension of the notion of connectedness to the category of C^* algebras. In precise terms, a traced unital C^* algebra (A, τ) is called integral if the range of the trace on $K_0(A)$ is contained in $\mathbf{Z}$.

Our main result, Theorem (V.13), gives necessary and sufficient conditions for an integral algebra to remain integral after we take its crossed product by an action of $\mathbf{Z}$. We show that this happens precisely when the rotation number map of the automorphism involved vanishes. We get the above result as a special case of Theorem (V.12) which is basically a formula for computing the range of a trace on the K_0 group of a crossed product algebra by $\mathbf{Z}$ in terms of the rotation number map of the automorphism involved. As one corollary we obtain a result of Rieffel, Pimsner and Voiculescu **[15, 18]** on the range of the trace on K_0 of irrational rotation algebras.

As indicated earlier, our definition of rotation number may be applied to homeomorphisms of compact topological spaces where an invariant measure is given. In this context we extend to compact spaces a result of Connes (Corollary 3 of **[1]**) on the nonexistence of proper projections in some crossed product algebras of the form $C(V) \times_\alpha \mathbf{Z}$ where V is a connected manifold for which $H^1(V, \mathbf{Z}) = 0$. We are thankful to the referee who pointed out that our initial result for CW-complexes could be extended to general compact spaces.

Another problem which we discuss (VIII.4) is the question of nonexistence of proper projections in the C^* algebras of torsion free groups, which are obtained as the semidirect product of free abelian groups and $\mathbf{Z}$, e.g., the discrete Heisenberg group. Although the class of groups mentioned above is much too small, we hope that our techniques may give some insight on the long standing conjecture according to which the result above holds for all torsion free groups. The next step should be, in our opinion, to prove that if a torsion free group G is such that its reduced C^* algebra contains no proper projections, then the same is true for a semidirect product of G by $\mathbf{Z}$.

We must mention that our work relies heavily on a paper by Paschke [**11**] as well as on the existence of the Pimsner-Voiculescu exact sequence for crossed products by $\mathbf{Z}$ [**14**]. Recent papers by Pimsner [**13**] and Packer [**25**] should also be noted as they are closely related to the present work. In [**13**] some of our results are generalized to crossed products by free groups.

Concerning our theory of determinants we should also mention a paper by P. de la Harpe and G. Skandalis [**6**] where they define a determinant which is closely related to ours, the difference being that our determinant is defined for all unitary matrices while theirs is defined only on a subgroup of the group of unitary matrices, namely the connected component of the identity. The price we pay is that our determinant is not unique. Advantages are that we can carry out a theory of determinants on algebras where a group action is given (see Chapter III) and this turns out to be related to the integrality of crossed product algebras (see (V.13.iv)).

The organization of the present work is as follows. In Chapter I we lay the groundwork for our theory of determinants which is exposed in Chapter II. Chapter III studies the behavior of determinants in the presence of a group action. Rotation numbers are defined and studied in Chapter IV. In Chapter V we use the ideas of the previous sections to arrive at our main results, which are (V.12) and (V.13). In Chapter VI we study the special case of commutative C^* algebras and give an alternative definition of rotation number. We also show how the two definitions of rotation number are related. In Chapters VII through IX we apply our main results to various classes of automorphisms of C^* algebras: almost periodic automorphisms, automorphisms induced by group automorphisms and automorphisms induced by affine transformations on topological groups.

We would now like to introduce some notation. If X is a compact topological space (always assumed to be Hausdorff) we denote by $C(X)$ the C^* algebra of all continuous complex valued functions on X.

If G is a topological group, we write $[G, G]$ for the commutator subgroup which is defined to be the closed subgroup generated by the set $\{ghg^{-1}h^{-1} : g, h \in G\}$. The connected component of the identity in G is denoted by G_0.

For any $*$-algebra A over the field of complex numbers, we let A_{sa} be the set of all self adjoint elements in A. For every natural number n we denote by $M_n(A)$ the $*$-algebra of all $n \times n$ matrices over A. If A is unital, that is, if A has an identity, $\mathrm{U}_n(A)$ stands for the group of all unitary $n \times n$ matrices over A. The identity $n \times n$ matrix and the zero element of $M_n(A)$ are denoted respectively by I_n and 0_n. If u is in $M_n(A)$ and v is in $M_m(A)$, we denote by $u \oplus v$ the $(n+m) \times (n+m)$ matrix over A containing u as the top left hand side $n \times n$ block, v as the bottom right hand side $m \times m$ block and zeros elsewhere.

A trace on a $*$-algebra A is a continuous linear map $\tau \colon A \to \mathbf{C}$ satisfying $\tau(ab) = \tau(ba)$ and $\tau(a^*) = \overline{\tau(a)}$ for all a and b in A. Given a trace τ on A we automatically assume that τ is extended to $M_n(A)$ for all n by the formula $\tau(a) = \sum_{1 \leq i \leq n} \tau(a_{i,i})$ for all $a = (a_{i,j})_{1 \leq i,j \leq n} \in M_n(A)$.

A trace is called positive if $\tau(a^*a) \geq 0$ for all a in A. If, moreover, $\tau(a^*a) > 0$ for $a \neq 0$ we say that τ is faithful. A trace defined on a unital algebra A is said to be normalized if $\tau(1) = 1$.

We shall assume our traces to be normalized whenever working with a unital algebra. Positiveness and faithfulness are assumed only where explicitly mentioned.

If τ is a fixed trace on A, we say that the pair (A, τ) is a traced algebra.

The K-theory groups of a given C^* algebra A are denoted by $K_0(A)$ and $K_1(A)$. If p is a self adjoint projection in some matrix algebra over A, we denote by $[p]_0$ its class in $K_0(A)$. Likewise, for any unitary matrix u over A the symbol $[u]_1$ stands for its class in $K_1(A)$.

A trace τ on A defines a group homomorphism (also denoted by τ) $\tau \colon K_0(A) \to \mathbf{R}$ via the formula $\tau([p]_0 - [q]_0) = \tau(p) - \tau(q)$ for all self adjoint projections p and q in some matrix algebra over A.

An automorphism of a $*$-algebra A is always assumed to preserve the $*$-operation. If α is such an automorphism we shall not introduce any extra notation for the induced automorphism on $M_n(A)$. That is, for $a = (a_{i,j})_{1 \leq i,j \leq n}$ in $M_n(A)$ we let $\alpha(a) = (\alpha(a_{i,j}))_{1 \leq i,j \leq n}$.

Our general references for C^* algebras are [**3**, **12**, **20**]. References for K-theory are [**2**, **7**, **9**, **10**, **24**].

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I. WINDING NUMBERS

In this chapter we develop the technical tools needed for our theory of determinants for C^* algebras.

Let A be a unital C^* algebra. We denote by $C(T, A)$ the C^* algebra of all continuous functions from the circle

$$T = \{z \in \mathbf{C} : |z| = 1\}$$

into A. Whenever it is convenient we will identify $C(T, A)$ with the algebra of periodic continuous functions from $\mathbf{R}$ into A with period 2π. We use the notation $C^\infty(T, A)$ to denote the subalgebra of $C(T, A)$ formed by C^∞ functions from T to A.

Let τ be a trace on A.

1. DEFINITION. If $n \in \mathbf{N}$, $n \geq 1$ and $u \in \mathrm{U}_n(C^\infty(T, A))$, we define the winding number of u with respect to τ by

$$\omega_\tau^n(U) = \frac{1}{2\pi i} \int_0^{2\pi} \tau\big(u'(t)u(t)^*\big)\, dt.$$

Therefore ω_τ^n is a map

$$\omega_\tau^n : \mathrm{U}_n(C^\infty(T, A)) \to \mathbf{C}.$$

Some properties of the winding number map are collected in the next

2. LEMMA. *Let $n \in \mathbf{N}$, $n \geq 1$ and $u \in \mathrm{U}_n(C^\infty(T, A))$.*

(i) *If $m > n$ and $v = u \oplus I_{m-n} \in \mathrm{U}_m(C^\infty(T, A))$ then $\omega_\tau^m(v) = \omega_\tau^n(u)$.*

(ii) *If $h \in M_n(C^\infty(T, A))_{\mathrm{sa}}$ then $e^{ih} \in \mathrm{U}_n(C^\infty(T, A))$ and $\omega_\tau^n(u) = \omega_\tau^n(ue^{ih})$.*

Proof. (i) Note that for all $t \in \mathbf{R}$

$$\begin{aligned} v'(t)v(t)^* &= \big(u'(t) \oplus 0_{m-n}\big)\big(u(t)^* \oplus I_{m-n}\big) \\ &= \big(u'(t)u(t)^*\big) \oplus 0_{m-n}. \end{aligned}$$

Therefore $\tau(v'(t)v(t)^*) = \tau(u'(t)u(t)^*)$ proving (i).

(ii) The fact that $e^{ih} \in \mathrm{U}_n(C^\infty(T, A))$ is a consequence of the fact that the exponential is a C^∞ map. We have

$$\begin{aligned}
\omega_\tau^n(ue^{ih}) &= \frac{1}{2\pi i}\int_0^{2\pi} \tau\left(\frac{d}{dt}\left(u(t)e^{ih(t)}\right)\left(u(t)e^{ih(t)}\right)^*\right) dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \tau\left(\left(u'(t)e^{ih(t)} + u(t)ih'(t)e^{ih(t)}\right)e^{-ih(t)}u(t)^*\right) dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \tau\left(u'(t)u(t)^*\right) dt + \frac{1}{2\pi i}\int_0^{2\pi} \tau\left(u(t)ih'(t)u(t)^*\right) dt \\
&= \omega_\tau^n(u) + \frac{1}{2\pi i}\int_0^{2\pi} i\tau\left(h'(t)\right) dt \\
&= \omega_\tau^n(U) + \frac{1}{2\pi}(\tau(h(2\pi)) - \tau(h(0))) = \omega_\tau^n(u).
\end{aligned}$$

Our goal is to construct out of ω_τ^n a map

$$\gamma\colon K_1(C(T, A)) \to \mathbf{R}.$$

The next result will be useful in that direction.

3. LEMMA. *Let* $n \in \mathbf{N}$, $n \geq 1$. *Then*

(i) $\mathrm{U}_n(C^\infty(T, A))$ *is dense in* $\mathrm{U}_n(C(T, A))$ *with respect to the norm topology.*

(ii) *For any* $u \in \mathrm{U}_n(C(T, A))$ *there is* $v \in \mathrm{U}_n(C^\infty(T, A))$ *such that the classes* $[u]_1$ *and* $[v]_1$ *in* $K_1(C(T, A))$ *are the same.*

(iii) *If* $u \in \mathrm{U}_n(C^\infty(T, A)) \cap \mathrm{U}_n(C(T, A))_0$ *there are elements* $h_1, \dots, h_m \in M_n(C^\infty(T, A))_{\mathrm{sa}}$ *such that* $u = e^{ih_1} \cdots e^{ih_n}$.

Proof. (i) it is well known that $C^\infty(T, A)$ is dense in $C(T, A)$ so that also $M_n(C^\infty(T, A))$ is dense in $M_n(C(T, A))$. As a consequence $\mathrm{GL}_n(C^\infty(T, A))$ is dense in $\mathrm{GL}_n(C(T, A))$ since the latter is open. The map

$$u \in \mathrm{GL}_n(C(T, A)) \to u(u^*u)^{1/2} \in \mathrm{U}_n(C(T, A))$$

is a continuous retraction and clearly leaves $\mathrm{U}_n(C^\infty(T, A))$ invariant. The image of $\mathrm{GL}_n(C^\infty(T, A))$ is then $\mathrm{U}_n(C^\infty(T, A))$, which therefore is dense in $\mathrm{U}_n(C(T, A))$.

(ii) Follows from (i).

(iii) Consider the subgroup S of $\mathrm{U}_n(C(T, A))$ defined by

$$S = \mathrm{U}_n(C^\infty(T, A)) \cap \mathrm{U}_n(C(T, A))_0.$$

With the norm topology S becomes a topological group. Using the fact that in a connected group a neighborhood of the identity is always a generating set, it is enough to prove the following statements:

(a) S is path connected.

(b) The set $\{e^{ih}: h \in M_n(C^\infty(T, A))_{\mathrm{sa}}\}$ is a neighborhood of the identity in S.

We prove (b) first. If $u \in S$ is such that $\|u - 1\| < 1$ the series

$$(1/i) \sum_{1 \leq p < \infty} (1/p)(1 - u)^p$$

is absolutely convergent and adds up to a self adjoint element $h \in M_n(C^\infty(T, A))$ satisfying $u = e^{ih}$.

To prove (a) let $u \in S$ and let $\{u_t: 0 \leq t \leq 1\}$ be a continuous path in $\mathrm{U}_n(C(T, A))$ with $u_0 = u$ and $u_1 = 1$. Take a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ with $\|u_{t_j} - u_{t_{j-1}}\| < 1/3$, $j = 1, 2, \ldots, k$ and by (i) let $v_j \in \mathrm{U}_n(C^\infty(T, A))$ be such that $\|v_j - u_{t_j}\| < 1/3$, $j = 0, 1, \ldots, k$ where $v_0 = u$, $v_k = 1$. Then $\|v_j - v_{j-1}\| < 1$ so $\|v_j(v_{j-1})^{-1}\| < 1$ and by (b) $v_j = e^{ih_j}v_{j-1}$ for some $h_j \in M_n(C^\infty(T, A))_{\mathrm{sa}}$.

The path $\{e^{ith_j}v_{j-1}: 0 \leq t \leq 1\}$ connects v_{j-1} and v_j and stays within $\mathrm{U}_n(C^\infty(T, A))$. Putting together all these paths we have connected v_0 to v_k, that is, u to 1, by a path in $\mathrm{U}_n(C^\infty(T, A))$ which certainly lies in $\mathrm{U}_n(C(T, A))_0$. In other words our path lies in S.

We are now ready to make the following

4. Definition. Given a traced unital C^* algebra (A, τ) we denote by γ the mapping

$$\gamma: K_1(C(T, A)) \to \mathbf{C}$$

given by $\gamma(x) = \omega_\tau^n(u)$ where n and u are chosen so that

$$u \in U_n(C^\infty(T, A))$$

and $[u]_1 = x$.

Note that (3.ii) provides such a u, while (2) shows that the definition of γ does not depend on the choice of u and n. Therefore γ is well defined.

A useful result in K-theory of C^* algebras states that $K_1(C(T, A))$ is isomorphic to $K_1(A)$ via an isomorphism

$$k: K_1(A) \oplus K_0(A) \to K_1(C(T, A))$$

satisfying $k([u]_1, [p]_0) = [u \otimes 1]_1 + [e^{itp}]_1$ where e^{itp} stands for the map $t \to e^{itp}$.

In our next proposition we investigate the relationship between k and γ.

5. PROPOSITION. *The following diagram is commutative*:

$$\begin{array}{ccc} K_1(C(T,A)) & \xrightarrow{\gamma} & \mathbf{C} \\ \uparrow k & & \nearrow_{0\oplus\tau} \\ K_1(A)\oplus K_0(A) & & \end{array}$$

Proof. Let $u \in \mathrm{U}_n(A)$, and let $p \in M_n(A)$ be a self adjoint projection. Then

$$\begin{aligned}\gamma(k([u]_1,[p]_0)) &= \gamma([u\oplus 1]_1 + [e^{itp}]_1) = \gamma([ue^{itp}]_1) \\ &= \omega_\tau^n(ue^{itp}) = \frac{1}{2\pi i}\int_0^{2\pi} \tau\left(\frac{d}{dt}(ue^{itp})e^{-itp}u^*\right)dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \tau(uipu^*)\,dt = \tau(p).\end{aligned}$$

6. COROLLARY. (i) *γ is a group homomorphism into the additive group of complex numbers*.

(ii) *The range of γ is $\tau(K_0(A))$*.

(iii) *If $h_1,\dots,h_m \in M_n(A)_{\mathrm{sa}}$ and $e^{ih_1}\cdots e^{ih_m} = 1$ then*

$$(1/2\pi)\tau\Big(\sum_{1\le j\le m} h_j\Big) \in \tau(K_0(A)).$$

Proof. (i) and (ii) follow immediately from (5). To prove (iii) let $u(t) = e^{ith_1/2\pi}\cdots e^{ith_m/2\pi}$, $t\in[0,2\pi]$. So $u \in \mathrm{U}_n(C^\infty(T,A))$ and we have

$$\begin{aligned}\gamma([u]_1) &= \omega_\tau^n(u) = \frac{1}{2\pi i}\int_0^{2\pi}\tau\big(u'(t)u(t)^*\big)\,dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi}\tau\Big(\sum_{1\le j\le m} e^{ith_1/2\pi}\cdots(ih_j/2\pi)e^{ith_j/2\pi}\cdots e^{ith_m/2\pi} \\ &\qquad\qquad \cdot e^{-ith_m/2\pi}\cdots e^{-ith_1/2\pi}\Big)\,dt \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\sum_{1\le j\le m}\tau(h_j) = \frac{1}{2\pi}\tau\Big(\sum_{1\le j\le m} h_j\Big).\end{aligned}$$

The result now follows from (ii).

A last technical result to be used later will occupy our attention in the remainder of this section.

7. LEMMA. *Let (A, τ) be a traced unital C^* algebra. Given u and v in $U_n(A)$ there are $h_1, \ldots, h_m \in M_{2n}(A)_{\mathrm{sa}}$ such that $(uvu^*v^*) \oplus I_n = e^{ih_1} \cdots e^{ih_m}$ and $\tau(\Sigma_{1 \le j \le m} h_j) = 0$.*

Proof. It is well known **[10]** that $v \oplus v^* \in \mathrm{U}_{2n}(A)_0$. So let $k_1, \ldots, k_r \in M_{2n}(A)_{\mathrm{sa}}$ satisfying $v \oplus v^* = e^{ik_1} \cdots e^{ik_r}$. Then

$$\begin{aligned} uvu^*v^* \oplus I_n &= (u \oplus I_n)(v \oplus v^*)(u^* \oplus I_n)(v^* \oplus v) \\ &= (u \oplus I_n)e^{ik_1} \cdots e^{ik_r}(u \oplus I_n)^* e^{-ik_r} \cdots e^{-ik_1} \\ &= e^{i(u \oplus I_n)k_1(u \oplus I_n)^*} \cdots e^{(u \oplus I_n)k_r(u \oplus I_n)^*} e^{-ik_r} \cdots e^{-ik_1}. \end{aligned}$$

If we put $h_j = (u \oplus I_n)k_j(u \oplus I_n)^*$ for $1 \le j \le r$ and $h_j = -k_{2r-j+1}$ for $r + 1 \le j \le 2r$ we have the desired conclusion.

