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LOCALIZATION IN THE CLASSIFICATION OF FLAT MANIFOLDS

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Two compact flat Riemannian manifolds are called comparable if each one is a covering space of the other in such a way that the covering maps are affine and both the compositions of the covering maps increase distance locally by a constant factor. Considering comparability classes instead of affine-equivalence classes corresponds to localizing the algebra in calculations.

Introduction. This paper is concerned with compact flat Riemannian manifolds, i.e. smooth compact manifolds with a Riemannian connection for which the Levi-Civita connection is flat. These are all quotients of Euclidean space $\mathbf{R}^n$ by a group of isometries Γ acting properly discontinuously. A continuous map between two such manifolds is called affine if it lifts to an affine map of $\mathbf{R}^n$. The rotational part of Γ, i.e. its image in $\mathrm{GL}(\mathbf{R}^n)$, is called the holonomy group of the manifold and is always finite. Charlap [**4**] showed that the affine-equivalence classes of these manifolds with given holonomy group G correspond bijectively with the isomorphism classes of a category $E_{\mathbf{Z}}(G)$ defined in terms of the integral representations of G. For the purpose of calculation it is convenient to localize the integral representations to get a category $\hat{E}(G)$. This will be seen to correspond to the following geometric notion.

DEFINITION. Two compact flat Riemannian manifolds B_1, B_2 are comparable if there exist affine covering maps θ_1: $B_1 \to B_2$, θ_2: $B_2 \to B_1$ such that $\theta_1 \circ \theta_2$ and $\theta_2 \circ \theta_1$ both increase distance locally by a factor m.

Section 1 covers the background material. In §2 we shall look at the endomorphisms of these manifolds and in §3 we shall prove the following.

THEOREM A. *There is a natural bijection between the isomorphism classes of $\hat{E}(G)$ and the comparability classes of compact flat Riemannian manifolds with holonomy group G.*

Section 4 investigates the extent to which the cohomology of the manifold is an invariant of the comparability class and §5 contains a calculation of the comparability classes when the holonomy group is the metacyclic group D_{pq}, p, q primes.

1. Background. The main theorem on the structure of compact flat Riemannian manifolds is the following one ([**1**], [**3**], [**11**]).

THEOREM. *If Γ is the fundamental group of a compact flat Riemannian manifold of dimension n then Γ fits into a short exact sequence*

$$(*) \qquad 0 \to M \to \Gamma \to G \to 1$$

in which

(i) *G is finite and*

(ii) *M is free-Abelian of rank n.*

G acts on M by conjugation and

(iii) *this action is faithful.*

Any two embeddings of Γ in Isom($\mathbf{R}^n$), the group of isometries of $\mathbf{R}^n$, are conjugate in Aff($\mathbf{R}^n$), the group of affine transformations of $\mathbf{R}^n$, and under such an embedding M corresponds to the subgroup of pure translations and G to the holonomy group.

Conversely, any abstract group which fits into an exact sequence with these properties can be realised as the fundamental group of a compact flat Riemannian manifold of dimension n, unique up to affine equivalence, and for each n there are only finitely many such groups up to isomorphism. In fact any finite group can occur as the holonomy group G of a compact flat Riemannian manifold for sufficiently large n [**1**].

Thus the task of determining the compact flat Riemannian manifolds up to affine equivalence is equivalent to that of determining the groups Γ which fit into a sequence $(*)$ and satisfy conditions (i), (ii) and (iii). First of all, recall that if R is a Dedekind ring and G a finite group then a finitely-generated RG-module which is R-projective is called an RG-lattice. In the sequence $(*)$ M is a $\mathbf{Z}G$-lattice and the sequence is determined by an element $\alpha \in H^2(G; M)$. We shall label the group Γ and also the corresponding manifold (up to affine isomorphism) by $B(M, \alpha)$, or $B(G; M, \alpha)$ if we wish to stress the role of G. There remains the question of whether different M or α can lead to the same Γ. This was considered in [**4**]: define $\alpha \in H^2(G; M)$ to be special if its restriction to each subgroup of prime order is non-zero. Then for any finite group G we can define a category $E_R(G)$ to have as objects pairs (M, α) where M is a faithful RG-lattice and $\alpha \in H^2(G; M)$ is special, and as morphisms pairs $(f, A)\colon (M, \alpha) \to (N, \beta)$ where A is an automorphism of G and to specify f we define another RG-lattice $\hat{A}(N)$ to be like N as an R-module but with a new action of G, denoted $*$,