8. COROLLARY. *Let (A, τ) be a traced unital C^* algebra and $u, v \in \mathrm{U}_n(A)$. Suppose we are given elements $k_1, \ldots, k_r \in M_n(A)_{\mathrm{sa}}$ such that $uvu^*v^* = e^{ik_1} \cdots e^{ik_r}$. Then*

$$\frac{1}{2\pi}\tau\Big(\sum_{1 \le j \le r} k_j\Big) \in \tau(K_0(A)).$$

Proof. Let $h_1, \ldots, h_m \in M_{2n}(A)$ be as in (7). We have

$$e^{ik_1 \oplus 0_n} \cdots e^{ik_r \oplus 0_n} = (uvu^*v^*) \oplus I_n = e^{ih_1} \cdots e^{ih_m}.$$

By (6.iii) we have

$$\frac{1}{2\pi}\tau\Big(\sum_{1 \le j \le r} k_j\Big) = \frac{1}{2\pi}\tau\Big(\sum_{1 \le j \le r} (k_j \oplus 0_n) - \sum_{1 \le j \le m} h_j\Big) \in \tau(K_0(A)).$$

As a consequence we obtain a result of Rieffel **[18]** on the range of the trace on K_0 of the irrational rotation algebras, as well as the rational ones. Recall that for every $\theta \in \mathbf{R}$ the algebra A_θ is defined to be the crossed product of $C(T)$ by $\mathbf{Z}$ where the action is by rotation by θ. For our purposes it is enough to know that A_θ contains two unitaries u and v such that $uv = e^{2\pi i\theta}vu$.

9. COROLLARY. *The range of the trace on $K_0(A_0)$ contains $\mathbf{Z} + \theta\mathbf{Z}$.*

Proof. If u and v are as described above, then $uvu^*v^* = e^{2\pi i\theta}$. By (8) we have $\theta = (1/2\pi)\tau(2\pi\theta) \in \tau(K_0(A_\theta))$.

Since it is obvious that $\mathbf{Z} \subseteq \tau(K_0(A_\theta))$ the result follows.

See (IX.12) for a proof that $\tau(K_0(A_\theta))$ is actually equal to $\mathbf{Z} + \theta\mathbf{Z}$.

II. Determinants

In studying the algebra of complex matrices one cannot avoid the use of traces and determinants. While the generalizations of the concept of a trace to the context of C^* algebras have proved to be extremely useful, the theory of determinants on C^* algebras is only now beginning to be studied (see [**6**] as well as (11)). Concerning von Neumann algebras we must mention a paper by B. Fuglede and R. V. Kadison on determinants for finite factors [**4**], but as we shall see (Theorem 10), it is more natural to study determinants in the realm of C^* algebras rather than for von Neumann algebras since the existence of too many projections in the algebra constitutes an obstruction to the existence of determinants with certain properties. We thus dedicate this chapter to the study of determinants on C^* algebras. The results we obtain here will prove to be crucial for the forthcoming sections.

Let A be a unital C^* algebra.

1. Definition. We denote by $\mathrm{U}(A)$ or by $\mathrm{U}_\infty(A)$ the unitary group of A defined to be the inductive limit of the sequence of groups

$$\mathrm{U}_1(A) \xrightarrow{i_1} \mathrm{U}_2(A) \xrightarrow{i_2} \cdots \mathrm{U}_n(A) \xrightarrow{i_n} \mathrm{U}_{n+1}(A) \xrightarrow{i_{n+1}} \cdots$$

where, for all n, i_n is defined by

$$i_n(u) = u \oplus I_1 \in \mathrm{U}_{n+1}(A), \qquad \forall u \in \mathrm{U}_n(A).$$

A concrete realization of $\mathrm{U}(A)$ may be obtained by taking the set of all $\infty \times \infty$ matrices u over A which agree with the $\infty \times \infty$ identity, $I_\infty = (\delta_{ij})_{i,j \in \mathbf{N}}$, except for finitely many entries and moreover satisfy $uu^* = u^*u = I_\infty$.

The product as well as the $*$ operation are defined as usual, observing that in the definition of the product, the infinite sums involved contain only a finite number of nonzero terms.

Let τ be a trace on A.

2. Definition. We say that a group homomorphism

$$\det : \mathrm{U}(A) \to T$$

is a determinant associated with the trace τ if for all $h \in M_n(A)_{\mathrm{sa}}$ one has

$$\det(e^{ih}) = e^{i\tau(h)}.$$

We now develop some concepts to be used in the study of determinants.

3. Definition. Let (A, τ) be a traced unital C^* algebra. For every integer $n \geq 1$ we let

$$\mathrm{SU}_n^\tau(A) = \left\{ e^{ih_1} \cdots e^{ih_k} : k \in \mathbf{N},\ h_1, \ldots, h_k \in M_n(A)_{\mathrm{sa}}, \right.$$

$$\left. \tau\left(\sum_{1 \leq j \leq k} h_j \right) = 0 \right\}$$

4. Proposition. *For all integers $n \geq 1$*

(i) $\mathrm{SU}_n^\tau(A)$ *is a normal subgroup of* $\mathrm{U}_n(A)$,

(ii) $\mathrm{SU}_n^\tau(A)$ *is connected with respect to the norm topology*,

(iii) $\mathrm{SU}_n^\tau(A)$ *is contained in* $\mathrm{U}_n(A)_0$ *and*

(iv) $\mathrm{SU}_{2n}^\tau(A)$ *contains* $[\mathrm{U}_n(A), \mathrm{U}_n(A)] \oplus I_n$.

Proof. (i) It is clear that $\mathrm{SU}_n^\tau(A)$ is a subgroup. If $u \in \mathrm{U}_n(A)$ and $v = e^{ih_1} \cdots e^{ih_k} \in \mathrm{SU}_n^\tau(A)$ where $\tau(\Sigma_{1 \leq j \leq k} h_j) = 0$, we have

$$uvu^* = e^{iuh_1u^*} \cdots e^{iuh_ku^*}.$$

Since $\tau(\Sigma_{1 \leq j \leq k} uh_ju^*) = \tau(\Sigma_{1 \leq j \leq k} h_j) = 0$ it follows that uvu^* is another element of $\mathrm{SU}_n^\tau(A)$, so $\mathrm{SU}_n^\tau(A)$ is normal.

(ii) Let $u \in \mathrm{SU}_n^\tau(A)$. Then there are $h_1 \cdots h_k \in M_n(A)_{\mathrm{sa}}$ with $\tau(\Sigma_{1 \leq j \leq k} h_j) = 0$ such that $u = e^{ih_1} \cdots e^{ih_k}$. The path $u_t = e^{ith_1} \cdots e^{ith_k}$, $t \in [0, 1]$, is contained in $\mathrm{SU}_n^\tau(A)$, is norm continuous and joins I_n to u. So $\mathrm{SU}_n^\tau(A)$ is connected.

(iii) Follows immediately from (ii).

(iv) Let $u, v \in \mathrm{U}_n(A)$. According to Lemma (I.7) there are elements $h_1, \ldots, h_m \in M_{2n}(A)_{\mathrm{sa}}$ such that $\tau(\Sigma_{1 \leq j \leq m} h_j) = 0$ and $(uvu^*v^*) \oplus I_n = e^{ih_1} \cdots e^{ih_m}$. Thus $(uvu^*v^*) \oplus I_n \in \mathrm{SU}_{2n}^\tau(A)$.

The question of whether $\mathrm{SU}_n^\tau(A)$ is closed in $\mathrm{U}_n(A)$ arises naturally at this point but it turns out not to be important for our purposes. A discussion of this question may be found in Appendix B.

Observe that for all $n \geq 1$ the mapping i_n defined in (1) carries $\mathrm{SU}_n^\tau(A)$ into $\mathrm{SU}_{n+1}^\tau(A)$. Therefore i_n passes to the quotient, giving a group homomorphism, still denoted by i_n

$$i_n : \mathrm{U}_n(A)/\mathrm{SU}_n^\tau(A) \to \mathrm{U}_{n+1}(A)/\mathrm{SU}_{n+1}^\tau(A).$$

Each of these quotients are groups by (4.i).

5. DEFINITION. We denote by $K_1^\tau(A)$ the inductive limit of the sequence

$$\mathrm{U}_1(A)/\mathrm{SU}_1^\tau(A) \xrightarrow{i_1} \cdots \mathrm{U}_n(A)/\mathrm{SU}_n^\tau(A) \xrightarrow{i_n} \mathrm{U}_{n+1}(A)/\mathrm{SU}_{n+1}^\tau(A) \xrightarrow{i_{n+1}} \cdots .$$

If $u \in \mathrm{U}_n(A)$ we use the symbol $[u]_1^\tau$ to denote the class of u in $K_1^\tau(A)$.

Given $x \in K_1^\tau(A)$, let $n \in \mathbf{N}$ and $u \in \mathrm{U}_n(A)$ be such that $x = [u]_1^\tau$ and put $\pi(x) = [u]_1 \in K_1(A)$.

6. PROPOSITION. (i) *$K_1^\tau(A)$ is an abelian group.*

(ii) *π is well defined and is a group homomorphism from $K_1^\tau(A)$ to $K_1(A)$ which moreover is surjective.*

(iii) *K_1^τ is a covariant functor from the category of traced unital C^* algebras with unital, trace preserving homomorphisms, to the category of abelian groups, and π is a natural transformation of functors from K_1^τ to K_1.*

Proof. The proof is essentially contained in (4). We simply observe that (i) follows from (4.iv) while (ii) is a consequence of (4.iii) except for the surjectivity which is trivial. We leave the proof of (iii) to the reader.

In our next step we consider the mapping

$$j : \lambda \in T \to [\lambda]_1^\tau \in K_1^\tau(A)$$

where we view λ as a unitary 1×1 matrix over A.

7. PROPOSITION. *The sequence*

$$0 \to \exp(2\pi i \tau(K_0(A))) \to T \xrightarrow{j} K_1^\tau(A) \xrightarrow{\pi} K_1(A) \to 0$$

is exact.

Proof. We already know that π is onto. Clearly $\pi \cdot j = 0$. To prove $\mathrm{Ker}(\pi) \subseteq \mathrm{Im}(j)$ let $u \in \mathrm{U}_n(A)$, $(n \in \mathbf{N})$ be such that $\pi([u]_1^\tau) = [u]_1 = 0$. So, for some $m \in \mathbf{N}$, $u \oplus I_m \in \mathrm{U}_{n+m}(A)_0$, and we can find elements $h_1, \ldots, h_k \in M_{n+m}(A)_{\mathrm{sa}}$ such that $u \oplus I_m = e^{ih_1} \cdots e^{ih_k}$. Let $\lambda = \exp(i\tau(\Sigma_{1 \le j \le k} h_j))$. Then $\lambda \in T$ and

$$\begin{aligned}(u \oplus I_m)&(\lambda \oplus I_{m+n-1})^{-1}\\ &= e^{ih_1} \cdots e^{ih_k} \exp\left(-i\left(\tau\left(\sum_{1 \le j \le k} h_j\right) \oplus 0_{m+n-1}\right)\right)\end{aligned}$$

which shows that $[u \oplus I_m]_1^\tau = [\lambda \oplus I_{m+n-1}]_1^\tau$, or just that $[u]_1^\tau = [\lambda]_1^\tau = j(\lambda)$, proving exactness at $K_1^\tau(A)$. Moving our attention now to T, let $t \in \tau(K_0(A))$. We will prove that $j(e^{2\pi it}) = 0$. Let $n \in \mathbf{N}$ and let $p, q \in M_n(A)$ be self adjoint projections with $\tau(p) - \tau(q) = t$. Note that $e^{2\pi ip}e^{-2\pi iq}e^{-2\pi i(t\oplus 0_{n-1})} \in \mathrm{SU}_n^\tau(A)$. But $e^{2\pi ip} = e^{-2\pi iq} = I_n$. Thus

$$j(e^{2\pi it}) = \left[e^{2\pi it}\right]_1^\tau = \left[e^{2\pi i(t\oplus 0_{n-1})}\right]_1^\tau = 0.$$

Conversely let $\lambda \in T$ be such that $j(\lambda) = 0$. We will prove that $\lambda \in \exp(2\pi i\tau(K_0(A)))$. Since $j(\lambda) = 0$, there will be an integer $n \geq 1$ and $h_1, \dots, h_k \in M_n(A)_{\mathrm{sa}}$ such that $\tau(\sum_{1\leq j\leq k} h_i) = 0$ and $\lambda \oplus I_{n-1} = e^{ih_1} \cdots e^{ih_k}$. Write $\lambda = e^{2\pi i\theta}$ for $\theta \in \mathbf{R}$, and note that

$$e^{2\pi i(\theta\oplus 0_{n-1})}e^{-ih_k} \cdots e^{-ih_1} = I_n.$$

Using corollary (I.6.iii) we have

$$\frac{1}{2\pi}\left(2\pi\theta - \tau\left(\sum_{1\leq i\leq k} h_i\right)\right) \in \tau(K_0(A)),$$

that is, $\theta \in \tau(K_0(A))$, proving that $\lambda \in \exp(2\pi i\tau(K_0(A)))$. This concludes the proof.

As a consequence we have

8. COROLLARY. *The sequence*

$$0 \to T \xrightarrow{j} K_1^\tau(A) \xrightarrow{\pi} K_1(A) \to 0$$

is exact if and only if $\tau(K_0(A)) \subseteq \mathbf{Z}$.

We thus arrive at an important point of our study. We will see that the equivalent conditions of the corollary above will play a central role in the theory of determinants. The following definition is intended to single out the class of C^* algebras to which we will direct our attention.

Unfortunately most von Neumann algebras will be ruled out. What follows, especially Theorem 10, should explain the difficulty Fuglede and Kadison [4] had in obtaining T valued determinants on von Neumann algebras.

9. DEFINITION. Let (A, τ) be a traced unital C^* algebra. We say that (A, τ) is integral if $\tau(K_0(A)) \subseteq \mathbf{Z}$.

See (VI.1) for a characterization of integral commutative C^* algebras.

10. THEOREM. *Let (A, τ) be a traced unital C^* algebra. Then A admits a determinant associated with τ if and only if (A, τ) is integral. In this case any determinant is continuous on $\mathrm{U}_n(A)$ for all $n \in \mathbf{N}$. Moreover, given a determinant $\det_0$ all other determinants are given by*

$$\det(u) = \det_0(u)\phi([u]_1), \qquad \forall u \in \mathrm{U}(A)$$

where ϕ is a fixed group homomorphism from $K_1(A)$ into T.

Proof. Suppose (A, τ) admits a determinant, denoted by det. Let $p \in M_n(A)$ be a self adjoint projection. Then $e^{2\pi i p} = I_n$ so $\det(e^{2\pi i p}) = \det(I_n) = 1$. Equivalently $e^{2\pi i \tau(p)} = 1$, which implies that $\tau(p) \in \mathbf{Z}$, and thus proves (A, τ) to be integral.

Conversely assume that (A, τ) is integral. By (8) the sequence

$$0 \to T \xrightarrow{j} K_1^\tau(A) \xrightarrow{\pi} K_1(A) \to 0$$

is exact. Because T is an injective group in the category of abelian groups [**19**, p. 184], the exact sequence above splits.

So let $\Delta\colon K_1^\tau(A) \to T$ be a group homomorphism satisfying $\Delta \cdot j = \mathrm{id}_T$. Define $\det\colon \mathrm{U}(A) \to T$ by the formula

$$\det(u) = \Delta([u]_1^\tau) \quad \text{for } u \in \mathrm{U}(A).$$

We claim that det is a determinant associated with τ. In fact let $h \in M_n(A)_{\mathrm{sa}}$. Set $v = e^{ih}$ and $\lambda = e^{i\tau(h)} \in T$. Then

$$(\lambda \oplus I_{n-1})v^{-1} = e^{i(\tau(h)\oplus 0_{n-1})}e^{-ih} \in \mathrm{SU}_n^\tau(A).$$

Therefore $[\lambda]_1^\tau = [\lambda \oplus I_{n-1}]_1^\tau = [v]_1^\tau$. It follows that

$$\det(v) = \Delta([v]_1^\tau) = \Delta([\lambda]_1^\tau) = \Delta(j(\lambda)) = \lambda.$$

In other words $\det(e^{ih}) = e^{i\tau(h)}$. This proves that det is in fact a determinant for A associated with τ.

To prove that any given determinant det is continuous we must prove that it is continuous on $\mathrm{U}_n(A)$ for all $n \in \mathbf{N}$.

The formula $\det(e^{ih}) = e^{i\tau(h)}$ can be expressed on a neighborhood of the identity matrix by $\det(u) = e^{\tau(\log(u))}$ as long as log is well defined. This proves that det is continuous on a neighborhood of the identity of $\mathrm{U}_n(A)$. Since det is a group homomorphism this proves it to be continuous on all of $\mathrm{U}_n(A)$.

Given two determinants $\det_0$ and det let g be defined on $\mathrm{U}(A)$ by $g(u) = \det(u)\det_0(u)^{-1}$. Clearly g is a group homomorphism into T. If $u \in \mathrm{U}(A)_0$ there will be some $n \in \mathbf{N}$ and $h_1, \ldots, h_k \in M_n(A)_{\mathrm{sa}}$ such that $u \in \mathrm{U}_n(A)_0$ and $u = e^{ih_1} \cdots e^{ih_k}$. By a simple computation one has $g(u) = 1$. Thus g factors through $\mathrm{U}(A)/\mathrm{U}(A)_0$ and this is precisely

$K_1(A)$. If $\phi: K_1(A) \to T$ is defined by $\phi([u]_1) = g(u)$ for all $u \in \mathrm{U}_n(A)$ we have

$$\det(u) = \det{}_0(u)\phi([u]_1) \qquad \forall u \in \mathrm{U}_n(A).$$

On the other hand if $\phi: K_1(A) \to T$ is any group homomorphism and $\det_0$ is a determinant, the map

$$u \in \mathrm{U}_n(A) \to \det{}_0(u)\phi([u]_1)$$

is a determinant for (A, τ) since for all $n \in \mathbf{N}$ and $h \in M_n(A)$ we have $\phi([e^{ih}]_1) = \phi(0) = 1$.

11. *Note*. The notion of determinant exposed in [**6**] is closely related to what we did above, the main difference being that our determinant is defined in the whole of $\mathrm{U}(A)$ while theirs is defined only on $\mathrm{U}(A)_0$. We shall see in what follows the consequences of having a determinant defined on all of $\mathrm{U}(A)$.

III. Invariant Determinants

In the last section we developed a theory of determinants for C^* algebras and gave a characterization of those traced unital C^* algebras admitting a determinant. Here we shall study determinants on C^* algebras where a group action is given. Throughout this section (A, τ) will denote a fixed traced unital C^* algebra which we will assume to be integral.

We shall denote by $\mathrm{Aut}(A, \tau)$ the group of trace preserving automorphisms of A.

Given a subgroup G of $\mathrm{Aut}(A, \tau)$, we consider the problem of finding a G-invariant determinant. That is, a determinant det satisfying $\det(\alpha(u)) = \det(u)$ for all $u \in \mathrm{U}(A)$ and $\alpha \in G$.

As a preliminary result we have:

1. Proposition. *If $u \in \mathrm{U}(A)_0$ and* det *is any determinant for (A, τ) then* $\det(\alpha(u)) = \det(u)$ *for any trace preserving automorphism α of A.*

Proof. Since $u \in \mathrm{U}(A)_0$ we may find $n \in \mathbf{N}$ such that $u \in U_n(A)_0$. Therefore there will be $h_1, \ldots, h_k \in M_n(A)$ such that $u = e^{ih_1} \cdots e^{ih_k}$. We then have, for any $\alpha \in \mathrm{Aut}(A, \tau)$

$$\det(\alpha(u)) = \det(e^{i\alpha(h_1)} \cdots e^{i\alpha(h_k)})$$
$$= e^{i\tau(\alpha(h_1))} \cdots e^{i\tau(\alpha(h_k))} = e^{i\tau(h_1)} \cdots e^{i\tau(h_k)} = \det(u).$$

This is evidence that the problem to be studied here is related to the possible values of a determinant outside $\mathrm{U}(A)_0$, and so it would be meaningless if one defined the determinant only on $\mathrm{U}(A)_0$ as it is done in **[6]**.