$$g^*m = A(g)m, \qquad g \in G,\ m \in M;$$

f is an RG-module homomorphism $M \to \hat{A}(N)$. There is an induced isomorphism $A_*\colon H^2(G; M) \to H^2(G; \hat{A}(N))$; we require $f_*(\alpha) = A_*(\beta)$.

THEOREM (*Charlap*). *The map B defines a bijection between the isomorphism classes of $E_{\mathbf{Z}}(G)$ and the affine-equivalence classes of compact flat Riemannian manifolds with holonomy group G.*

This reduces the geometric problem of classifying the compact flat Riemannian manifolds with given holonomy group to an algebraic one involving $\mathbf{Z}G$-lattices M and $H^2(G; M)$. Charlap **[4]** gave the solution for $G = C_p$, the cyclic group of order p, which is one of the few cases where all the $\mathbf{Z}G$-lattices are known.

The advantage of the comparability classes is that we only need to consider the genus of the $\mathbf{Z}G$-lattice M, where two $\mathbf{Z}G$-lattices M, N are in the same genus if they are isomorphic when localized at any prime p (we write $M_p \cong N_p$) or equivalently when completed at any prime p ($\hat{M}_p \cong \hat{N}_p$). See **[7]**. These are much easier to handle.

2. Construction of automorphisms. In order to decide whether two manifolds $B(G; M, \alpha_1)$, $B(G; M, \alpha_2)$ are affinely equivalent we have to be able to construct automorphisms of M. We shall use the following proposition which applies to any ring Λ.

PROPOSITION. *Let $0 \to L \xrightarrow{r} M \xrightarrow{s} N \to 0$ be an exact sequence of Λ-modules determined by $\phi \in \mathrm{Ext}_\Lambda(N, L)$ and let α be an endomorphism of L, β one of N such that for some $\psi \in \mathrm{Ext}_\Lambda(N, L)$, $\alpha_*\psi = \psi\beta_* = \phi$. Then there is an endomorphism γ of M which makes the following diagram commute.*

$$\begin{array}{ccccccccc} 0 & \to & L & \xrightarrow{r} & M & \xrightarrow{s} & N & \to & 0 \\ & & \alpha\downarrow & & \gamma\downarrow & & \beta\downarrow & & \\ 0 & \to & L & \xrightarrow{r} & M & \xrightarrow{s} & N & \to & 0 \end{array}$$

Proof. Hilton and Stammbach **[9]** give an interpretation of the functor Ext in terms of extensions, not only for the modules but also for the morphisms.

Let ψ determine the extension $0 \to L \xrightarrow{r'} M' \xrightarrow{s'} N \to 0$. By definition of α_* there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \xrightarrow{r'} & M' & \xrightarrow{s'} & N & \to & 0 \\ & & \alpha\downarrow & & \delta\downarrow & & \mathrm{id}\downarrow & & \\ 0 & \to & L & \to & M & \to & N & \to & 0 \end{array}$$

in which the left-hand square is a push-out. By definition of β_* there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \xrightarrow{r} & M & \xrightarrow{s} & N & \to & 0 \\ & & \mathrm{id}\downarrow & & \varepsilon\downarrow & & \beta\downarrow & & \\ 0 & \to & L & \xrightarrow{r'} & M' & \xrightarrow{s'} & N & \to & 0 \end{array}$$

in which the right-hand square in a pull-back. Take $\gamma = \delta\varepsilon$. □