Let G be a subgroup of $\mathrm{Aut}(A, \tau)$. There is a natural action of G on $K_1(A)$ given by

$$\alpha_*([u]_1) = [\alpha(u)]_1 \qquad \forall u \in \mathrm{U}(A),\ \alpha \in G$$

as well as on the Pontryagin dual $\widehat{K_1(A)}$ of $K_1(A)$ (with discrete topology). The action on $\widehat{K_1(A)}$ is defined by

$$\big(\alpha(\phi)\big)(x) = \phi\big(\alpha_*^{-1}(x)\big)$$

for all $\alpha \in G$, $\phi \in \widehat{K_1(A)}$ and $x \in K_1(A)$.

2. Definition. Let det be a determinant for (A, τ) and let G be a subgroup of $\mathrm{Aut}(A, \tau)$. We denote by $\bar{\zeta}$ the mapping

$$\bar{\zeta} : G \to \widehat{K_1(A)}$$

given by $\bar{\zeta}(\alpha)([u]_1) = \det(\alpha^{-1}(u^*)u)$ for all $\alpha \in G$ and $u \in \mathrm{U}(A)$.

In case we need to make clear which group of automorphisms and which determinant we are using in the definition of $\bar{\zeta}$, we will use the notation $\bar{\zeta}^G_{\det}$.

The next proposition will show that our definition carries no ambiguity.

3. Proposition. *Given* det *and* G *as above*

(i) $\bar{\zeta}(\alpha)$ *indeed belongs to* $\widehat{K_1(A)}$ *for all* $\alpha \in G$,

(ii) $\bar{\zeta}$ *satisfies the* 1-*cocyle identity* $\bar{\zeta}(\alpha\beta) = \alpha(\bar{\zeta}(\beta)) + \bar{\zeta}(\alpha)$ $\forall \alpha, \beta \in G$ *and*

(iii) *if* det′ *is another determinant for* (A, τ) *with associated* 1-*cocycle* $\bar{\zeta}'$, *then* $\bar{\zeta}' - \bar{\zeta}$ *is a coboundary in the sense that there is* $\phi \in \widehat{K_1(A)}$ *such that* $(\bar{\zeta}' - \bar{\zeta})(\alpha) = \alpha(\phi) - \phi$ $\forall \alpha \in G$.

Proof. In order to prove (i) we need to verify the following statements for all $u, v \in \mathrm{U}(A)$ and $\alpha \in G$:

(a) If $[u]_1 = [v]_1$, then $\det(\alpha^{-1}(u^*)u) = \det(\alpha^{-1}(v^*)v)$.

(b) $\det(\alpha^{-1}((uv)^*uv) = \det(\alpha^{-1}(u^*)u)\det(\alpha^{-1}(v^*)v)$.

From (a) it will follow that $\bar{\zeta}(\alpha)$ is a well-defined map from $K_1(A)$ to T, and (b) will show it to be a homomorphism.

Assume that $u, v \in \mathrm{U}_n(A)$ ($n \in \mathbf{N}$) and suppose that $h_1, \ldots, h_k$ are elements in $M_n(A)_{\mathrm{sa}}$ satisfying $u = ve^{ih_1} \cdots e^{ih_k}$. Then

$$\begin{aligned}\det\big(\alpha^{-1}(u^*)u\big) &= \det\Big(e^{-i\alpha^{-1}(h_k)} \cdots e^{-i\alpha^{-1}(h_1)}\alpha^{-1}(v^*)ve^{ih_1} \cdots e^{ih_k}\Big)\\ &= \det\big(\alpha^{-1}(v^*)v\big).\end{aligned}$$

This proves (a).

As an immediate consequence of the fact that det is a homomorphism taking values in a commutative group we get (b).

In order to prove (ii) let $\alpha, \beta \in G$. For all $[u]_1 \in K_1(A)$ we have

$$\begin{aligned}\bar{\zeta}(\alpha\beta)([u]_1) &= \det\big(\beta^{-1}\big(\alpha^{-1}(u^*)\big)u\big)\\ &= \det\big(\beta^{-1}\big(\alpha^{-1}(u^*)\big)\alpha^{-1}(u)\alpha^{-1}(u^*)u\big)\\ &= \bar{\zeta}(\beta)\big(\alpha_*^{-1}([u]_1)\big)\ \bar{\zeta}(\alpha)([u]_1)\\ &= \big(\alpha\big(\bar{\zeta}(\beta)\big)\big)([u]_1)\ \bar{\zeta}(\alpha)([u]_1) = \big(\alpha\big(\bar{\zeta}(\beta)\big) + \bar{\zeta}(\alpha)\big)([u]_1),\end{aligned}$$

proving (ii).

Finally we prove (iii). Let $\phi\colon K_1(A) \to T$ be defined by $\phi([u]_1) = \det'(u)\det(u^*)$ for all $u \in \mathrm{U}(A)$. We claim that ϕ is well defined. In fact if $v \in \mathrm{U}(A)$ and $[v]_1 = [u]_1$, then there will be some integer n and $h_1, \ldots, h_k \in M_n(A)_{\mathrm{sa}}$ such that $u, v \in \mathrm{U}_n(A)$ and $u = ve^{ih_1} \cdots e^{ih_k}$. So

$$\begin{aligned}\det'(u)\det(u^*) &= \det'(v)e^{i\tau(h_1)} \cdots e^{i\tau(h_k)}e^{-i\tau(h_k)} \cdots e^{-i\tau(h_1)}\det(v^*)\\ &= \det'(v)\det(v^*).\end{aligned}$$

The fact that ϕ is multiplicative follows from the multiplicativity of determinants. Therefore for all $\alpha \in G$ and $u \in \mathrm{U}(A)$

$$\begin{aligned}(\bar{\zeta} - \bar{\zeta}')(\alpha)([u]_1) &= \det\big(\alpha^{-1}(u^*)u\big)\big(\det'\big(\alpha^{-1}(u^*)u\big)\big)^{-1}\\ &= \det'\big(\alpha^{-1}(u)\big)\det\big(\alpha^{-1}(u^*)\big)\ \big(\det'(u)\det(u^*)\big)^{-1}\\ &= \phi\big(\alpha_*^{-1}([u]_1)\big)\phi([u]_1)^{-1} = \big(\alpha(\phi) - \phi\big)([u]_1).\end{aligned}$$

This completes the proof.

We may now give the following

4. Definition. The class of $\bar{\zeta}_{\det}^G$ in $H^1(G, \widetilde{K_1(A)})$ is denoted by $\zeta(G)$ and is called the G-invariant determinant obstruction.

Observe that by (3.iii) $\zeta(G)$ does not depend on the determinant used in its definition.

Before we state the main result of this section, we study how $\zeta(G)$ changes with G.

5. PROPOSITION. *If* $G_1 \subseteq G_2 \subseteq \mathrm{Aut}(A,\tau)$ *then* $\zeta(G_1)$ *is the image of* $\zeta(G_2)$ *under the restriction homomorphism*:

$$r: H^1\big(G_2, \widehat{K_1(A)}\big) \to H^1\big(G_1, \widehat{K_1(A)}\big).$$

Proof. Let det be any determinant for (A,τ). It is enough to prove that $\bar{\zeta}_{\det}^{G_1}$ is the restriction of $\bar{\zeta}_{\det}^{G_2}$ to G_1 and this is obvious.

We thus see that for any $G \subseteq \mathrm{Aut}(A,\tau)$ $\zeta(G)$ is the "restriction" of $\zeta(\mathrm{Aut}(A,\tau))$ to G. It is remarkable that for any integral algebra (A,τ) the process above singles out in a canonical way an element in $H^1(\mathrm{Aut}(A,\tau), \widehat{K_1(A)})$.

The following is the main result of this chapter. It justifies the name given to the G-invariant determinant obstruction.

6. THEOREM. *Let* (A,τ) *be an integral unital* C^* *algebra and let G be a group of trace preserving automorphisms of A. The following are equivalent*:

(i) (A,τ) *admits a G-invariant determinant.*

(ii) $\zeta(G) = 0$ *in* $H^1(G, \widehat{K_1(A)})$.

Proof. Assume (i) and thus let det be a G-invariant determinant. It is clear that $\bar{\zeta}_{\det}^{G} = 0$ and thus $\zeta(G) = 0$. Conversely let det be any determinant for (A,τ). If $\zeta(G) = 0$ then $\bar{\zeta}_{\det}^{G}$, which we denote simply by $\bar{\zeta}$, is a coboundary. Pick $\phi \in \widehat{K_1(A)}$ such that $\bar{\zeta}(\alpha) = \alpha(\phi) - \phi$ for all $\alpha \in G$ and define $\det' : \mathrm{U}(A) \to T$ by $\det'(u) = \det(u)\phi([u]_1)$. It is clear that $\det'$ is a determinant for (A,τ). Moreover, for any $\alpha \in G$ and $u \in \mathrm{U}(A)$

$$\det\big(\alpha^{-1}(u^*)u\big) = \bar{\zeta}(\alpha)([u]_1) = \phi\big(\alpha_*^{-1}([u]_1)\phi([u]_1)\big)^{-1},$$

Equivalently

$$\det(u)\phi([u]_1) = \det\big(\alpha^{-1}(u)\big)\phi\big([\alpha^{-1}(u)]_1\big)$$

which proves that $\det'$ is G-invariant.

7. PROPOSITION. *If* $\det_0$ *is a G-invariant determinant for* (A,τ) *then any other G-invariant determinant* det *is of the form*

$$\det(u) = \det{}_0(u)\phi([u]_1), \qquad \forall u \in \mathrm{U}(A)$$

where $\phi \in \widehat{K_1(A)}$ *is G-invariant.*

Proof. By (II.10) we know that any (not necessarily G-invariant) determinant det is of the form

$$\det(u) = \det_0(u)\phi([u]_1), \qquad \forall u \in \mathrm{U}(A)$$

where $\phi \in \widehat{K_1(A)}$. For $\alpha \in G$ we have

$$\phi(\alpha_*([u]_1)) = \det(\alpha(u))\det_0(u)^{-1}$$

from what we see that det is G-invariant if and only if ϕ is G-invariant.

IV. Rotation Numbers

If α is any orientation preserving homeomorphism of the circle it is possible to define as in [**8**, **16**] its rotation number. In this chapter we generalize this notion to automorphisms of integral C^* algebras. Specializing this concept for commutative algebras we obtain a notion of rotation numbers for homeomorphisms of any compact connected topological space with an invariant probability measure (see Chapter VI).

We should note that what we do here is somewhat related to the mass flow homomorphism introduced by Schwartzman in [**22**].

The concept of rotation number applies only to integral algebras. With this in mind we let (A, τ) be a unital integral C^* algebra, considered fixed throughout this chapter. We also fix a trace-preserving automorphism of A, denoted by α.

1. Definition. The rotation number map of α with respect to the trace τ is the group homomorphism

$$\rho_\alpha^\tau : K_1(A)^\alpha \to T$$

defined as follows. Its domain $K_1(A)^\alpha$ is the subgroup of fixed points for the action of α on $K_1(A)$, i.e.

$$K_1(A)^\alpha = \{x \in K_1(A) : \alpha_*(x) = x\}.$$

For each $x \in K_1(A)^\alpha$ we put

$$\rho_\alpha^\tau = \det\bigl(\alpha(u^*)u\bigr),$$

where $u \in \mathrm{U}(A)$ is such that $[u]_1 = x$ and det is a determinant for A.

The next proposition shows there is no ambiguity in our definition.

2. Proposition. *The rotation number map*

(i) *is well defined,*

(ii) *is a group homomorphism and*

(iii) *does not depend on the determinant used in its definition.*

Proof. (i) and (ii) follow from (a) and (b) in the proof of (III.3) by simply replacing α by α^{-1}. In order to verify (iii) let $\det'$ be another determinant and $u \in \mathrm{U}(A)$ be such that $[u]_1 \in K_1(A)^\alpha$. By (II.10)

$$\det'\big(\alpha(u^*)u\big) = \det\big(\alpha(u^*)u\big)\phi\big([\alpha(u^*)u]_1\big)$$

for some $\phi \in \widehat{K_1(A)}$ but since $[\alpha(u^*)u]_1 = 0$ the result follows.

We should note that although the rotation number map has some similarities with the map $\bar{\zeta}$ defined in the previous section, it will play a very different role in what follows.

A natural question to ask is how the rotation number map of the composition of two automorphisms behaves with respect to each of the factors. The answer to this question is proved in the next

3. Proposition. *Let α and β be trace-preserving automorphisms of the unital integral C^* algebra (A, τ) and let $\gamma = \alpha\beta$. Denote by $K_1(A)^{\alpha,\beta}$ the intersection of $K_1(A)^\alpha$ and $K_1(A)^\beta$. Then $K_1(A)^{\alpha,\beta} \subseteq K_1(A)^\gamma$ and*

$$\rho_\gamma^\tau|_{K_1(A)^{\alpha,\beta}} = \rho_\alpha^\tau|_{K_1(A)^{\alpha,\beta}} \cdot \rho_\beta^\tau|_{K_1(A)^{\alpha,\beta}}.$$

The multiplication above is to be understood pointwise.

Proof. It is clear that $K_1(A)^{\alpha,\beta} \subseteq K_1(A)^\gamma$. If $x \in K_1(A)^{\alpha,\beta}$ let $u \in \mathrm{U}(A)$ be such that $[u]_1 = x$. If det is a determinant for (A, τ) we have

$$\begin{aligned}\rho_\gamma^\tau(x) &= \det\big(\gamma(u^*)u\big) = \det\big(\alpha(\beta(u^*))\alpha(u)\alpha(u^*)u\big)\\ &= \det\big(\alpha(\beta(u^*)u)\big)\det\big(\alpha(u^*)u\big).\end{aligned}$$

Now note that $\beta(u^*)u = e^{ih_1} \cdots e^{ih_k}$ for some $h_1, \dots, h_k \in M_n(A)_{\mathrm{sa}}$ and some $n \in \mathbf{N}$. It follows that

$$\begin{aligned}\det\big(\alpha(\beta(u^*)u)\big) &= \det\big(e^{i\alpha(h_1)} \cdots e^{i\alpha(h_k)}\big)\\ &= e^{i\tau(h_1)} \cdots e^{i\tau(h_k)} = \det\big(\beta(u^*)u\big).\end{aligned}$$

Thus

$$\rho_\gamma^\tau(x) = \det\big(\beta(u^*)u\big)\det\big(\alpha(u^*)u\big) = \rho_\alpha^\tau(x)\rho_\beta^\tau(x).$$

As a consequence we get

4. Corollary. *If α is a trace-preserving automorphism of (A, τ) and $n \geq 1$ is an integral number, we have*

$$\rho_{\alpha^n}^\tau|_{K_1(A)^\alpha} = \big(\rho_\alpha^\tau\big)^n.$$

Concluding this section we give a few examples in which one can effectively compute rotation numbers. This should show that the computations involved are quite easy once we have a good description of the algebra, its trace and the given automorphism.

5. EXAMPLE. (Rotations on the circle.) Let $A = C(T)$, the algebra of continuous complex valued functions on the circle. The normalized Haar measure on T defines via integration a trace on A:

$$\tau(f) = \int_T f(z)\,dz, \qquad f \in A.$$

Let θ be a real number and define the "rotation by θ" automorphism of A by $\alpha(f)(z) = f(e^{-i\theta}z)$ for all $f \in C(T)$, $z \in T$.

It is easy to show that (A, τ) is integral (see (VI.1)) and that α is trace-preserving. We compute the rotation number map of α with respect to τ. First of all observe that $\alpha_* = \mathrm{id}$ on $K_1(A)$ because α is homotopic to the identity on A so that

$$K_1(A)^\alpha = K_1(A) = \{[z^n]_1 : n \in \mathbf{Z}\} \simeq \mathbf{Z},$$

where z is the unitary in A representing the inclusion map of T into $\mathbf{C}$.

We have $\alpha(z^*)z = e^{i\theta}$, so that $\rho_\alpha^\tau([z]_1) = \det(e^{i\theta}) = e^{i\theta}$. Consequently $\rho_\alpha^\tau([z^n]_1) = e^{in\theta}\ \forall n \in \mathbf{Z}$.

6. EXAMPLE. (Translations on the 2-torus.) Let $A = C(T^2)$, let τ be the trace associated to the Haar measure on T^2 and let

$$\alpha(f)(z,w) = f(e^{-i\theta}z, e^{-i\phi}w), \quad \forall f \in C(T^2), \quad (z,w) \in T$$

where θ and ϕ are fixed real numbers. Again it is easy to prove that (A, τ) is integral and α is trace-preserving. As before $\alpha_* = \mathrm{id}$ on $K_1(A)$ so that

$$K_1(A)^\alpha = K_1(A) = \{[z^n w^m]_1 : n, m \in \mathbf{Z}\} \simeq \mathbf{Z}^2$$

where z and w represent the two canonical coordinate functions on the 2-torus.

We have $\alpha(z^*)z = e^{i\theta}$ while $\alpha(w^*)w = e^{i\phi}$. Therefore $\rho_\alpha^\tau([z]_1) = e^{i\theta}$ and $\rho_\alpha^\tau([w]_1) = e^{i\phi}$. It follows that ρ_α^τ is the map

$$[z^n w^m]_1 \in K_1(A) \to e^{in\theta}e^{im\phi} \in T.$$

7. EXAMPLE (Twist of the annulus.) Let X be the annulus,

$$X = \{z \in \mathbf{C} : 1 \leq |z| \leq 2\}$$

and $A = C(X)$. The Lebesgue measure of X is 3π so that

$$\tau(f) = \frac{1}{3\pi}\int_X f(x+iy)\,dx\,dy \qquad \forall f \in A$$

is a normalized trace on A which makes (A, τ) integral. Let $\theta:[1,2] \to \mathbf{R}$ be any continuous function and define the θ-twist to be the automorphism of A given by

$$\alpha(f)(z) = f(e^{-i\theta(|z|)}z), \qquad \forall z \in X, f \in C(X).$$

Integration with polar coordinates shows that α leaves τ-invariant. Since α is homotopic to the identity we have, as in the examples above, $\alpha_* = \text{id}$ and

$$K_1(A)^\alpha = K_1(A) = \{[u^n]_1 : n \in \mathbf{Z}\} \simeq \mathbf{Z},$$

where u is the unitary in A given by $u(z) = z|z|^{-1}$. We have

$$(\alpha(u^*)u)(z) = e^{i\theta(|z|)}.$$

So

$$\rho_\alpha^\tau([u]_1) = \det(e^{i\theta(|z|)}) = \exp\left(\frac{i}{3\pi}\int_X \theta(|z|)\, dz\right).$$

To make it more concrete assume $\theta(r) = 2\pi r$. Then

$$\int_X \theta(|z|)\, dz = \int_0^{2\pi}\int_1^2 \theta(r) r\, dr\, d\theta = 2\pi\int_1^2 2\pi r^2 dr = 4\pi^2\frac{7}{3}.$$

Thus $\rho_\alpha^\tau([u]_1) = e^{i\pi(28/9)}$. Whence $\rho_\alpha^\tau([u^n]_1) = e^{i\pi(28n/9)}$.

V. Crossed Products

We devote this chapter to the study of the range of the trace on K_0 groups of crossed product algebras by $\mathbf{Z}$.

As we shall see, this problem is deeply related to the theory of rotation numbers described in Chapter IV. In fact it was our interest in crossed product algebras and their traces which led us to the definition of rotation numbers.

We begin by briefly defining crossed product algebras. We refer the reader to **[12]** for an extensive treatment on that subject.

Given a unital C^* algebra A and an action α of $\mathbf{Z}$ on A, the crossed product algebra is a C^* algebra denoted by $A \times_\alpha \mathbf{Z}$, or simply by $A \times \mathbf{Z}$ if the action is understood, which is generated by a copy of A and a unitary L satisfying the following conditions

(a) For all $a \in A$ and $n \in \mathbf{Z}$ we have $L^n a L^{-n} \in A$ and $L^n a L^{-n} = \alpha_n(a)$.

(b) It is universal with this propery, i.e. given any C^* algebra B containing A and a unitary L' satisfying (a) there is a unique $*$-homomorphism ϕ: $A \times \mathbf{Z} \to B$ such that $\phi(a) = a$ for any $a \in A$ and $\phi(L) = L'$.

It follows that the subset of $A \times \mathbf{Z}$ given by

$$\left\{ \sum_{n \in \mathbf{Z}} a_n L^n : a_n \in A \; \forall n \in \mathbf{Z}, \; \sum_{n \in \mathbf{Z}} \|a_n\| < \infty \right\}$$

is a dense $*$-subalgebra.

If τ is a trace on A which is invariant under the action α, the formula $\tilde{\tau}(\sum_{n \in \mathbf{Z}} a_n L^n) = \tau(a_0)$ defines a trace on the dense subalgebra mentioned above. One can prove that $\tilde{\tau}$ is bounded so it extends to the whole of $A \times \mathbf{Z}$ giving a trace which we shall denote also by τ as no confusion will arise. This extended trace is sometimes called the dual trace.

An important tool which we shall use is the Pimsner-Voiculescu exact sequence for K-theory of crossed product algebras by $\mathbf{Z}$ [**14**]. Given a C^* algebra A and an action α of $\mathbf{Z}$ on A as above, it asserts that there is an exact sequence

1.

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*^{-1}} & K_0(A) & \to & K_0(A \times_\alpha \mathbf{Z}) \\ \exp \uparrow & & & & \downarrow \partial \\ K_1(A \times_\alpha \mathbf{Z}) & \leftarrow & K_1(A) & \xleftarrow{1-\alpha_*^{-1}} & K_1(A) \end{array}$$

Among all the maps involved, the most important one for our purposes is the right hand side vertical map indicated by ∂.

When doing computations involving ∂ we will make use of a result of Paschke [**11**] proving that the K-theory of a crossed product algebra by $\mathbf{Z}$ is isomorphic to the K-theory of the mapping torus with a shift in the grading index. Paschke's result will turn out to be our main tool, so we briefly describe it.

Let A be a unital C^* algebra and α be an automorphism of A. We thus get an action of $\mathbf{Z}$ on A by taking powers of α. Note that all actions of $\mathbf{Z}$ on A are of this form.

2. Definition. The mapping torus of the pair (A, α) is the C^* algebra $T_\alpha(A)$ consisting of all A valued continuous functions f on the interval $[0,1]$ satisfying $f(1) = \alpha(f(0))$.

3. Theorem (*W. Paschke* [**11**], *A. Connes* [**1**].) *There are isomorphisms*

$$k_0 : K_0(T_\alpha(A)) \to K_1(A \times \mathbf{Z}),$$
$$k_1 : K_1(T_\alpha(A)) \to K_0(A \times \mathbf{Z}).$$

It is k_1 which will be most useful in our study. Unfortunately the formula for k_1 in **[11]** is too cumbersome. Our technique will be to make use of a sort of "suspension" argument which will enable us to use k_0 instead of k_1. A description of k_0 is in order (see **[11]**).

4. *Description of* k_0. Let $n \in \mathbf{N}$, $n \geq 1$, and let $p \in M_n(T_\alpha(A))$ be a self adjoint projection. We may view p as a continuous function $p:[0,1] \to M_n(A)$ such that $p(t)$ is a self adjoint projection for all $t \in [0,1]$ and $p(1) = \alpha(p(0))$. According to **[11]** there is a continuous path of unitaries, $\{w(t): 0 \leq t \leq 1\}$, such that $w(0) = 1$, $p(t) = w(t)p(0)w(t)^*$ for all t in $[0,1]$, $(L \otimes I_n)^* w(1)$ commutes with $p(0)$ and we have

$$k_0([p]_0) = \left[(L \otimes I_n)^* w(1) p(0) + I_n - p(0)\right]_1.$$

In the following we explain how our "suspension" argument works. Consider the pair $(A \otimes C(T), \alpha \otimes 1)$ and recall that

$$(A \otimes C(T)) \times_{\alpha\otimes 1} \mathbf{Z} \simeq (A \times_\alpha \mathbf{Z}) \otimes C(T) \quad \text{and}$$

$$T_{\alpha\otimes 1}(A \otimes C(T)) \simeq T_\alpha(A) \otimes C(T).$$

Using the identifications above we arrive at a description of k_1 in terms of k_0. Consider the diagram

5.

$$\begin{array}{ccc} K_0(T_\alpha(A) \otimes C(T)) & \xrightarrow{k_0} & K_1((A \otimes \mathbf{Z}) \otimes C(T)) \\ \text{ind}\uparrow & & \downarrow \pi \\ K_1(T_\alpha(A)) & \xrightarrow{k_1} & K_0(A \times \mathbf{Z}) \end{array}$$

where π is the projection corresponding to the natural decomposition

$$K_1((A \times \mathbf{Z}) \otimes C(T)) \simeq K_0(A \times \mathbf{Z}) \oplus K_1(A \times \mathbf{Z})$$

and ind is the inclusion corresponding to the natural decomposition.

$$K_0(T_\alpha(A) \otimes C(T)) \simeq K_1(T_\alpha(A)) \oplus K_0(T_\alpha(A)).$$

This diagram is commutative because it actually defines k_1 in **[11]**.

Although an explicit formula for k_1 is provided in **[11]** we shall avoid it and use (5) together with the formula for k_0 described in (4) whenever we need to compute k_1. We believe that this procedure will make things clearer than a direct use of the formula for k_1 as given in **[11]**.

Let us pause for a moment to give a concrete formula for ind.

6. *Description of ind.* (See [1].) For any unital C^* algebra B the map

$$\operatorname{ind} : K_1(B) \to K_0(B \otimes C(T))$$

may be defined as follows: for $u \in \mathrm{U}_n(B)$

$$\operatorname{ind}([u]_1) = [p]_0 - [p_0]_0$$

where $p, p_0 \in M_{2n}(B \otimes C(T))$ are self adjoint projections given by

$$p_0(s) = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \quad \forall s \in [0, 2\pi] \quad \text{and}$$

$$p(s) = R(s)\begin{bmatrix} u^* & 0 \\ 0 & I_n \end{bmatrix} R(s)^* \begin{bmatrix} I_n & 0 \\ 0 & 0_n \end{bmatrix} R(s) \begin{bmatrix} u & 0 \\ 0 & I_n \end{bmatrix} R(s)^*$$

$$\forall s \in [0, 2\pi]$$

where

$$R(s) = \begin{bmatrix} \cos(s/4) I_n & -\sin(s/4) I_n \\ \sin(s/4) I_n & \cos(s/4) I_n \end{bmatrix} \quad \forall s \in [0, 2\pi]$$

or any other path of unitaries joining $\left[\begin{smallmatrix} I_n & 0 \\ 0 & I_n \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} 0 & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$.

The following result, which is implicitly in [11], relates k_1 to ∂ and will be helpful when computations involving ∂ are performed.

Let $\operatorname{val} : T_\alpha(A) \to A$ be defined by $\operatorname{val}(f) = f(0)$ for all $f \in T_\alpha(A)$.

7. Theorem. (*Paschke.*) *The following diagram is commutative.*

$$\begin{tikzcd} K_1(T_\alpha(A)) \arrow[r, "k_1"] \arrow[dr, "\operatorname{val}_*"'] & K_0(A \times \mathbf{Z}) \arrow[d, "\partial"] \\ & K_1(A) \end{tikzcd}$$

This concludes our preparations. We may now start the main argument.

Our first major step will be the computation of $\tau \cdot k_1$. Putting together (I.5) and (5) we get the following commuting diagram

$$\begin{tikzcd} K_0(T_\alpha(A) \otimes C(T)) \arrow[r, "k_0"] & K_1((A \times \mathbf{Z}) \otimes C(T)) \arrow[r, "\gamma"] \arrow[d, "\pi"] & \mathbf{R} \\ K_1(T_\alpha(A)) \arrow[u, "\operatorname{ind}"] \arrow[r, "k_1"] & K_0(A \times \mathbf{Z}) \arrow[ur, "\tau"'] & \end{tikzcd}$$

so that $\tau \cdot k_1 = \gamma \cdot k_0 \cdot \mathrm{ind}$. We use the right hand side to perform our computations. This will be accomplished step by step in two lemmas.

8. *Note*. The algebra $T_\alpha(A) \otimes C(T)$ will be identified with the subalgebra of $C([0,1] \times [0,2\pi], A)$ formed by those elements x satisfying

$$x(t,0) = x(t,2\pi) \quad \forall t \in [0,1]$$

$$x(1,s) = \alpha(x(0,s)) \quad \forall s \in [0,2\pi].$$

9. Lemma. *Let $p = p(t,s)$ be a self adjoint projection in $T_\alpha(A) \otimes C(T)$. Also let $w : [0,1] \to A \otimes C(T)$ be a continuous path of unitaries satisfying $w(0) = 1$ and $p(t,\cdot) = w(t)p(0,\cdot)w(t)^*$. With the notation $w(t,s) := w(t)(s)$ the conditions above are equivalent to*

$$w(0,s) = 1, \qquad 0 \le s \le 2\pi$$

$$p(t,s) = w(t,s)p(0,s)w(t,s)^*, \quad 0 \le s \le 2\pi, 0 \le t \le 1.$$

Assume that $s \to w(1,s)$ and $s \to p(0,s)$ are C^∞ maps. Then

$$\gamma(k_0(p)) = \frac{1}{2\pi i}\int_0^{2\pi} \tau\left(w(1,s)^* \frac{\partial w}{\partial s}(1,s)p(0,s)\right) ds.$$

Proof. Let $v \in (A \times \mathbf{Z}) \otimes C(T)$ be given by

$$v(s) = L^* w(1,s)p(0,s) + 1 - p(0,s).$$

By definition (see (4)) $[v]_1 = k_0([p]_0)$ and we have

$$\frac{dv}{ds}(s) = L^* \frac{\partial w}{\partial s}(1,s)p(0,s) + L^* w(1,s)\frac{\partial p}{\partial s}(0,s) - \frac{\partial p}{\partial s}(0,s).$$

Therefore

$$\begin{aligned}\frac{dv}{ds}(s)v(s)^* = L^* &\frac{\partial w}{\partial s}(1,s)p(0,s)w(1,s)^* L \\ &+ L^* w(1,s)\frac{\partial p}{\partial s}(0,s)(1 - p(0,s)) \\ &+ L^* w(1,s)\frac{\partial p}{\partial s}(0,s)p(0,s)w(1,s)^* L \\ &- \frac{\partial p}{\partial s}(0,s)(1 - p(0,s)) - \frac{\partial p}{\partial s}(0,s)p(0,s)w(1,s)^* L.\end{aligned}$$

Applying τ, we have

$$\begin{aligned}\tau\left(\frac{dv}{ds}(s)v^*(s)\right) &= \tau\left(\frac{\partial w}{\partial s}(1,s)p(0,s)w(1,s)^*\right)\\ &\quad+\tau\left(L^*w(1,s)\frac{\partial p}{\partial s}(0,s)(1-p(0,s))\right)\\ &\quad+\tau\left(\frac{\partial p}{\partial s}(0,s)p(0,s)\right)\\ &\quad-\tau\left(\frac{\partial p}{\partial s}(0,s)(1-p(0,s))\right)\\ &\quad-\tau\left(\frac{\partial p}{\partial s}(0,s)p(0,s)w(1,s)^*L\right).\end{aligned}$$

Note that the second and last terms above vanish because $\tau(L^*a) = \tau(aL) = 0$ for all $a \in A$. Also observe that differentiating the expression $p(0,s)^2 = p(0,s)$ with respect to s we obtain

$$\frac{\partial p}{\partial s}(0,s)p(0,s) + p(0,s)\frac{\partial p}{\partial s}(0,s) = \frac{\partial p}{\partial s}(0,s).$$

It follows that

$$\frac{\partial p}{\partial s}(0,s)(1-p(0,s)) = p(0,s)\frac{\partial p}{\partial s}(0,s),$$

from which we see that the third and fourth terms above cancel each other, and we are left with

$$\tau\left(\frac{dv}{ds}(s)v(s)^*\right) = \tau\left(w(1,s)^*\frac{\partial w}{\partial s}(1,s)p(0,s)\right).$$

If we now use the definition of γ as in (I.4) we get the conclusion.

10. Lemma. *Let $u \in T_\alpha(A)$. For all $(t,s) \in [0,1] \times [0,2\pi]$ put*

$$\Omega(t,s) = R(s)\begin{bmatrix} u(t)^* & 0 \\ 0 & 1\end{bmatrix}R(s)^*$$

where $R(s)$ is as in (6).

Let $p(t,s) = \Omega(t,s)\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\Omega(t,s)^*$ and

$$w(t,s) = \Omega(t,s)\begin{bmatrix} u(st/2\pi)^*u(0) & 0 \\ 0 & u(st/2\pi)u(0)^*\end{bmatrix}\Omega(0,s)^*$$

for all $(t,s) \in [0,1] \times [0,2\pi]$. Then

(i) $w(t,\cdot) \in U_2(A \otimes C(T))\ \forall t \in [0,1]$.

(ii) $w(0,\cdot) = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$,

(iii) $p(t,s) = w(t,s)p(0,s)w(t,s)^*\ \forall (t,s) \in [0,1] \times [0,2\pi]$.

Proof. We verify (i) by proving that $w(t,0) = w(t,2\pi)$ for all $t \in [0,1]$. We have

$$w(t,0) = \Omega(t,0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Omega(0,0)^* = \begin{bmatrix} u(t)^* & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u(0) & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} u(t)^*u(0) & 0 \\ 0 & 1 \end{bmatrix}.$$

While

$$w(t,2\pi) = \Omega(t,2\pi)\begin{bmatrix} u(t)^*u(0) & 0 \\ 0 & u(t)u(0)^* \end{bmatrix}\Omega(0,2\pi)^*$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & u(t)^* \end{bmatrix}\begin{bmatrix} u(t)^*u(0) & 0 \\ 0 & u(t)u(0)^* \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & u(0) \end{bmatrix}$$
$$= \begin{bmatrix} u(t)^*u(0) & 0 \\ 0 & 1 \end{bmatrix}.$$

To prove (ii) we simply compute:

$$w(0,s) = \Omega(0,s)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Omega(0,s)^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Another computation gives (iii):

$$w(t,s)p(0,s)w(t,s)^*$$
$$= \Omega(t,s)\begin{bmatrix} u(st/2\pi)^*u(0) & 0 \\ 0 & u(st/2\pi)u(0)^* \end{bmatrix}\Omega(0,s)^*\Omega(0,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\cdot\Omega(0,s)^*\Omega(0,s)\begin{bmatrix} u(0)^*u(st/2\pi) & 0 \\ 0 & u(0)u(st/2\pi)^* \end{bmatrix}\Omega(t,s)^*$$
$$= \Omega(t,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega(t,s)^* = p(t,s).$$

The following completes our computation of $\tau \cdot k_1$.

11. THEOREM. *Let $n \in \mathbf{N}$, $n \geq 1$ and let $u \in \mathrm{U}_n(T_\alpha(A))$. Viewing u as a function $u\colon [0,1] \to \mathrm{U}_n(A)$, assume that u is C^∞. Then*

$$\tau(k_1([u]_1)) = \frac{1}{2\pi i}\int_0^1 \tau\big(u'(t)^*u(t)\big)\,dt.$$

Proof. We already know that

$$\tau(k_1([u]_1)) = \gamma(k_0(\mathrm{ind}([u]_1))).$$

By replacing A by $M_n(A)$ we may assume that $n = 1$. Let p, $p_0 \in M_2(T_\alpha(A) \otimes C(T))$ be given by

$$p(t,s) = \Omega(t,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega(t,s)^* \quad \text{and}$$

$$p_0(t,s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \forall (t,s) \in [0,1] \times [0, 2\pi]$$

where $\Omega(t,s)$ is defined in terms of u as in (10). By (6) $\mathrm{ind}([u]_1) = [p]_0 - [p_0]_0$. Thus

$$\tau(k_1([u]_1)) = \gamma(k_0([p]_0)) - \gamma(k_0([p_0]_0)).$$

We use Lemma (9) to compute the right hand side. With respect to p_0 note that if $w_0(t,s) = [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$ for all $(t,s) \in [0,1] \times [0, 2\pi]$ then w_0 satisfies the hypothesis of (9). We then conclude that $\gamma(k_0([p_0]_0)) = 0$. As for p, let $w = w(t,s)$ be as in (10). It satisfies the hypothesis of (9) with respect to p so we have

$$\gamma(k_0([p]_0)) = \frac{1}{2\pi i}\int_0^{2\pi} \tau\left(w(1,s)^* \frac{\partial w}{\partial s}(1,s) p(0,s)\right) ds.$$

The integrand equals

$$\tau\left(\frac{\partial w}{\partial s}(1,s) p(0,s) w(1,s)^*\right).$$

Let us first compute $p(0,s)\ w(1,s)^*$. We have

$$\begin{aligned} p(0,s)w(1,s)^* &= \Omega(0,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega(0,s)^*\Omega(0,s) \\ &\quad \cdot \begin{bmatrix} u(0)^*u(s/2\pi) & 0 \\ 0 & u(0)u(s/2\pi)^* \end{bmatrix}\Omega(1,s)^* \\ &= \Omega(0,s)\begin{bmatrix} u(0)^*u(s/2\pi) & 0 \\ 0 & 0 \end{bmatrix}\Omega(1,s)^*. \end{aligned}$$