Such an interpretation also works for $H^2(G; M)$. In particular consider the endomorphism of M which is just multiplication by a constant m. If m is prime to $|G|$ then it does not affect $H^2(G; M)$ so for any group Γ which is an extension of G by M we get a monomorphism γ satisfying

$$\begin{array}{ccccccccc} 0 & \to & M & \to & \Gamma & \to & G & \to & 1 \\ & & m\downarrow & & \gamma\downarrow & & \mathrm{id}\downarrow & & \\ 0 & \to & M & \to & \Gamma & \to & G & \to & 1. \end{array}$$

If M is a faithful $\mathbf{Z}G$-lattice then Γ can be embedded in $\mathrm{Aff}(\mathbf{R}^n)$ and $\mathrm{im}\,\gamma$ must be conjugate to Γ in $\mathrm{Aff}(\mathbf{R}^n)$ so if Γ is torsion-free we get an affine map from the corresponding flat manifold to itself which evidently increases distances by a factor m.

DEFINITION. An endomorphism of a flat manifold which increases all distances by a factor m is called expanding of degree m.

THEOREM. *A flat manifold has an expanding endomorphism of degree m if and only if m is prime to the order of the holonomy group.*

Proof. We have shown the existence above. On the other hand any expanding map of degree m leads to a diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & \Gamma & \to & G & \to & 1 \\ & & m\phi\downarrow & & \gamma\downarrow & & A\downarrow & & \\ 0 & \to & M & \to & \Gamma & \to & G & \to & 1 \end{array}$$

with A an automorphism of G and ϕ an automorphism of M. The extension is determined by $\alpha \in H^2(G; M)$ and the diagram implies $m\phi_*\alpha A_*^{-1} = \alpha$, but α is special so for any prime p dividing $|G|$ let C be a cyclic subgroup of order p; $\operatorname{res}_C^G \alpha \neq 0$ so $m\phi_* \operatorname{res}_{A^{-1}(C)} \alpha \neq 0$ so p does not divide m, i.e. m is prime to $|G|$. □

REMARK. Epstein and Shub [8] construct expanding endomorphisms of degree $k|G| + 1$, $k \in \mathbf{Z}$, $k \geq 0$.

3. Comparability. Recall that we defined two compact flat Riemannian manifolds B_1, B_2 to be comparable if there exist affine covering maps θ_1: $B_1 \to B_2$, θ_2: $B_2 \to B_1$ such that $\theta_1 \circ \theta_2$ and $\theta_2 \circ \theta_1$ are both expanding maps.

If $B_1 = B(G_1; M_1, \alpha_1)$, $B_2 = B(G_2; M_2, \alpha_2)$ and B_1 and B_2 are comparable then we have a diagram

$$\begin{array}{ccccccccc} 0 & \to & M_1 & \to & \pi_1(B_1) & \to & G_1 & \to & 1 \\ & & {\scriptstyle \phi_1}\downarrow & & \downarrow & & {\scriptstyle A_1}\downarrow & & \\ 0 & \to & M_2 & \to & \pi_1(B_2) & \to & G_2 & \to & 1 \\ & & {\scriptstyle \phi_2}\downarrow & & \downarrow & & {\scriptstyle A_2}\downarrow & & \\ 0 & \to & M_1 & \to & \pi_1(B_1) & \to & G_1 & \to & 1. \end{array}$$

$\phi_2 \circ \phi_1$ is multiplication by an integer m, prime to p, and $A_2 \circ A_1$ is an automorphism of G_1. Also $A_1 \circ A_2$ must be an automorphism of G_2 so A_1 and A_2 are isomorphisms.

$$\phi_{1*}\alpha_1 = \alpha_2 A_{2*}, \quad \phi_{2*}\alpha_2 = \alpha_1 A_{2*}$$

so

$$m\alpha_1 = \alpha_1 A_{2*} A_{1*}.$$

For any prime p dividing $|G|$ let C be a cyclic subgroup of order p, $\operatorname{res}_C^G \alpha_1 \neq 0$ so $m \operatorname{res}_{(A_2 A_1)^{-1}C}^G \alpha_1 \neq 0$ so p does not divide m, i.e. m is prime to $|G|$.

Thus if we localize at any prime p in $|G|$ then, in the category $E_{\mathbf{Z}_p}(G)$, $(\phi_2\phi_{1,p}, A_2A_1)$ is an isomorphism as is $(\phi_1\phi_{2,p}, A_1A_2)$, so $(\phi_{1,p}, A_1)$ is mono and epi and hence $\phi_{1,p}$ is an isomorphism of $M_{1,p}$ with $M_{2,p}$ if we identify G_1 with G_2 using A_1. M_1 and M_2 must be in the same genus under this identification and ϕ_1 induces an isomorphism

$$\phi_{1*}: H^2(G_1; M_1) \to H^2(G_1; M_2) \quad \text{with } \phi_{1*}\alpha_1 = \alpha_2.$$

DEFINITION. Suppose $B_1 = B(G_1; M_1, \alpha_1)$, $B_2 = B(G_2; M_2, \alpha_2)$. We shall say that B_1 and B_2 are algebraically comparable if and only if for some identification $G_1 \cong G_2 = G$, say, M_1 is in the same genus as M_2 and for each prime p dividing $|G|$ there is an isomorphism ψ_p: $\hat{M}_{1,p} \to \hat{M}_{2,p}$ such that if $\alpha_{i,p}$ is the image of α_i in $H^2(G; \hat{M}_{i,p})$ then $\psi_{p*}\alpha_{1,p} = \alpha_{2,p}$.

We shall need the following lemma which is based on one of Roiter [cf. **7**, p. 645], see also [**5**].

LEMMA A. *Let M and N be* $\mathbf{Z}G$*-lattices in the same genus, so for each p dividing* $|G|$ *there are isomorphisms* $\psi_p\colon \hat{M}_p \to \hat{N}_p$. *Then there is a homomorphism* $\phi\colon M \to N$ *such that* $\hat{\phi}_{p*} = \psi_{p*}\colon H^*(G; \hat{M}_p) \to H^*(G; \hat{N}_p)$ *and there is an exact sequence*

$$0 \to M \xrightarrow{\phi} N \to T \to 0$$

with T finite of order prime to $|G|$.

Proof. $\hat{\mathbf{Z}}_p \otimes \mathrm{Hom}_{\mathbf{Z}G}(M, N) \cong \mathrm{Hom}_{\hat{\mathbf{Z}}_pG}(\hat{M}_p, \hat{N}_p)$. $\mathrm{Hom}_{\mathbf{Z}G}(M, N)$ is densely embedded in $\mathrm{Hom}_{\hat{\mathbf{Z}}_pG}(\hat{M}_p, \hat{N}_p)$ under the p-adic topology so for each p dividing $|G|$ we may choose $u_p \in \mathrm{Hom}(M, N)$ with $\hat{u}_p \equiv \psi_p \bmod p^r$ where r is large enough that p^r does not divide $|G|$. By the Chinese Remainder Theorem choose integers α_p such that

$$\alpha_p \equiv 1 \bmod p^r, \quad \alpha_p \equiv 0 \bmod p'^{r'}, \qquad p' \neq p$$

where p' varies the over all primes dividing $|G|$ except p, and $p'^{r'} \nmid |G|$. Then $\phi = \Sigma_{p\,||G|}\alpha_p u_p$ has the correct effect on cohomology since $\phi_p \equiv u_p \bmod p^r$ and $H^*(G; -)_p$ is annihiliated by p^r. Similarly we can get $\theta \in \mathrm{Hom}(N, M)$ and

$$\hat{\theta}_p\hat{\phi}_p \equiv 1 \bmod p^r, \quad \hat{\phi}_p\hat{\theta}_p = 1 \bmod p^r.$$

By Nakayama's lemma, $\hat{\theta}_p\hat{\phi}_p$ is onto $\hat{M}_p$ so $\hat{\theta}_p\hat{\phi}_p$ must be an automorphism of $\hat{M}_p$ and in particular $\ker \hat{\phi}_p = 0$. Hence $(\ker \phi)_p = 0$ and since $\ker \phi$ is torsion-free, $\ker \phi = 0$. Let $T = \operatorname{coker} \phi$, so

$$0 \to M \xrightarrow{\phi} N \to T \to 0.$$

Since $\hat{\phi}_p$ is an isomorphism, $\hat{T}_p = 0$, so T is finite and $|T|$ is prime to $|G|$. □