Therefore

$$\tau\left(\frac{\partial w}{\partial s}(1,s)p(0,s)w(1,s)^*\right)$$

$$= \tau\left\{\left(\frac{\partial\Omega}{\partial s}(1,s)\begin{bmatrix} u(s/2\pi)^*u(0) & 0 \\ 0 & u(s/2\pi)u(0)^* \end{bmatrix}\Omega(0,s)^*\right.\right.$$

$$+ (1/2\pi)\Omega(1,s)\begin{bmatrix} u'(s/2\pi)^*u(0) & 0 \\ 0 & u'(s/2\pi)u(0)^* \end{bmatrix}\Omega(0,s)^*$$

$$\left. + \Omega(1,s)\begin{bmatrix} u(s/2\pi)^*u(0) & 0 \\ 0 & u(s/2\pi)u(0)^* \end{bmatrix}\frac{\partial\Omega}{\partial s}(0,s)^*\right)$$

$$\left. \cdot\Omega(0,s)\begin{bmatrix} u(0)^*u(s/2\pi) & 0 \\ 0 & 0 \end{bmatrix}\Omega(1,s^*)\right\}$$

$$= (1/2\pi)\tau\left\{\Omega(1,s)\begin{bmatrix} u'(s/2\pi)^*u(s/2\pi) & 0 \\ 0 & 0 \end{bmatrix}\Omega(1,s)^*\right\}$$

$$+ \tau\left\{\frac{\partial\Omega}{\partial s}(1,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega(1,s)^*\right\} + \tau\left\{\frac{\partial\Omega}{\partial s}(0,s)^*\Omega(0,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}.$$

We now observe that since $u \in T_\alpha(A)$ we have $u(1) = \alpha(u(0))$. Therefore

$$\Omega(1,s) = R(s)\begin{bmatrix} u(1)^* & 0 \\ 0 & 1 \end{bmatrix}R(s)^*$$

$$= \alpha\left\{R(s)\begin{bmatrix} u(0)^* & 0 \\ 0 & 1 \end{bmatrix}R(s)^*\right\} = \alpha(\Omega(0,s)).$$

Thus

$$\tau\left\{\frac{\partial\Omega}{\partial s}(1,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega(1,s)^*\right\} = \tau\left\{\alpha\left(\frac{\partial\Omega}{\partial s}(0,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega(0,s)^*\right)\right\}$$

$$= \tau\left\{\frac{\partial\Omega}{\partial s}(0,s)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega(0,s)^*\right\}.$$

So

$$\begin{aligned}
&\tau\left(\frac{\partial w}{\partial s}(1,s)p(0,s)w(1,s)^*\right)\\
&= (1/2\pi)\tau\left(\Omega(1,s)\begin{bmatrix} u'(s/2\pi)^*u(s/2\pi) & 0\\ 0 & 0\end{bmatrix}\Omega(1,s)^*\right)\\
&\quad+\tau\left(\frac{\partial\Omega}{\partial s}(0,s)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\Omega(0,s)^*\right)+\tau\left(\Omega(0,s)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\frac{\partial\Omega}{\partial s}(0,s)^*\right)\\
&= (1/2\pi)\tau\big(u'(s/2\pi)^*u(s/2\pi)\big)+\tau\left(\frac{d}{ds}\left(\Omega(1,s)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\Omega(1,s)^*\right)\right)\\
&= (1/2\pi)\tau\big(u'(s/2\pi)^*u(s/2\pi)\big)+\frac{d}{ds}\tau\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)\\
&= (1/2\pi)\tau\big(u'(s/2\pi)^*u(s/2\pi)\big).
\end{aligned}$$

The computation above gives us

$$\gamma(k_0([p]_0)) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{2\pi}\tau\left(u'\left(\frac{s}{2\pi}\right)^*u\left(\frac{s}{2\pi}\right)\right)ds$$

and, after the change of variables $s = 2\pi t$ we have

$$\gamma(k_0([p]_0)) = \frac{1}{2\pi i}\int_0^1\tau\big(u'(t)^*u(t)\big)\,dt.$$

The proof is now complete.

In our next result we make full use of the technical facts developed in this chapter, condensing everything into a simple commuting square.

12. THEOREM. *Let (A,τ) be an integral unital C^* algebra and α be a trace-preserving automorphism of A. The following diagram is commutative*

$$\begin{array}{ccc}
K_0(A\times_\alpha\mathbf{Z}) & \xrightarrow{\tau} & \mathbf{R}\\
\partial\downarrow & & \downarrow\pi\\
K_1(A)^\alpha & \xrightarrow[\rho_\alpha^\tau]{} & T
\end{array}$$

where π denotes the map $t\in\mathbf{R}\to e^{2\pi i t}\in T$.

Proof. First observe that by exactness of the diagram in (1) the range of ∂ is precisely $K_1(A)^\alpha$. Next note that for $x\in K_0(A\times\mathbf{Z})$ we have $\pi(\tau(x)) = 1$ whenever $\partial(x) = 0$. In fact, if $\partial(x) = 0$ then x is in the image of the map

$$K_0(A)\to K_0(A\times\mathbf{Z})$$

(see (1)), and it follows that $\tau(x) \in \tau(K_0(A)) = \mathbf{Z}$, so $\pi(\tau(x)) = 1$. Therefore the formula

$$\rho'(\partial(x)) = e^{2\pi i \tau(x)}$$

defines a map ρ': $K_1(A)^\alpha \to T$ making the diagram below commutative.

$$\begin{array}{ccc} K_0(A \times \mathbf{Z}) & \xrightarrow{\tau} & \mathbf{R} \\ \partial \downarrow & & \downarrow \pi \\ K_1(A)^\alpha & \xrightarrow[\rho']{} & T \end{array}$$

Our proof will be complete once we prove that $\rho' = \rho_\alpha^\tau$.

Let $x \in K_1(A)^\alpha$ and pick $n \in \mathbf{N}$ and $u \in U_n(A)$ such that $x = [u]_1$. From $\alpha_*(x) = x$ it follows that there is $m \geq n$ and elements $h_1, \ldots, h_k \in M_m(A)_{\text{sa}}$ such that

$$\left(u\alpha(u)^*\right) \oplus I_{m-n} = e^{ih_1} \cdots e^{ih_k}.$$

Replacing A by $M_m(A)$, we may assume that $m = n = 1$ and

$$u\alpha(u)^* = e^{ih_1} \cdots e^{ih_k}.$$

Define $\tilde{u}(t) = e^{-th_n} \cdots e^{-tk_1}u$, and notice that $\tilde{u}(0) = u$ and $\tilde{u}(1) = \alpha(u)$. Thus $\tilde{u}$ represents a unitary element in $T_\alpha(A)$ whose image under 'val' is u.

At this moment we need to recall (7) in order to write the following commuting diagram

$$\begin{array}{ccccc} K_1(T_\alpha(A)) & \xrightarrow{k_1} & K_0(A \times \mathbf{Z}) & \xrightarrow{\tau} & \mathbf{R} \\ & \text{val}_* \searrow & \partial \downarrow & & \downarrow \pi \\ & & K_1(A)^\alpha & \xrightarrow{\rho'} & T \end{array}$$

We have $\rho'(x) = \rho'(\text{val}_*([\tilde{u}]_1)) = \pi(\tau(k_1([\tilde{u}]_1)))$ and if we use the formula for $\tau \cdot k_1$ given in (11) we get

$$\begin{aligned} \tau\left(k_1([\tilde{u}]_1)\right) &= \frac{1}{2\pi i}\int_0^1 \tau\left(\tilde{u}'(t)^*\tilde{u}(t)\right) dt \\ &= \frac{1}{2\pi i}\int_0^1 \tau\Bigg(\sum_{1 \leq j \leq k} u^* e^{ith_1} \cdots ih_j e^{ith_j} \cdots e^{ith_k} \\ &\qquad\qquad \cdot e^{-ith_k} \cdots e^{-ith_1}u\Bigg) dt \\ &= \frac{1}{2\pi}\int_0^1 \tau\Bigg(\sum_{1 \leq j \leq k} h_j\Bigg) dt = \frac{1}{2\pi}\tau\Bigg(\sum_{1 \leq j \leq k} h_j\Bigg). \end{aligned}$$

If we now fix a determinant det for (A, τ) we get

$$\rho'(x) = \exp\left(i\tau\left(\sum_{1 \le j \le k} h_j\right)\right) = \det(e^{ih_1} \cdots e^{ih_k})$$

$$= \det(u\alpha(u^*)) = \det(\alpha(u^*)u) = \rho_\alpha^\tau(x).$$

Recall that by (IV.2.iii) the rotation number map is independent of the choice of a determinant. This completes the proof.

We close this chapter with a result that summarizes most of our work from this and the previous chapters.

13. THEOREM. *Let (A, τ) be an integral unital C^* algebra and let α be a trace-preserving automorphism of A. The following are equivalent*:

(i) *ρ_α^τ is the trivial homomorphism.*

(ii) *$A \times_\alpha \mathbf{Z}$ is integral with respect to the dual trace.*

(iii) *If $\langle \alpha \rangle$ denotes the subgroup of* $\mathrm{Aut}(A, \tau)$ *generated by α then the $\langle \alpha \rangle$-invariant determinant obstruction vanishes, i.e.* $\zeta(\langle \alpha \rangle) = 0$.

(iv) *A admits an α-invariant determinant.*

Proof. (i) $\to$ (ii) By (12) we have $\pi \cdot \tau = \rho_\alpha^\tau \cdot \partial = 1$ so for any $x \in K_0(A \times \mathbf{Z})$ we have $e^{2\pi i \tau(x)} = 1$. Thus $\tau(x) \in \mathbf{Z}$.

(ii) $\to$ (iv) Let det be a determinant for $A \times \mathbf{Z}$. By restriction we get a determinant for A and if $u \in \mathrm{U}_n(A)$ for some $n \in \mathbf{N}$ we have

$$\det(\alpha(u)) = \det\big((L \otimes I_n)u(L \otimes I_n)^*\big)$$

$$= \det(L \otimes I_n)\det(u)\det(L \otimes I_n)^{-1} = \det(u).$$

(iv) $\to$ (i) Let det be an α-invariant determinant for A. Given $x = [u]_1 \in K_1(A)^\alpha$ we have

$$\rho_\alpha^\tau(x) = \det(\alpha(u^*)u) = 1$$

whence (i).

(iii) $\leftrightarrow$ (iv) This is precisely Theorem (III.6).

VI. COMMUTATIVE C^* ALGEBRAS

In this chapter we specialize the theory developed so far to commutative algebras. We have tried to make it as self contained as possible since all concepts involved are fairly elementary and do not require much of what was done up to now. We give an alternative definition of rotation number and exploit its relationship with the previously defined one. This will also be used in our applications, namely the ones involving commutative algebras.

Let X be a compact Hausdorff space and consider the algebra $C(X)$. Given a (not necessarily positive) regular Borel measure μ on X with $\mu(X) = 1$, the formula

$$\tau_\mu(f) = \int_X f(x)\, d\mu(x), \qquad f \in C(X)$$

defines a trace on $C(X)$. It is well known that any traced unital commutative C^* algebra is of the form $(C(X), \tau_\mu)$ for some X and μ as above. In our next proposition we characterize the integral ones.

1. PROPOSITION. *Given X and μ as above a necessary and sufficient condition for $(C(X), \tau_\mu)$ to be integral is that for any subset $C \subseteq X$ which is both open and closed, one must have $\mu(C) \in \mathbf{Z}$.*

Proof. (Sufficiency) Suppose $p = (p_{ij}) \in M_n(C(X))$ is a self adjoint projection. Then

$$\begin{aligned}\tau_\mu(p) &= \sum_{1 \le i \le n} \tau_\mu(p_{ii}) = \sum_{1 \le i \le n} \int_X p_{ii}(x)\, d\mu(x) \\ &= \int_X \operatorname{tr}(p(x))\, d\mu(x).\end{aligned}$$

Let for every $k = 0, 1, \ldots, n$

$$C_k = \{x \in X \colon \operatorname{tr}(p(x)) = k\}.$$

Clearly $\{C_k \colon 0 \le k \le n\}$ is a partition of X in open-closed sets. It follows that

$$\tau_\mu(p) = \sum_{0 \le k \le n} \int_{C_k} k\, d\mu(x) = \sum_{0 \le k \le n} k\mu(C_k) \in \mathbf{Z}$$

thus proving $(C(X), \tau_\mu)$ to be integral.

(Necessity) Given that $(C(X), \tau_\mu)$ is integral, let $C \subseteq X$ be an open-closed subset. Let $p \colon X \to \mathbf{C}$ be the characteristic function of C. Then p is a self adjoint projection in $C(X)$ and by our hypothesis

$$\mu(C) = \tau_\mu(p) \in \mathbf{Z}.$$

Of course it follows that for any compact connected space X and for every probability measure μ on X, the pair $(C(X), \tau_\mu)$ is an integral algebra. In what follows we will assume this is the case. We therefore fix a compact connected space X and a probability measure μ on X for the rest of this chapter.

2. Definition. We will denote by $[X, T]$ the group of homotopy classes of continuous mappings from X to T with the group operation given by pointwise multiplication.

Any homeomorphism α: $X \to X$ induces a group automorphism $\alpha_{\#}$: $[X, T] \to [X, T]$ by the formula $\alpha_{\#}[u] = [u \cdot \alpha^{-1}]$ where $[u]$ indicates the homotopy class of u in $[X, T]$ for any continuous u: $X \to T$.

It is routine to verify that $[X, T]$ is an abelian group and that $\alpha_{\#}$ is an automorphism.

Let α: $X \to X$ be a homeomorphism which moreover fixes μ, i.e. $\mu(\alpha(E)) = \mu(E)$ for all Borel subsets E of X. Denote by $[X, T]^{\alpha}$ the subgroup of $[X, T]$ formed by the fixed points of $\alpha_{\#}$ i.e.

$$[X, T]^{\alpha} = \{c \in [X, T] : \alpha_{\#}(c) = c\}.$$

Let u: $X \to T$ be continuous and assume $[u] \in [X, T]^{\alpha}$. That is to say that

$$[\alpha(u^{-1})u] = \alpha_{\#}(-[u]) + [u] = 0.$$

It follows that the mapping $x \to u(\alpha^{-1}(x))^{-1}u(x)$ is homotopic to a constant, and thus liftable to the universal cover of T. Equivalently there is a continuous function h: $X \to \mathbf{R}$ such that

$$u(\alpha^{-1}(x))^{-1}u(x) = e^{ih(x)} \quad \forall x \in X.$$

3. Definition. We will denote by $\tilde{R}^{\mu}_{\alpha}(u)$ or simply by $\tilde{R}_{\alpha}(u)$ the number

$$\tilde{R}_{\alpha}(u) = \exp\left(i\int_X h(x)\, d\mu(x)\right) \in T.$$

4. Lemma. *Given u and v in $[X, T]^{\alpha}$*

(i) *$\tilde{R}_{\alpha}(u)$ does not depend on the lifting h,*

(ii) *$\tilde{R}_{\alpha}(uv) = \tilde{R}_{\alpha}(u)\tilde{R}_{\alpha}(v)$ and*

(iii) *if $[u] = [v]$ then $\tilde{R}_{\alpha}(u) = \tilde{R}_{\alpha}(v)$.*

Proof. (i) Assume h_1 and h_2 are continuous functions from X to $\mathbf{R}$ and

$$u(\alpha^{-1}(x))^{-1}u(x) = e^{ih_1(x)} = e^{ih_2(x)} \quad \forall x \in X.$$

Then for all x in X $h_1(x) - h_2(x) \in 2\pi\mathbf{Z}$. Since X is assumed to be connected and $h_1(x) - h_2(x)$ is a continuous function of x, there will be an integer n such that $h_1(x) - h_2(x) = 2\pi n$ for all x in X. Therefore

$$\exp\left(i\int_X h_1(x)\,d\mu(x)\right) = \exp\left(i\int_X (h_2(x) + 2\pi n)\,d\mu(x)\right)$$

$$= \exp\left(i\int_X h_2(x)\,d\mu(x)\right)$$

proving (i).

(ii) Let h and k be continuous functions from X to $\mathbf{R}$ satisfying

$$u\left(\alpha^{-1}(x)\right)^{-1}u(x) = e^{ih(x)} \quad \text{and}$$

$$v\left(\alpha^{-1}(x)\right)^{-1}v(x) = e^{ik(x)}.$$

So

$$\left((uv)\left(\alpha^{-1}(x)\right)\right)^{-1}((uv)(x)) = e^{ih(x)}e^{ik(x)} = e^{i(h(x)+k(x))}.$$

Therefore we have

$$\tilde{R}_\alpha(uv) = \exp\left(i\int_X (h(x) + k(x))\,d\mu(x)\right)$$

$$= \exp\left(i\int_X h(x)\,d\mu(x)\right)\exp\left(i\int_X k(x)\,d\mu(x)\right)$$

$$= \tilde{R}_\alpha(u)\tilde{R}_\alpha(v).$$

(iii) Let $w = uv^{-1}$. Then $u = wv$ so $\tilde{R}_\alpha(u) = \tilde{R}_\alpha(w)\tilde{R}_\alpha(v)$ and it is enough to prove that $\tilde{R}_\alpha(w) = 1$. Since $[u] = [v]$ by hypothesis, it follows that $[w] = 0$. So w is homotopic to a constant which implies that

$$w(x) = e^{il(x)} \quad \forall x \in X$$

for some continuous function l: $X \to \mathbf{R}$. We then have

$$w\left(\alpha^{-1}(x)\right)^{-1}w(x) = e^{-il(\alpha^{-1}(x))}e^{il(x)} = e^{i(l(x)-l(\alpha^{-1}(x)))}$$

thus

$$\tilde{R}_\alpha(w) = \exp\left(i\int_X l(x)\,d\mu(x)\right)\exp\left(-i\int_X l\left(\alpha^{-1}(x)\right)d\mu(x)\right) = 1$$

because μ is invariant under α. The proof is now complete.

In view of this last result we may give the following

5. Definition. The (commutative) rotation number map of α with respect to μ is the group homomorphism

$$R_\alpha\colon [X,T]^\alpha \to T$$

defined by $R_\alpha([u]) = \tilde{R}_\alpha(u)$ for all $[u]$ in $[X,T]^\alpha$. If we need to make the measure μ explicit we will use the notation R_α^μ.

In order to set the technical tools to relate R_α to our previous definition of rotation number, we introduce some extra notation.

6. Definition. Given a commutative unital C^* algebra A we denote by T_n and Det_n the maps from $M_n(A)$ to A defined for every $n \times n$ matrix $a = (a_{i,j})_{1 \le i,j \le n}$ by

$$T_n(a) = \sum_{1 \le i \le n} a_{i,i} \quad \text{and}$$

$$\mathrm{Det}_n(a) = \sum_\sigma \mathrm{sgn}(\sigma) \prod_{1 \le i \le n} a_{i,\sigma(i)}$$

where the sum is over all permutations σ of n elements.

7. Lemma. *For all h in $M_n(A)$ we have*

$$\mathrm{Det}_n(e^h) = e^{T_n(h)}.$$

Proof. Represent A as $C(Y)$ for some space Y. Then

$$\mathrm{Det}_n(e^h)(y) = \det(e^{h(y)}) = e^{\mathrm{tr}(h(y))} = e^{(T_n(h)(y))} = e^{T_n(y)}(y) \quad \forall y \in Y.$$

8. Definition. We denote by Det_* the mapping

$$\mathrm{Det}_*\colon K_1(C(X)) \to [X,T]$$

defined by $\mathrm{Det}_*([u]_1) = [\mathrm{Det}_n(u)]$ for all u in $\mathrm{U}_n(C(X))$.

9. Proposition. Det_* *is well defined and is a group homomorphism.*

Proof. If u is in $\mathrm{U}_n(C(X))$ and $m > n$ it is clear that $\mathrm{Det}_n(u) = \mathrm{Det}_m(u \oplus I_{m-n})$.