Now if we let $m = |T|$, $mN \subset \phi(M)$, so we can define $\phi'\colon N \to M$ by $\phi'(n) = \phi^{-1}(mn)$, $n \in N$. On cohomology ϕ_* is iso and $\phi'_*\phi_* = m$ is iso so ϕ'_* is iso. By the proposition of §2 and the discussion afterwards, ϕ and ϕ' lead to covering maps $\theta\colon B_1 \to B_2$, $\theta'\colon B_2 \to B_1$ which show that B_1 and B_2 are comparable. We have shown:

THEOREM B. *Algebraic comparability is equivalent to comparability for flat manifolds.*

It is now clear how to define the category $\hat{E}(G)$ in order to make Theorem A valid. The objects are the same as those of $E_{\mathbf{Z}}(G)$ but the morphisms must be those occurring in the definition of algebraically comparable. That is, a morphism $(M, \alpha) \to (N, \beta)$ is now a pair $(\{f_p\}, A)$ where A is an automorphism of G and for each p which divides G there is a $\hat{\mathbf{Z}}G$-module automorphism $f_p\colon \hat{M}_p \to \hat{A}(\hat{N}_p)$ such that $f_{p*}(\alpha_p) = A_*(\beta_p)$.

REMARKS. (a) Since there are exist non-isomorphic $\mathbf{Z}G$-lattices in the same genus, for example when G is cyclic of order 23, the construction yields examples of manifolds which cover each other in a non-trivial way. Charlap [4] has examples which become affinely equivalent after taking the product with a circle.

(b) It is perhaps interesting to see what happens if we weaken the condition in the definition of comparable that the compositions should be expanding. A lot of the integral representation theory is lost: for example let $B(G; M, \alpha)$ be a flat manifold and let S, T be two $\mathbf{Z}G$-lattices which are isomorphic as $\mathbf{Q}G$-spaces, but not even in the same genus. Then $B(G; M \oplus S, \alpha \oplus 0)$ and $B(G; M \oplus T, \alpha \oplus 0)$ both cover each other (since S is isomorphic to a submodule of T of finite index and vice versa).

4. The cohomology of a flat manifold. It is interesting to see how the cohomology of a flat manifold $B(G; M, \alpha)$ depends on M and α. We shall calculate $H^*9B)$ using the spectral sequence of the extension $0 \to M \to \pi_1(B) \to G \to 1$, (since B is an Eilenberg-MacLane space $K(\pi_1(B), 1)))$. For coefficients in a ring R,

$$E_2^{ij}(R) = H^i(G; H^j(M; R)) = H^i(G; \Lambda^j H^1(M; R)).$$

PROPOSITION. *The Betti numbers of $B(G; M, \alpha)$ depend only on M.*

Proof. $E_2^{ij}(\mathbf{Q}) = 0$ unless $i = 0$.

$$H_j(B; \mathbf{Q}) \cong E_2^{0j}(\mathbf{Q}) \cong (\Lambda^j(M \otimes \mathbf{Q}))^G. \qquad \square$$

PROPOSITION. *The E_2 terms of the spectral sequence (with any coefficients) depend only on the genus of M.*

Proof. The torsion-free part is taken care of as above. As for the torsion, $H^1(M; R) \cong \operatorname{Hom}(M, R)$ and $H^i(G; N)_p \cong H^i(G; N_p)$ and

$$\operatorname{Hom}(M, R) \otimes \mathbf{Z}_p \cong \operatorname{Hom}(M_p, R \otimes \mathbf{Z}_p). \qquad \square$$

THEOREM. *If two flat manifolds B_1, B_2 are comparable then the cohomology groups $H^*(B_1; \mathbf{Z})$, $H^*(B_2; \mathbf{Z})$ are isomorphic. If R is a subring of $\mathbf{Q}$ with all primes not in a finite set inverted or R is a field then there is an affine map f: $B_1 \to B_2$ which induces an isomorphism of rings f^*: $H^*(B_2; R) \cong H^*(B_1; R)$.*

Proof. Suppose $R \subset \mathbf{Q}$ and all primes not in $|B|$ are invertible in R. We shall show that θ^*: $H^*(B_2; R) \to H^*(B_1; R)$ is an isomorphism when θ: $B_1 \to B_2$ is a covering map constructed in the proof that algebraically comparable implies comparable. Let ϕ: $M_1 \to M_2$ be the induced map on $\mathbf{Z}G$-lattices; it is sufficient to show that this gives an isomorphism ϕ^*: $H^1(M_2; R) \to H^1(M_1; R)$ (since $H^j(M_i; R) \cong \Lambda^j H^1(M_i; R)$). But, by construction, $\operatorname{coker}\phi$ is finite of order prime to $|G|$ and so from the exact sequence

$$0 \to \underset{\displaystyle 0}{\underset{\parallel}{\operatorname{Hom}(\operatorname{coker}\phi, R)}} \to \operatorname{Hom}(M_2, R) \to \operatorname{Hom}(M_1, R) \to \underset{\displaystyle 0}{\underset{\parallel}{\operatorname{Ext}(\operatorname{coker}\phi, R)}}$$

we see that this is the case ($H^1(M; R) \cong \operatorname{Hom}(M, R)$). For other R we must examine the proof of Lemma A to see that $\operatorname{coker}\phi$ can be made coprime to an additional finite set of primes.

The only torsion that can occur in $H^*(B; -)$ is at primes in $|G|$ since the only E_2^{**} terms which are not annihilated by $|G|$ are the E_2^{0*} and there are no differentials with image in $E^{0}{}^*{}_*$. Thus we can deduce the result for $R = \mathbf{Z}$ from the case with the primes not in $|G|$ inverted. □

REMARKS. (a) A version of this theorem for G of prime order was proved by Charlap and Vasquez [**5**], who also calculate the groups $H^*(B; \mathbf{Z})$ in this case.

(b) It is not known whether the rings $H^*(B; \mathbf{Z})$ must be isomorphic.

5. Metacyclic groups. Let D_{pq} be a metacyclic group of order pq, with p, q distinct primes and q dividing $p - 1$:

$$1 \to C_p \to D_{pq} \to C_q \to 1,$$
$$D_{pq} = gp\langle x, y \mid x^p = y^q = 1, yxy^{-1} = x^r \rangle,$$

where r is a primitive qth root of 1 mod p. We shall find the comparability classes of flat manifolds with holonomy group isomorphic to D_{pq}. The genera of the indecomposable D_{pq}-lattices were determined by Pu [**10**] and are given in [7, pp. 747–751], whose description we follow here.

Let $R = \mathbf{Z}[\zeta]$, where ζ is a primitive pth root of 1. Define an automorphism σ of R over $\mathbf{Z}$ by $\sigma(\zeta) = \zeta^r$ and let D_{pq} act on R by

$$xr = \zeta r,$$
$$yr = \sigma(R), \qquad r \in R.$$

Thus R is a $\mathbf{Z}G$-lattice and so is the ideal $P = (1 - \zeta)R$ or any power of it, P^i. There are also the indecomposable lattices for the factor C_q, in particular $\mathbf{Z}$, $\mathbf{Z}H$ and its augmentation ideal S, where $H = \langle y \rangle$.

If we localize at q the indecomposable $\mathbf{Z}_q D_{pq}$-lattices are as follows (we shall calculate the cohomology groups later, but include them here for convenience).