If $\{u_t\colon 0 \le t \le 1\}$ is a continuous path in $\mathrm{U}_n(C(X))$ then $\mathrm{Det}_n(u_t)$ is a homotopy between $\mathrm{Det}_n(u_0)$ and $\mathrm{Det}_n(u_1)$ so $[\mathrm{Det}_n(u_0)] = [\mathrm{Det}_n(u_1)]$ proving that Det_* is well defined. The formula $\mathrm{Det}_n(uv) = \mathrm{Det}_n(u)\mathrm{Det}_n(v)$ is easily verified for all u and v in $\mathrm{U}_n(C(X))$ and proves that Det_* is a group homomorphism.

10. Proposition. *If α is any homeomorphism of X and $z \in K_1(C(X))$ then*

$$\mathrm{Det}_*(\alpha_*(z)) = \alpha_\#(\mathrm{Det}_*(z)).$$

In other words Det_* *is covariant under the action of homeomorphisms of X.*

Proof. If u is a unitary $n \times n$ matrix over $C(X)$ and $z = [u]_1$ then

$$\mathrm{Det}_*(\alpha_*(z)) = \left[\mathrm{Det}_n(\alpha(u))\right] = \left[\alpha(\mathrm{Det}_n(u))\right] = \alpha_\#(\mathrm{Det}_*(z)).$$

We are now ready to state the main result of this chapter relating the notion of commutative rotation number to the one defined in Chapter IV.

11. Theorem. *Let X be a compact connected topological space, let α be a homeomorphism of X and let μ be an invariant probability measure. The following diagram is commutative.*

$$\begin{array}{ccc} K_1(C(X))^\alpha & \xrightarrow{\rho_\alpha^{\tau_\mu}} & T \\ {\scriptstyle \mathrm{Det}_*}\downarrow & \nearrow_{R_\alpha^\mu} & \\ [X,T]^\alpha & & \end{array}$$

Proof. First note that Det_* is indeed a map between the indicated groups by (10). Let $[u]_1 \in K_1(C(X))^\alpha$ where $u \in \mathrm{U}_n(C(X))$. Replacing n by a suitable $m \geq n$ and u by $u \oplus I_{m-n}$ we may assume that there are $h_1, \ldots, h_r$ in $M_m(C(X))$ such that $\alpha(u^*)u = e^{ih_1} \cdots e^{ih_r}$. We simply compute $\rho_\alpha^{\tau_\mu}([u]_1)$ and $R_\alpha^\mu(\mathrm{Det}_*([u]_1))$ to check they agree. Before computing the latter we must write $\alpha(\mathrm{Det}_m(u))^{-1}\mathrm{Det}_m(u)$ as an exponential. We have

$$\alpha(\mathrm{Det}_m(u))^{-1}\mathrm{Det}_m(u) = \mathrm{Det}_m\big(\alpha(u^*)u\big)$$

$$= \mathrm{Det}_m(e^{ih_1} \cdots e^{ih_r}) = e^{iT_m(h_1)} \cdots e^{iT_m(h_r)} = \exp\left(iT_m\left(\sum_{1 \leq k \leq r} h_k\right)\right).$$

Thus according to (5) we have

$$R_\alpha^\mu(\mathrm{Det}_*([u]_1)) = R_\alpha^\mu(\left[\mathrm{Det}_m(u)\right]) = \tilde{R}_\alpha^\mu(\mathrm{Det}_m(u))$$

$$= \exp\left(i\int_X T_m\left(\sum_{1 \leq k \leq r} h_k\right)(x)\, d\mu(x)\right).$$

To compute $\rho_\alpha^{\tau_\mu}([u]_1)$ let $\det_{\tau_\mu}$ be a determinant associated with τ_μ as in (II). So

$$\begin{aligned}\rho_\alpha^{\tau_\mu}([u]_1) &= \det_{\tau_\mu}(\alpha(u^*)u) = \det_{\tau_\mu}(e^{ih_1}\cdots e^{ih_r}) \\ &= e^{i\tau_\mu(h_1)}\cdots e^{\tau_\mu(h_r)} = \exp\left(i\tau_\mu\left(\sum_{1\le k\le r} h_k\right)\right) \\ &= \exp\left(i\int_X T_m\left(\sum_{1\le k\le r} h_k\right)(x)\,d\mu(x)\right)\end{aligned}$$

completing the proof.

One of the interesting features of our last result is the fact that $\rho_\alpha^{\tau_\mu}$ factors through $[X,T]^\alpha$. Based on this we obtain the following generalization of Corollary (3) of [**1**].

12. Corollary. *Let X, α and μ be as above and assume that the first Čech cohomology group $\check{H}^1(X,\mathbf{Z})$ is zero. Then*

(i) $\rho_\alpha^{\tau_\mu}$ *is the trivial homomorphism,*

(ii) *the crossed product algebra $C(X)\times_\alpha \mathbf{Z}$ is integral with respect to its natural trace and*

(iii) $(C(X),\tau_\mu)$ *admits an α-invariant determinant.*

Proof. Following [**23**, pp. 323 ff.] one may prove that $[X,T] = \check{H}^1(X,\mathbf{Z}) = 0$. The result then follows from (11) and (V.13).

VII. Almost Periodic Automorphisms

With this chapter we start a sequence of applications of the theory developed above, especially Theorems (V.12) and (V.13) which in a way summarize the main ideas exposed up to now. Our goal will be to give examples for which (V.12) or (V.13) can be applied giving new information in a more or less concrete setting.

For our first application we consider almost periodic automorphisms of integral algebras. In one of our results we shall give sufficient conditions for the crossed product by $\mathbf{Z}$ to be integral.

In order to fix the objects to be studied here we let A be a unital C^* algebra and α be a $*$-automorphism of A.

1. Definition. We say that α is almost periodic if for every a in A its orbit is relatively compact in the norm topology. In other words the set $\{\alpha^n(a) : n \in \mathbf{Z}\}$ must have compact closure for all a in A.

2. Lemma. *If α is an almost periodic automorphism of A then*

(i) id $\otimes$ α *is an almost periodic automorphism of $M_n(A)$ for all $n \in \mathbf{N}$ and*

(ii) *for every a in A, $n \in \mathbf{N}$ and $\varepsilon > 0$ there is $m > n$ such that $\|\alpha^m(a) - a\| < \varepsilon$. Roughly speaking, the orbit of any a in A comes close to a infinitely often.*

Proof. The first statement is obvious. As for (ii) let B be the open ball centered at a with radius ε. We clearly have

$$\{\alpha^n(a): n \in \mathbf{Z}\} \subseteq \bigcup_{n \in \mathbf{Z}} \alpha^n(B)$$

and if we observe that the distance between the set $\{\alpha^n(a): n \in \mathbf{Z}\}$ and the complement of $\bigcup_{n \in \mathbf{Z}} \alpha^n(B)$ is greater than ε we see that

$$\overline{\{\alpha^n(a): n \in \mathbf{Z}\}} \subseteq \bigcup_{n \in \mathbf{Z}} \alpha^n(B).$$

By compactness there is a finite set F of integers such that

$$\overline{\{\alpha^n(a): n \in \mathbf{Z}\}} \subseteq \bigcup_{k \in F} \alpha^k(B).$$

For some $k \in F$ we must therefore have $\|\alpha^n(a) - \alpha^k(a)\| < \varepsilon$ for infinitely many integers n. The conclusion now follows by taking $m = n - k$ or $m = k - n$, whichever is positive, for a suitable n.

Let us now introduce the notion of fixed point for an automorphism of a C^* algebra.

3. Definition. Let ϕ be a *-homomorphism of A into the complex numbers. We say that ϕ is a fixed point for α if $\phi \cdot \alpha = \phi$.

We should observe that the expression "fixed point" refers to the action of α on the set of complex homomorphisms of A rather than the action on A itself.

Note also that since many C^* algebras have no complex homomorphisms at all, our definition has a limited range of applications. But if A is commutative, say $A = C(X)$ for some compact space X, and if α is induced by a homeomorphism β of X, then the notion above corresponds to the standard notion of fixed point for the action of β on X.

Suppose ϕ is a complex homomorphism of A which is a fixed point for α in the sense of (2). There are two relatively canonical ways to extend ϕ to the crossed product algebra $A \times_\alpha \mathbf{Z}$. On the one hand we can view ϕ as an invariant trace on A and extend it as one usually extends invariant traces. This procedure yields a trace

$$\phi_0\colon A \times \mathbf{Z} \to \mathbf{C}$$

satisfying $\phi_0(\Sigma_n a_n L^n) = \phi(a_0)$ where L is again the unitary in $A \times \mathbf{Z}$ implementing the action. As a trace, ϕ_0 extends as usual to $M_n(A \times \mathbf{Z})$ for all n and we keep the same notation as we have been doing so far. Adopting a different point of view, we may consider ϕ as being part of the covariant 1-dimensional representation of the C^*-dynamical system $(A, \alpha, \mathbf{Z})$ given by (ϕ, t) where ϕ is the 1-dimensional representation of the algebra A and t is the trivial representation of the group $\mathbf{Z}$. By the universal property of the crossed product it follows that there is a unique 1-dimensional representation of $A \times \mathbf{Z}$, here denoted by ϕ_1, satisfying

$$\phi_1\Big(\sum_n a_n L^n\Big) = \sum_n \phi(a_n) t(L)^n = \sum_n \phi(a_n).$$

It should be stressed that ϕ_0 and ϕ_1 are never the same since $\phi_0(L) = 0$ and $\phi_1(L) = 1$.

Another map obtained from ϕ which we will use is the extension of ϕ to any matrix algebra over A when ϕ is viewed as a trace. Precisely, we let for all $n \geq 1$ $\bar{\phi}\colon M_n(A) \to \mathbf{C}$ be defined by $\bar{\phi}(a) = \Sigma_{1 \leq i \leq n} \phi(a_{i,i})$ for every $n \times n$ matrix $a = (a_{i,j})_{1 \leq i,j \leq n}$ in $M_n(A)$. As a trace, $\bar{\phi}$ induces a map (also denoted by $\bar{\phi}$) on $K_0(A)$

$$\bar{\phi}\colon K_0(A) \to \mathbf{R}$$

in the usual way. Observe that this map is equal to the map $\phi_*\colon K_0(A) \to K_0(\mathbf{C})$ once $K_0(\mathbf{C})$ is identified with $\mathbf{Z}$ as usual. It follows that the range of $\bar{\phi}$ on $K_0(A)$ is equal to $\mathbf{Z}$.

Observe that ϕ_0, ϕ_1 and $\bar{\phi}$ when restricted to A (or $M_1(A)$) are all equal to the original ϕ. Whenever we are required to choose one of the above notations for ϕ on A we should pick the notation that seems most appropriate in each special case. For instance, when we use (as we shall do) a convex combination of a given trace τ on A and ϕ considered as a trace we will use the notation $(1 - t)\tau + t\bar{\phi}$ instead of $(1 - t)\tau + t\phi$ even if it makes no difference.

4. THEOREM. *Let (A, τ) be a traced unital C^* algebra and let α be a trace-preserving automorphism of A. Assume that α is almost periodic and has a fixed point ϕ satisfying $\bar{\phi} = \tau$ on $K_0(A)$. Then the equivalent conditions of* (V.13) *are satisfied and $\bar{\phi}_1 = \tau$ on $K_0(A \times_\alpha \mathbf{Z})$. Here $\bar{\phi}_1$ is defined as $\bar{\phi}$ when A is replaced by $A \times \mathbf{Z}$ and ϕ is replaced by ϕ_1.*

Proof. We first notice that the hypothesis that $\bar{\phi} = \tau$ on $K_0(A)$ implies that (A, τ) is integral. Next we prove (V.13.i).

Let $u \in \mathrm{U}_n(A)$ be such that $[u]_1 \in K_1(A)^\alpha$. We must show that $\rho_\alpha^\tau([u]_1) = 1$. Replacing n by a larger integer, we may assume that there are elements $h_1, \dots, h_l$ in $M_n(A)_{\mathrm{sa}}$ such that

$$\alpha(u^*)u = e^{ih_1} \cdots e^{ih_l}.$$

In order to simplify our notation we define the following symbols:

$$h := (h_1, \dots, h_l),$$

$$e^{ih} := e^{ih_1} \cdots e^{ih_l},$$

$$\sum h := h_1 + \cdots + h_l,$$

$$\alpha(h) := (\alpha(h_1), \dots, \alpha(h_l)).$$

Given $\varepsilon > 0$ with $\varepsilon < 1$, choose $m \geq 1$ according to (2.ii) such that

$$\|\alpha^m(u) - u\| < \varepsilon;$$

thus

$$\|\alpha^m(u^*)u - 1\| < \varepsilon.$$

We may therefore pick k in $M_n(A)_{\mathrm{sa}}$ such that $\alpha^m(u^*)u = e^{ik}$ with $\|k\| < \nu(\varepsilon)$ where $\nu = \nu(\varepsilon)$ is some positive valued function such that $\lim_{\varepsilon \to 0} \nu(\varepsilon) = 0$. This is simply because the exponential map is a local homeomorphism. Writing

$$\alpha^m(u^*)u = \alpha^{m-1}(\alpha(u^*)u) \cdots \alpha(\alpha(u^*)u)\alpha(u^*)u$$

we get

$$\alpha^m(u^*)u = e^{i\alpha^{m-1}(h)} \cdots e^{i\alpha(h)}e^{ih} = e^{ik}.$$

By (I.6.iii) we have that for any trace T on A with respect to which A is integral

$$T\left(\sum \alpha^{m-1}(h)\right) + \cdots + T\left(\sum h\right) - T(k) \in 2\pi\mathbf{Z}.$$

How many traces with this property are at hand? Certainly τ and $\bar{\phi}$ are among them but because τ and $\bar{\phi}$ agree on $K_0(A)$, any convex combination of these is just as good. So for every $t \in [0, 1]$ let τ_t be the trace on A defined by

$$\tau_t = (1 - t)\tau + t\bar{\phi}.$$

It follows that

$$\tau_t\left(\sum \alpha^{m-1}(h)\right) + \cdots + \tau_t\left(\sum h\right) - \tau_t(k) \in 2\pi\mathbf{Z}$$

must be constant, or

$$\tau\left(\sum \alpha^{m-1}(h)\right) + \cdots + \tau\left(\sum h\right) - \tau(k)$$
$$= \bar{\phi}\left(\sum \alpha^{m-1}(h)\right) + \cdots + \bar{\phi}\left(\sum h\right) - \bar{\phi}(k).$$

Since both τ and $\bar{\phi}$ are α-invariant, the expression above becomes

$$(\dagger) \qquad m\tau\left(\sum h\right) - \tau(k) = m\bar{\phi}\left(\sum h\right) - \bar{\phi}(k).$$

Before we continue, we need to prove that $\bar{\phi}(\Sigma h) \in 2\pi\mathbf{Z}$. To do this apply id $\otimes$ ϕ to both sides of the expression

$$\alpha(u^*)u = e^{ih_1} \cdots e^{ih_l}$$

to get

$$I_n = e^{i(\mathrm{id}\otimes\phi)(h_1)} \cdots e^{i(\mathrm{id}\otimes\phi)(h_l)} \quad \text{in } M_n(\mathbf{C}).$$

Applying the standard determinant it follows that

$$1 = e^{i\,\mathrm{tr}((\mathrm{id}\otimes\phi)(h_1))} \cdots e^{i\,\mathrm{tr}((\mathrm{id}\otimes\phi)(h_l))}$$

where tr is the standard, non normalized trace on $M_n(\mathbf{C})$. We then have

$$1 = e^{i\,\mathrm{tr}((\mathrm{id}\otimes\phi)(\Sigma h))} = e^{i\bar{\phi}(\Sigma h)}$$

so that $\bar{\phi}(\Sigma h) \in 2\pi\mathbf{Z}$ as claimed.

The expression ($\dagger$) above gives

$$\left|\tau\left(\sum h\right) - \bar{\phi}\left(\sum h\right)\right| = (1/m)\left|\tau(k) - \bar{\phi}(k)\right|$$
$$\leq (1/m)\left(\|\tau\| + \|\bar{\phi}\|\right)\|k\| \leq (1/m)\left(\|\tau\| + n\right)\nu(\varepsilon)$$
$$\leq \left(\|\tau\| + n\right)\nu(\varepsilon).$$

The left hand side does not depend on ε while the right hand side has limit zero for $\varepsilon \to 0$, so we conclude that

$$\tau\left(\sum h\right) = \bar{\phi}\left(\sum h\right) \in 2\pi\mathbf{Z}.$$

Now pick any determinant det for A associated with τ. Then

$$\rho_\alpha^\tau([u]_1) = \det\left(\alpha(u^*)u\right) = e^{i\tau(\Sigma h)} = 1.$$

This proves condition (i) of (V.13) and therefore also (ii) through (iv).

Next we must prove that $\bar{\phi}_1 = \tau$ on $K_0(A \times \mathbf{Z})$. To do this we use once more the traces τ_t defined earlier. Observe that all the hypotheses assumed for τ are satisfied by τ_t. Thus by what we have just proved we conclude that $A \times \mathbf{Z}$ is integral with respect to (the natural extension to $A \times \mathbf{Z}$ of) τ_t. It is easily seen that

$$\tau_t = (1 - t)\tau + t\phi_0$$

on $A \times \mathbf{Z}$, by direct computation on the set of elements of the form aL^k for $a \in A$ and $k \in \mathbf{Z}$.

Let $p \in M_n(A \times \mathbf{Z})$ be a self adjoint projection. Then by the observation above $\tau_t(p) \in \mathbf{Z}$ for all $t \in [0,1]$. More explicitly

$$(1 - t)\tau(p) + t\phi_0(p) \in \mathbf{Z},$$

from which we deduce that $\tau(p) = \phi_0(p)$. It is now enough to show that $\phi_0(p) = \bar{\phi}_1(p)$.

Recall that the circle group T acts on $A \times \mathbf{Z}$ by the dual action $\hat{\alpha}$. For all $a \in A$, $n \in \mathbf{Z}$ we have

$$\hat{\alpha}_\lambda(aL^n) = a\lambda^n L^n \qquad \forall \lambda \in T.$$

Consider the mapping $E\colon A \times \mathbf{Z} \to A$ defined by

$$E(x) = \int_T \hat{\alpha}_\lambda(x)\, d\lambda, \qquad \forall x \in A \times \mathbf{Z}.$$

A simple computation shows that for all $a \in A$ and $n \in \mathbf{Z}$ with $n \neq 0$ one has $E(a) = a$ and $E(aL^n) = 0$. One can also prove that E is a conditional expectation, as defined in [**17**], from $A \times \mathbf{Z}$ onto A. Note that $\phi_0 = \phi_1 \cdot E$. It follows that

$$\begin{aligned}
\phi_0(p) &= \sum_{1 \le i \le n} \phi_0(p_{i,i}) = \sum_{1 \le i \le n} \phi_1(E(p_{i,i})) \\
&= \sum_{1 \le i \le n} \phi_1\left(\int_T \hat{\alpha}_\lambda(p_{i,i})\, d\lambda\right) \\
&= \int_T \sum_{1 \le i \le n} \phi_1(\hat{\alpha}_\lambda(p_{i,i}))\, d\lambda \\
&= \int_T \operatorname{tr}\Big(\big(\mathrm{id}_{M_n} \otimes (\phi_1 \cdot \hat{\alpha}_\lambda)\big)(p)\Big)\, d\lambda.
\end{aligned}$$

The argument of 'tr' above is a projection in $M_n(\mathbf{C})$ varying continuously with the parameter λ, so its trace must be constant. We conclude that with $\lambda = 1$

$$\begin{aligned}\phi_0(p) &= \operatorname{tr}\bigl((\mathrm{id}_{M_n} \otimes (\phi_1 \cdot \hat{\alpha}_1))(p)\bigr)\\ &= \operatorname{tr}\bigl((\mathrm{id}_{M_n} \otimes \phi_1)(p)\bigr) = \sum_{1 \le i \le n} \phi_1(p_{i,i}) = \bar{\phi}_1(p).\end{aligned}$$

The proof is now complete.

5. COROLLARY. *Let (A, τ) be a traced unital C^* algebra and α be a trace-preserving automorphism of A. Assume that α is almost periodic and that for some integer n, α^n has a fixed point ϕ (i.e. a periodic point for α) satisfying $\bar{\phi} = \tau$ on $K_0(A)$. Then the range of (the natural extension to $A \times_\alpha \mathbf{Z}$ of) τ on $K_0(A \times_\alpha \mathbf{Z})$ is contained in $(1/n)\mathbf{Z}$.*

Proof. The conclusions of (4) hold for α^n. In particular $\rho^\tau_{\alpha^n} = 1$. By (IV.4) we have $(\rho^\tau_\alpha)^n = 1$ so the range of ρ^τ_α is contained in the set of nth roots of unity. By (V.12) the conclusion follows.

In the special case of commutative algebras we obtain the following

6. COROLLARY. *Let X be a connected compact topological space with a probability measure μ. Also let α be a homeomorphism of X which leave μ invariant, and assume that the set of all powers of α, $\{\alpha^n\colon n \in \mathbf{Z}\}$, is equicontinuous (with respect to the unique uniform structure that a compact space X admits). If α has a periodic point (in the standard sense) with period n and if τ is the extension to $C(X) \times_\alpha \mathbf{Z}$ of the trace on $C(X)$ given by integration against μ, we have*

$$\tau(K_0(C(X) \times_\alpha \mathbf{Z})) \subseteq (1/n)\mathbf{Z}.$$

Proof. We should note that we are also denoting by α the induced automorphism on $C(X)$. The proof will of course consist of checking the hypothesis of (5).

From the Arzela-Ascoli theorem it follows that α is almost periodic as an automorphism of $C(X)$.

Let $q \in X$ be such that $\alpha^n(q) = q$ and let ϕ be the complex homomorphism of $C(X)$ given by evaluation at q. We will prove that $\bar{\phi} = \tau$ on $K_0(C(X))$. Adopting the vector bundle point of view, note that for any complex vector bundle F over X the map $\bar{\phi}$ gives the dimension of the fiber F_q while τ gives the common dimension of all fibers since X is connected. It is then clear that $\bar{\phi} = \tau$. This concludes the proof.

7. *Note*. In order to show the necessity of the equicontinuity hypothesis assumed in (6) or the almost periodicity assumed in (4) and (5) we recall the example of the twist of the annulus (IV.7) in which there are fixed points but where the conclusions of (4), (5) or (6) are not satisfied.

VIII. Automorphisms of Connected Groups

In our second application of (V.13) we consider a topological group G which is supposed to be compact and connected and we fix an automorphism α of G. If we let $A = C(G)$, the group automorphism α induces a $*$-automorphism on A (which we still denote by α) according to

$$\alpha(f)(t) = f\big(\alpha^{-1}(t)\big) \qquad \forall t \in G,\ f \in C(G).$$

Let μ be the normalized Haar measure on G and let τ be the trace on A given by integration against μ.

It is well known that a compact group has a unique normalized Haar measure, from which it follows that μ is invariant under the group automorphism α. Consequently τ is invariant under the algebra automorphism α.

Recalling Proposition (VI.1) and also the fact that G is assumed connected, one sees that A is integral with respect to τ. We are then in a position to ask whether the triple (A, τ, α) satisfies the equivalent conditions of (V.13). It is our goal in this section to give an affirmative answer to this question.

We begin by introducing three maps from $C(G)$ to $C(G \times G)$ (equivalently from A to $A \otimes A$) which will prove to be of great importance.

1. Definition. For every $f \in C(G)$ define $\Delta(f)$, $i_1(f)$ and $i_2(f)$ in $C(G \times G)$ by

$$\Delta(f)(x, y) = f(xy),$$

$$i_1(f)(x, y) = f(x) \quad \text{and}$$

$$i_2(f)(x, y) = f(y) \qquad \forall (x, y) \in G \times G.$$

The reader should have no difficulty in checking that Δ, i_1 and i_2 define $*$-homomorphisms from $C(G)$ to $C(G \times G)$.

2. Lemma. *Let $u \in U_1(C(G))$. Then*

$$\Delta_*([u]_1) = i_{1*}([u]_1) + i_{2*}([u]_1)$$

in $K_1(C(G \times G))$.

Proof. Let $v(x, y) = u(xy)\overline{u(x)u(y)}$ for all (x, y) in $G \times G$ so that v is a map $v\colon G \times G \to T$. We claim that v induces the trivial map at the level of fundamental groups. In fact, let $\gamma \in \Pi_1(G \times G)$ and suppose it is represented by a continuous loop $(x(t), y(t))_{0 \leq t \leq 1}$. Recall that for any two loops in a topological group the pointwise product has a homotopy class equal to the sum (in Π_1) of the individual classes. It follows that the class of the loop $v(x(t), y(t))$ is the sum of the classes of $u(x(t)y(t))$, $\overline{u(x(t))}$ and $\overline{u(y(t))}$ in $\Pi_1(T)$.

Consider the maps

$$\begin{aligned} &p\colon G \times G \to G, \\ &q_1\colon G \times G \to G \quad \text{and} \\ &q_2\colon G \times G \to G \end{aligned}$$

given by $p(x, y) = xy$, $q_1(x, y) = x$ and $q_2(x, y) = y$ for all (x, y) in $G \times G$.

With this notation the statement above becomes

$$v_*(\gamma) = u_*(p_*(\gamma)) - u_*(q_{1*}(\gamma)) - u_*(q_{2*}(\gamma))$$

in $\Pi_1(T)$.

The same observation on pointwise product of loops we made earlier gives us

$$p_*(\gamma) = q_{1*}(\gamma) + q_{2*}(\gamma)$$

in $\Pi_1(G)$. Therefore

$$v_*(\gamma) = u_*(q_{1*}(\gamma) + q_{2*}(\gamma)) - u_*(q_{1*}(\gamma)) - u_*(q_{2*}(\gamma)) = 0$$

proving our claim. We conclude that v lifts to the universal cover of the circle, which is equivalent to saying that there is a continuous real valued function h on $G \times G$ such that

$$v(x, y) = e^{ih(x,y)} \qquad \forall (x, y) \in G \times G.$$

We then have

$$\Delta(u) i_1(u)^{-1} i_2(u)^{-1} = e^{ih}.$$

So $\Delta_*([u]_1) = i_{1*}([u]_1) + i_{2*}([u]_1)$ in $K_1(C(G \times G))$.

We should say a few words about the hypothesis made above that $u \in U_1(C(G))$. It looks plausible to expect (2) to hold also for $u \in U_n(C(G \times G))$ for all n, but contrary to our intuition this is not so. We have found a counter example already for $n = 2$ and $G = T^3$. A brief discussion of this example is included in appendix B. Nevertheless we will be able to use (2) in a very efficient way. What will cover our apparent deficiency will be the existence of the algebra valued determinant defined in (VI.6).

We now present the main result of this section.

3. THEOREM. *Let G be a compact connected topological group. Let α denote a given automorphism of G, as well as the induced automorphism of $C(G)$. Denote by τ the trace on $C(G)$ given by integration against the normalized Haar measure on G. Then the triple $(C(G), \tau, \alpha)$ satisfies the equivalent conditions of* (V.13). *In particular $C(G) \times_\alpha \mathbf{Z}$ is integral with respect to its natural trace and $(C(G), \tau)$ admits an α-invariant determinant.*

Proof. It is enough to prove (V.13.iv). Using (VI.1) together with (II.10) for the algebra $C(G \times G)$ with the trace τ_2 given by normalized Haar measure on $G \times G$, we conclude that $(C(G \times G), \tau_2)$ admits a determinant, say det.

For all $u \in \mathrm{U}_n(C(G))$, $n \in \mathbf{N}$ let

$$\delta(u) = \det\left(i_1(\mathrm{Det}_n(u))i_2(\mathrm{Det}_n(u))\Delta(\mathrm{Det}_n(u))^{-1}\right)$$

where Det_n is defined in (VI.6). We shall prove that δ is an α-invariant determinant for $C(G)$. It is certainly a group homomorphism into T.

Let $h \in M_n(C(G))_{\mathrm{sa}}$. We have

$$\begin{aligned}\delta(e^{ih}) &= \det\left(i_1(e^{iT_n(h)})i_2(e^{iT_n(h)})\Delta(e^{-iT_n(h)})\right)\\ &= \det\left(\exp\left(i\left(i_1(T_n(h)) + i_2(T_n(h)) - \Delta(T_n(h))\right)\right)\right)\\ &= \exp\left(i\tau_2\left(i_1(T_n(h)) + i_2(T_n(h)) - \Delta(T_n(h))\right)\right).\end{aligned}$$

We leave for the reader to verify that i_1, i_2 and Δ are trace-preserving. So

$$\begin{aligned}\delta(e^{ih}) &= \exp\left(i\left(\tau(T_n(h)) + \tau(T_n(h)) - \tau(T_n(h))\right)\right)\\ &= \exp\left(i\tau(T_n(h))\right) = e^{i\tau(h)}.\end{aligned}$$

In order to complete our proof we must verify that δ is left invariant by α. Let $n \in \mathbf{N}$ and $u \in \mathrm{U}_n(C(G))$. Put $v = \mathrm{Det}_n(u)$ and use (2) to write

$$i_1(v)i_2(v)\Delta(v)^{-1} = e^{ih}$$

for some $h \in C(G \times G)_{\mathrm{sa}}$. We have

$$\delta(u) = \det\left(i_1(v)i_2(v)\Delta(v)^{-1}\right) = \det(e^{ih}) = e^{i\tau_2(h)}.$$

We now compute $\delta(\alpha(u))$. Since $\alpha(v) = \mathrm{Det}_n(\alpha(u))$ we have

$$\delta(\alpha(u)) = \det\big(i_1(\alpha(v))i_2(\alpha(v))\Delta(\alpha(v))^{-1}\big).$$

Denote by $\alpha \otimes \alpha$ the automorphism of $C(G \times G)$ given by

$$\alpha \otimes \alpha(f)(x, y) = f\big(\alpha^{-1}(x), \alpha^{-1}(y)\big).$$

It is routine to verify that

(a) $$\alpha \otimes \alpha \cdot i_1 = i_1 \cdot \alpha,$$

(b) $$\alpha \otimes \alpha \cdot i_2 = i_2 \cdot \alpha \quad \text{and}$$

(c) $$\alpha \otimes \alpha \cdot \Delta = \Delta \cdot \alpha.$$

That is i_1, i_2 and Δ are α-equivariant. One should note that multiplicativity of α as a group automorphism is used in (c) and nowhere else. It is nevertheless crucial for our argument.

We then have

$$\begin{aligned}\delta(\alpha(u)) &= \det\Big(\alpha \otimes \alpha\big(i_1(v)i_2(v)\Delta(v)^{-1}\big)\Big)\\ &= \det\big(\alpha \otimes \alpha(e^{ih})\big) = e^{i\tau_2(\alpha\otimes\alpha(h))} = e^{i\tau_2(h)}\end{aligned}$$

since τ_2 is certainly $\alpha \otimes \alpha$-invariant. By comparison we see that $\delta(u) = \delta(\alpha(u))$ so that (V.13.iv) is verified. This concludes the proof.

4. Corollary. *Let H be the group obtained as the semidirect product of a free abelian group by $\mathbf{Z}$. Then $C^*(H)$ with its canonical trace τ is integral. Moreover $C^*(H)$ contains no non-trivial idempotents.*

Proof. Assume that $H = \mathbf{Z}^n \times_\alpha \mathbf{Z}$ where n is an integer or equals $+\infty$, and α is an automorphism of $\mathbf{Z}^n$. It is well known that

$$C^*(H) \cong C\big(\widehat{\mathbf{Z}^n}\big) \times_\alpha \mathbf{Z}.$$

It then follows from (3) that $C^*(H)$ is integral.

If $p \in C^*(H)$ is an idempotent, we have on the one hand $\tau(p) \in \mathbf{Z}$, and on the other $\tau(p) \in [0,1]$. Thus $\tau(p) = 0$ or $\tau(p) = 1$.

Since H is an amenable group, $C^*(H)$ coincides with $C_r^*(H)$, the reduced C^* algebra of H [**12**]. A simple computation shows that the canonical trace is faithful on $C_r^*(\Gamma)$ for any discrete group Γ so that τ is faithful on $C^*(H)$.

It follows that if $\tau(p) = 0$ we must have $p = 0$. In case $\tau(p) = 1$ we have $\tau(1 - p) = 0$ which implies that $p = 1$.

We should note that (4) applies to the discrete Heisenberg group, the group of 3×3 upper triangular integer matrices with ones in the diagonal. The reason being that it can be described as the semidirect product $\mathbf{Z}^2 \times_\alpha \mathbf{Z}$ where α is the automorphism of $\mathbf{Z}^2$ represented as an element of $GL_2(\mathbf{Z})$ by $\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$.

We may also use (4) to obtain a collection of torsion free groups whose C^* algebra have no non-trivial idempotents.

IX. Translations and Affine Homeomorphisms of Connected Groups

We again consider a fixed connected compact topological group G as in (VIII), but now we concentrate our attention on a different class of automorphisms of $C(G)$. We first study the automorphisms of this algebra induced by translations on G, and then, adding the results of (VIII), we study affine homeomorphisms (see (8) ff.).

1. Definition. For every g in G we denote by λ_g the map $\lambda_g\colon t \in G \to gt \in G$. We also denote by λ_g the induced automorphism of $C(G)$ i.e. $\lambda_g(f)(t) = f(g^{-1}t)$ for all t in G and f in $C(G)$.

In what follows we adopt the topologists' convention according to which $K_1(C(G))$ is denoted by $K^1(G)$.

Since G is connected, for any g in G the group homeomorphism λ_g is homotopic to the identity map. The same clearly applies to the $*$-automorphism λ_g of $C(G)$. This said, we see that at the level of K_1 groups λ_{g*} is the identity map, so that $K^1(G)^{\lambda_g}$ (see (IV.1)) is equal to the whole of $K^1(G)$ and the rotation number map has $K^1(G)$ as its domain of definition. We may thus give the following

2. Definition. We denote by b the map

$$b\colon K^1(G) \times G \to T$$

defined by $b(x, g) = \rho^\tau_{\lambda_g}(x)$, where τ is, as usual, the trace on $C(G)$ associated with normalized Haar measure on G.

3. Proposition. (i) *For all g in G the map $x \in K^1(G) \to b(x, g) \in T$ is a group homomorphism.*

(ii) *For all x in $K^1(G)$ the map $g \in G \to b(x, g) \in T$ is a continuous group homomorphism, i.e. a character of G.*

Proof. (i) Follows from the fact that $\rho^\tau_{\lambda_g}$ is a group homomorphism (see (IV.2)).

(ii) Fix $x \in K^1(G)$ and let $u \in \mathrm{U}_n(C(G))$, $(n \in \mathbf{N})$ be such that $x = [u]_1$. Also let det be a determinant for $C(G)$ associated with τ. We then have for all g in G

$$b(x, g) = \det\big(\lambda_g(u^*)u\big).$$

Continuity of $b(x, g)$ with respect to the second variable is a consequence of (II.10).

If h is another element in G we have

$$\begin{aligned} b(x, gh) &= \det\big(\lambda_{gh}(u^*)u\big) = \det\big(\lambda_g\big(\lambda_h(u^*)u\big)\lambda_g(u^*)u\big) \\ &= \det\big(\lambda_g\big(\lambda_h(u^*)u\big)\big)\det\big(\lambda_g(u^*)u\big) \\ &= \det\big(\lambda_h(u^*)u\big)\det\big(\lambda_g(u^*)u\big) \end{aligned}$$

by (III.1) since $\lambda_h(u^*)u \in \mathrm{U}_n(C(G))_0$. It follows that $b(x, gh) = b(x, g)b(x, h)$ completing the proof.

The outcome of our last proposition is that b is a bi-character on $K^1(G) \times G$. We should note that the process above gives characters of G for each element of $K^1(G)$. Of course not all groups G have non-trivial characters, so b may well be the trivial bi-character. In this case we have the following result.

4. Proposition. *Let G be as above and assume that G is perfect, i.e. $G = \overline{[G, G]}$. Let τ be the trace on $C(G)$ associated with normalized Haar measure. Then for every g in G the triple $(C(G), \tau, \lambda_g)$ satisfies the equivalent conditions of* (V.13).

Proof. From the fact that G is perfect it follows that G admits no non-trivial characters. Thus $b = 1$, or equivalently $\rho^{\tau}_{\lambda_g}(x) = 1$ for all g in G and x in $K^1(G)$. It follows that (V.13.i) holds. This completes the proof.

We now introduce two new mappings.

5. Definition. Let

$$P\colon K^1(G) \to \mathrm{Hom}(G, T) \quad \text{and} \quad J\colon \mathrm{Hom}(G, T) \to K^1(G)$$

be defined as follows. For x in $K^1(G)$ we let

$$P(x)(g) = \rho^{\tau}_{\lambda_g}(x) \qquad \forall g \in G.$$

For χ in $\mathrm{Hom}(G,T)$ we view χ as a unitary element in $C(G)$ and put $J(\chi) = [\chi]_1$.

It is an easy exercise to verify that P and J are group homomorphisms.

6. PROPOSITION. *Given J and P as above we have $P \cdot J = \mathrm{id}_{\mathrm{Hom}(G,T)}$.*

Proof. Let $\chi \in \mathrm{Hom}(G,T)$. Then for every g in G we have

$$P \cdot J(\chi)(g) = \rho^{\tau}_{\lambda_g}([\chi]_1) = \det\big(\lambda_g(\chi^*)\chi\big).$$

Now observe that for all $t \in G$

$$\big(\lambda_g(\chi^*)\chi\big)(t) = \chi\big(\lambda_{g^{-1}}(t)\big)^{-1}\chi(t) = \chi(g^{-1}t)^{-1}\chi(t) = \chi(g).$$

Therefore $P \cdot J(\chi)(g) = \chi(g)$ so that $P \cdot J(\chi) = \chi$.

An immediate but relevant consequence is:

7. COROLLARY. *J embeds* $\mathrm{Hom}(G,T)$ *as a complemented subgroup of $K^1(G)$ for which P is a left inverse.*

We omit the proof, which is elementary.

Yet another consequence of (6) is that any character of G is of the form $b(x,\cdot)$ for some x in $K^1(G)$.

We now start our study of affine homeomorphisms. Some of the results to be obtained shortly, specifically (11), hold for translations as well, so that we are not leaving aside our interest in translations.

8. DEFINITION. Let η be a homeomorphism of G. We say that η is an affine homeomorphism if there are $g \in G$ and $\alpha \in \mathrm{Aut}(G)$ such that $\eta = \lambda_g \cdot \alpha$. In this case we also denote by η the automorphism of $C(G)$ given by $\eta(f)(t) = f(\eta^{-1}(t))\ \forall t \in G,\ f \in C(G)$.

9. LEMMA. *Given $\eta = \lambda_g \cdot \alpha$ as above*

(i) *$\eta_* = \alpha_*$ as automorphisms of $K^1(G)$,*

(ii) *the fixed point subgroups $K^1(G)^\eta$ and $K^1(G)^\alpha$ are the same and*

(iii) *the rotation number map ρ^τ_η is the restriction of $\rho^\tau_{\lambda_g}$ to $K^1(G)^\eta$.*

Proof. As noted earlier $\lambda_{g*} = \mathrm{id}_{K^1(G)}$ because G is connected, whence $\eta_* = \alpha_*$. This proves (i) and also (ii). In order to prove (iii) we use (VIII.3) and fix an α-invariant determinant det for $(C(G),\tau)$.

Let $x \in K^1(G)^\eta$ with $x = [u]_1$ for some $u \in \mathrm{U}_n(C(G))$, $n \in \mathbf{N}$. We have

$$\begin{aligned}\rho_\eta^\tau(x) &= \det\big(\eta(u^*)u\big) = \det\big(\lambda_g\big(\alpha(u^*)\big)u\big)\\ &= \det\big(\lambda_g\big(\alpha(u^*)u\big)\big)\det\big(\lambda_g(u^*)u\big)\\ &= \det\big(\lambda_g\big(\alpha(u^*)u\big)\big)\rho_{\lambda_g}^\tau(x).\end{aligned}$$

It is now enough to check that $\det(\lambda_g(\alpha(u^*)u)) = 1$. To do this first recall that since $[u]_1 \in K^1(G)^\alpha$ we have that (perhaps for a larger n) $\alpha(u^*)u \in \mathrm{U}_n(C(G))_0$. We may thus apply (III.1) to conclude that

$$\det\big(\lambda_g\big(\alpha(u^*)u\big)\big) = \det\big(\alpha(u^*)u\big)$$

and since det is α-invariant

$$\det\big(\alpha(u^*)u\big) = \det\big(\alpha(u)\big)^{-1}\det(u) = 1.$$

Given $\alpha \in \mathrm{Aut}(G)$, we have seen how α acts on $C(G)$ as well as on $K^1(G)$. We shall also use the natural action of α on $\mathrm{Hom}(G,T)$. For $\chi \in \mathrm{Hom}(G,T)$ we let $\alpha(\chi)(t) = \chi(\alpha^{-1}(t))$ for all t in G. It is clear that this becomes an automorphism of $\mathrm{Hom}(G,T)$ (which we still denote by α).

10. Lemma. *P and J are equivariant with respect to the action of* $\mathrm{Aut}(G)$ *on* $K^1(G)$ *and* $\mathrm{Hom}(G,T)$. *In other words given* $\alpha \in \mathrm{Aut}(G)$ *we have* $P \cdot \alpha_* = \alpha \cdot P$ *and* $J \cdot \alpha = \alpha_* \cdot J$.

Proof. Let $\alpha \in \mathrm{Aut}(G)$ and $\chi \in \mathrm{Hom}(G,T)$. Then

$$\begin{aligned}J(\alpha(\chi)) &= J(\chi \cdot \alpha^{-1}) = [\chi \cdot \alpha^{-1}]_1 = [\alpha(\chi)]_1\\ &= \alpha_*([\chi]_1) = \alpha_*(J(\chi)).\end{aligned}$$

Thus $J \cdot \alpha = \alpha_* \cdot J$.

Next observe that for $\alpha \in \mathrm{Aut}(G)$ and t, $g \in G$

$$(\lambda_g \cdot \alpha)(t) = g\alpha(t) = \alpha\big(\alpha^{-1}(g)t\big) = \big(\alpha \cdot \lambda_{\alpha^{-1}(g)}\big)(t),$$

i.e. $\lambda_g \cdot \alpha = \alpha \cdot \lambda_{\alpha^{-1}(g)}$. Thus for $[u]_1 \in K^1(G)$ we have for all $g \in G$

$$\begin{aligned}P(\alpha_*([u]_1))(g) &= \rho_{\lambda_g}^\tau(\alpha_*([u]_1)) = \det\big(\lambda_g\big(\alpha(u^*)\big)\alpha(u)\big)\\ &= \det\big(\alpha\big(\lambda_{\alpha^{-1}(g)}(u^*)u\big)\big).\end{aligned}$$

By (III.1) and the fact that $\lambda_{\alpha^{-1}(g)}(u^*)u \in \mathrm{U}_n(C(G))_0$ for some $n \in \mathbf{N}$ we have

$$P(\alpha_*([u]_1))(g) = \det\bigl(\lambda_{\alpha^{-1}(g)}(u^*)u\bigr) = \rho^\tau_{\lambda_{\alpha^{-1}(g)}}([u]_1)$$
$$= P([u]_1)(\alpha^{-1}(g)) = \alpha(P([u]_1))(g).$$

Therefore $P \cdot \alpha_* = \alpha \cdot P$.

The following is the main result of this section. Since translations are a special case of affine homeomorphisms, it applies to the former type of homeomorphisms as well.

11. THEOREM. *Let G be a compact connected topological group and $\eta = \lambda_g \cdot \alpha$ where $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. Also let τ be the trace on $C(G)$ given by Haar measure on G.*

(i) *The following diagram commutes*

$$\begin{tikzcd}
\mathrm{Hom}(G,T)^\alpha \arrow[dr, "J"] \arrow[drr, bend left, "\cdot(g)"] \arrow[dd, equal] & & \\
& K^1(G)^\alpha \arrow[r, "\rho^\tau_\eta"] \arrow[dl, "P"] & T \\
\mathrm{Hom}(G,T)^\alpha \arrow[urr, "\cdot(g)"'] & &
\end{tikzcd}$$

where $\mathrm{Hom}(G,T)^\alpha$ denotes the subgroup of fixed points with respect to the action of α and $\cdot(g)$ denotes evaluation of characters on g.

(ii) *The range of ρ^τ_η is equal to $\{\chi(g)\colon \chi \in \mathrm{Hom}(G,T)^\alpha\}$.*

(iii) *The range of the natural trace on $K_0(C(G) \times_\eta \mathbf{Z})$ is equal to*

$$\{t \in \mathbf{R}\colon e^{2\pi i t} = \chi(g) \textit{ for some } \chi \in \mathrm{Hom}(G,T)^\alpha\}.$$

Proof. (i) First note that the equivariance of P and J as proved in (10) ensures that the maps in our diagram go into the indicated groups. Now (6) yields the commutativity of the left hand side triangle. To prove commutativity of the upper right hand side triangle let $\chi \in \mathrm{Hom}(G,T)^\alpha$. Then using any determinant for $(C(G), \tau)$ we have

$$\rho^\tau_\eta(J(\chi)) = \det\bigl(\eta(\chi^*)\chi\bigr) = \det\bigl(\lambda_g(\alpha(\chi^*))\chi\bigr).$$

Note that for all $t \in G$

$$\begin{aligned}\left(\lambda_g(\alpha(\chi^*))\chi\right)(t) &= \chi\left(\alpha^{-1}(\lambda_{g^{-1}}(t))\right)^{-1}\chi(t)\\ &= \chi(g^{-1}t)^{-1}\chi(t) = \chi(g).\end{aligned}$$

Thus

$$\rho_\eta^\tau(J(\chi)) = \chi(g).$$

Finally, if $[u]_1 \in K^1(G)^\alpha$ then

$$P([u]_1)(g) = \rho_{\lambda_g}^\tau([u]_1) = \rho_\eta^\tau([u]_1) \qquad \text{by (9.iii)}.$$

To prove (ii) note that since P is surjective

$$\mathrm{Range}(\rho_\eta^\tau) = \left(P\left(K_1(G)^\alpha\right)\right)(g) = \left(\mathrm{Hom}(G,T)^\alpha\right)(g)$$

proving (ii).

In view of (V.12) we see that (iii) follows from (ii), completing the proof.

As a direct consequence of our last theorem we prove a result of Rieffel, Pimsner and Voiculescu [**18, 15**] on the range of the trace on K_0 of the irrational (as well as rational) rotation C^* algebra.

12. Example. Let $\theta \in \mathbf{R}$ and denote by A_θ the algebra $C(T) \times_{\lambda_g} \mathbf{Z}$ where $g = e^{2\pi i\theta} \in T$. Let τ be the trace on $C(T)$ given by Haar measure on T. We may then apply (11) with $G = T$, $g = e^{2\pi i\theta}$ and $\alpha = \mathrm{id}_G$. The conclusion is then that

$$\tau\left(K_0(A_\theta)\right) = \left\{t \in \mathbf{R}\colon e^{2\pi it} = \left(e^{2\pi i\theta}\right)^n \text{ for some } n \in \mathbf{Z}\right\} = \mathbf{Z} + \theta\mathbf{Z}.$$

Appendix A. For any traced C^* algebra (A,τ) and $n \in \mathbf{Z}$ we have defined $\mathrm{SU}_n^\tau(A)$ in (II.3) to be a subgroup of $U_n(A)$. A natural question to ask is whether it is closed in $U_n(A)$. Using K-theory methods we can give a satisfactory answer for $n = \infty$.

Let $\mathrm{SU}_\infty^\tau(A)$ be defined by

$$\mathrm{SU}_\infty^\tau(A) = \bigcup_{n\in\mathbf{N}} \mathrm{SU}_n^\tau(A).$$

In this way $\mathrm{SU}_\infty^\tau(A)$ becomes a subgroup of $\mathrm{U}_\infty(A)$.

Theorem $\mathrm{SU}_\infty^\tau(A)$ *is closed in* $\mathrm{U}_\infty(A)$ *if and only if* $\tau(K_0(A))$ *is closed in* $\mathbf{R}$.

Proof. Suppose $\tau(K_0(A))$ is not closed in $\mathbf{R}$. Then we may find two sequences of self adjoint projections in $\bigcup_{m\geq 1}M_m(A)$, say $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$, such that $\tau(p_n)-\tau(q_n)$ converges to some $t\in\mathbf{R}-\tau(K_0(A))$. Let $u_n=e^{2\pi i(\tau(p_n)-\tau(q_n))}$ and view u_n as an element of $\mathrm{U}_1(A)$. We claim that $u_n\in\mathrm{SU}_\infty^\tau(A)$ for all n. In fact

$$u_n=e^{2\pi i(\tau(p_n)-\tau(q_n))}=e^{2\pi i(\tau(p_n)-\tau(q_n))}e^{-2\pi i p_n}e^{2\pi i q_n}$$

(we should note that since we are working in the inductive limit group we are allowed to take the product of unitary matrices of different sizes) and

$$\tau\big(2\pi(\tau(p_n)-\tau(q_n))-2\pi p_n+2\pi q_n\big)=0.$$

It is clear that u_n converges to $e^{2\pi it}$. But $e^{2\pi it}\notin\mathrm{SU}_\infty^\tau(A)$. Otherwise there would be $m\in\mathbf{N}$ and $h_1,\ldots,h_k\in M_m(A)_{\mathrm{sa}}$ such that

$$e^{2\pi it}=e^{ih_1}\cdots e^{ih_k}$$

and $\tau(h_1+\cdots+h_k)=0$. By (I.6.i) we would have $t\in\tau(K_0(A))$ which is a contradiction.

Conversely assume that $\tau(K_0(A))$ is closed in $\mathbf{R}$. In order to prove that $\mathrm{SU}_\infty^\tau(A)$ is closed in $\mathrm{U}_\infty(A)$ it is enough to show it is closed in $\mathrm{U}_\infty(A)_0$ since $\mathrm{U}_\infty(A)_0$ is closed in $\mathrm{U}_\infty(A)$.

Recalling how is the inductive limit topology defined we see that we just have to show that $\mathrm{SU}_\infty^\tau(A)\cap\mathrm{U}_n(A)_0$ is closed in $\mathrm{U}_n(A)_0$ for all $n\geq 1$. Let $u\in\mathrm{U}_n(A)_0-\mathrm{SU}_\infty^\tau(A)$. Write $u=e^{ih_1}\cdots e^{ih_k}$ where $h_1,\ldots,h_k\in M_n(A)_{\mathrm{sa}}$. We claim that $(1/2\pi i)\tau(h_1+\cdots+h_k)\notin\tau(K_0(A))$. In fact if

$$(1/2\pi i)\tau(h_1+\cdots+h_k)=\tau(p)-\tau(q)$$

for some self adjoint projections p and q in some matrix algebra over A we would have

$$u=e^{ih_1}\cdots e^{ih_k}e^{-2\pi ip}e^{2\pi iq}$$

which would imply that $u\in\mathrm{SU}_\infty^\tau(A)$.

Let d be the distance from $(1/2\pi)\tau(h_1+\cdots+h_k)$ to $\tau(K_0(A))$. We claim that the set $\{ue^{ih}\colon h\in M_n(A)_{\mathrm{sa}},\ |\tau(h)|<2\pi d\}$ (which is clearly a neighborhood of u in $\mathrm{U}_n(A)_0$) has no intersection with $\mathrm{SU}_\infty^\tau(A)$. Otherwise there would be $h\in M_n(A)_{\mathrm{sa}}$ with $|\tau(h)|<2\pi d$ and $l_1,\ldots,l_r\in M_m(A)_{\mathrm{sa}}$, for some $m\geq n$, with $\tau(l_1+\cdots+l_r)=0$ and

$$ue^{ih}=e^{ih_1}\cdots e^{ih_k}e^{ih}=e^{il_1}\cdots e^{il_r}.$$

We would have by (I.6.i)

$$(1/2\pi)\big(\tau(h_1+\cdots+h_k)+\tau(h)\big)\in\tau(K_0(A));$$

hence

$$\operatorname{dist}((1/2\pi)\tau(h_1 + \cdots + h_k), \tau(K_0(A))) \leq (1/2\pi)|\tau(h)| < d$$

which is a contradiction.

Appendix B. Let G be a compact connected topological group. In (VIII.2) we proved that the formula

$$(\dagger) \qquad \Delta_*([u]_1) = i_{1*}([u]_1) + i_{2*}([u]_1)$$

holds for any u in $U_1(C(G))$.

We shall now give an example for $G = T^3$ to show that $(\dagger)$ does not hold in general for u in $U_2(C(T^3))$. While we skip most of the computations, we shall describe more or less precisely the element $u \in U_2(C(T^3))$ for which $(\dagger)$ fails, as well as the outcome of both sides of $(\dagger)$ when applied to our u.

Recall that $K_0(C(T^3)) = K_1(C(T^3)) = \mathbf{Z}^4$. A set of generators for $K_0(C(T^3))$ is given by the elements 1, p_x^y, p_y^z and p_z^x where 1 denotes the element of $K_0(C(T^3))$ corresponding to the free 1-dimensional module. In order to define the three remaining elements we consider the mapping

$$\operatorname{ind}\colon K_1(C(T)) \to K_0(C(T) \otimes C(T)) = K_0(C(T \times T))$$

defined in (V.6). If $[w]_1$ denotes the standard generator of $K_1(C(T))$ we let $b = \operatorname{ind}([w]_1)$. One may call b the Bott element.

Let w_1 and w_2 denote the coordinate functions of T^2 and let x, y and z denote the coordinate functions of T^3.

Consider the $*$-monomorphisms

$$i_{x,y}, i_{y,z}, i_{z,x}\colon C(T^2) \to C(T^3)$$

given by

$$\begin{aligned} i_{x,y}(w_1) &= x, & i_{x,y}(w_2) &= y, \\ i_{y,z}(w_1) &= y, & i_{y,z}(w_2) &= z, \\ i_{z,x}(w_1) &= z, & i_{z,x}(w_2) &= x. \end{aligned}$$

We define $p_x^y = i_{x,y*}(b)$, $p_y^z = i_{y,z*}(b)$ and $p_z^x = i_{z,x*}(b)$. One can prove that 1, p_x^y, p_y^z and p_z^x form a basis for $K_0(C(T^3))$. We now describe a basis for $K_1(C(T^3))$. Three of its four elements are just $[x]_1$, $[y]_1$, and $[z]_1$. In order to define the fourth we observe that there is a canonical isomorphism

$$K_1(C(T^2)) \oplus K_0(C(T^2)) \cong K_1(C(T^3)).$$

The fourth generator of $K_1(C(T^3))$ is just the image of the Bott element under the isomorphism above. We denote it by $[u]_1$. It is possible to show that $[u]_1$ may be represented by a unitary element $u \in U_2(C(T^3))$ but it has no representative in $U_1(C(T^3))$.

Since we will be working with the map Δ_* on $K_1(C(T^3))$ we must also consider the group $K_1(C(T^6))$ which turns out to be isomorphic to the 32nd power of **Z**. In order to avoid dealing with thirty two coordinates we use the isomorphism

$$J\colon K_1\big(C(T^3)\big) \otimes K_0\big(C(T^3)\big) \oplus K_0\big(C(T^3)\big) \otimes K_1\big(C(T^3) \to K_1\big(C(T^6)\big)\big)$$

constructed in [**21**].

If we let D be defined by $J^{-1} \cdot \Delta_*$ we arrive at the commuting diagram

$$\begin{tikzcd} K_1\big(C(T^3)\big) \arrow[r, "D"] \arrow[dr, "\Delta_*"'] & K_1\big(C(T^3)\big) \otimes K_0\big(C(T^3)\big) \quad \oplus \quad K_0\big(C(T^3)\big) \otimes K_1\big(C(T^3)\big) \arrow[d, "J"] \\ & K_1\big(C(T^6)\big) \end{tikzcd}$$

We also let $I_1 = J^{-1} \cdot i_{1_*}$ and $I_2 = J^{-1} \cdot i_{2_*}$ so similar diagrams may be drawn for I_1 and I_2.

Therefore, in order to compare Δ_* and $i_{1*} + i_{2*}$, we may compare D and $I_1 + I_2$. The following table gives the values of the maps D and $I_1 + I_2$ on the generators of $K_1(C(T^3))$.

	$I_1 + I_2$	D
$[x]_1$	$([x]_1 \otimes 1, 1 \otimes [x]_1)$	$([x]_1 \otimes 1, 1 \otimes [x]_1)$
$[y]_1$	$([y]_1 \otimes 1, 1 \otimes [y]_1)$	$([y]_1 \otimes 1, 1 \otimes [y]_1)$
$[z]_1$	$([z]_1 \otimes 1, 1 \otimes [z]_1)$	$([z]_1 \otimes 1, 1 \otimes [z]_1)$
$[u]_1$	$([u]_1 \otimes 1, 1 \otimes [u]_1)$	$([u]_1 \otimes 1, 1 \otimes [u]_1)$ $+([x]_1 \otimes p_y^z, p_y^z \otimes [x]_1)$ $+([y]_1 \otimes p_z^x, p_z^x \otimes [y]_1)$ $+([z]_1 \otimes p_x^y, p_x^y \otimes [z]_1)$

We thus see that $D([u]_1) \neq (I_1 + I_2)([u]_1)$ and therefore

$$\Delta_*([u]_1) \neq i_{1*}([u]_1) + i_{2*}([u]_1).$$

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