$\mathbf{Z}_q D_{pq}$-lattice M	$H^2(D_{pq}; M)$
$\mathbf{Z}_q$	$\mathbf{Z}/q$
S_q	0
$\mathbf{Z}_q H$	0
R_q	0

Alternatively if we complete at p the indecomposable lattices are

$\hat{\mathbf{Z}}_p D_{pq}$-lattice M		$H^2(D_{pq}; M)$
$\hat{\mathbf{Z}}_i$	$0 \le i \le q-1$	$\begin{cases} \mathbf{Z}/p, & i = 1, \\ 0, & \text{otherwise} \end{cases}$
$\hat{P}^i$	$0 \le i \le q-1$	0
$\hat{\mathbf{Z}}_i^G$	$0 \le i \le q-1$	0

$\hat{\mathbf{Z}}_i$ is a copy of $\hat{\mathbf{Z}}_p$ on which x acts as 1 and y acts as θ^i, where θ is the primitive qth root of 1 in $\hat{\mathbf{Z}}_p$ with $\theta \equiv r \bmod (p\mathbf{Z}_p)$. $\hat{\mathbf{Z}}_i^G$ is induced from $\hat{\mathbf{Z}}_i$ restricted to H and is also the unique non-split extension of $\hat{\mathbf{Z}}_i$ by $\hat{P}^{i+1}$

$$(\dagger) \qquad 0 \to \hat{P}^{i+1} + \hat{\mathbf{Z}}_i^G \to \hat{\mathbf{Z}}_i \to 0.$$

LEMMA. (*See* [2].) *Let N be a normal subgroup of G with $|N|$ prime to $|G/N|$. Then*

$$H^r(G; M) \cong H^r(G/N; M^N) \oplus H^r(N; M)^{G/N}.$$

Proof. Use the spectral sequence for $1 \to N \to G \to G/N \to 1$.

REMARK. If the action of N on M is trivial, so $H^2(N; M) \cong H^1(N; \mathbf{Q} \otimes M/M) \cong \mathrm{Hom}(N, \mathbf{Q} \otimes M/M)$, then the action of G on $H^2(N; M)$ is given by

$$({}^g f)(x) = gf(g^{-1}xg), \quad f \in \mathrm{Hom}(N, \mathbf{Q} \otimes M/M), \quad x \in N,\ g \in G$$

(see [2]).

The cohomology can now be calculated using the formula for the cohomology of a cyclic group. Observe that if M is $\mathbf{Z}_q D_{pq}$-lattice then $H^2(D_{pq}; M)$ has only q-torsion so $H^2(D_{pq}; M) \cong H^2(D_{pq}/C_p; M^{C_p})$. Hence the results for $\mathbf{Z}_q$, S_q and $\mathbf{Z}_q H$. Also, $R_q^{C_p} = 0$.

Over $\hat{\mathbf{Z}}_p$, $H^2(D_{pq}; M) \cong H^2(C_p; M)^H$.

$$H^2(C_p; P^i) = 0.$$

$H^2(D_{pq}; \hat{\mathbf{Z}}_i^G) \cong H^2(H; \hat{\mathbf{Z}}_i) \cong 0$ since $\mathbf{Z}_i^G$ is induced from $\mathbf{Z}_i$.

$H^2(C_p; \hat{\mathbf{Z}}_i) \cong \operatorname{Hom}\left(C_p, \left(\hat{\mathbf{Q}}_p \otimes_{\hat{\mathbf{Z}}_p} \hat{\mathbf{Z}}_i\right)/\hat{\mathbf{Z}}_i\right) \cong \operatorname{Hom}\left(C_p, \hat{\mathbf{Z}}_i/p\hat{\mathbf{Z}}_i\right)$.

Let $f \in \operatorname{Hom}(C_p, \hat{\mathbf{Z}}_i/p\hat{\mathbf{Z}}_i)$; then if $f = {}^y f$,

$$\begin{aligned} f(x) &= yf\left(y^{-1}xy\right) = \theta^i f(x^s) \quad \text{where } sr \equiv 1 \bmod p \\ &= s\theta^i f(x), \end{aligned}$$

so $i = 1$.

Representatives of the $2^q + 2^{q-1} + q + 2$ indecomposable genera of the D_{pq}-lattices are as follows.

lattice M	M_q	$\hat{M}_p$	$H^2(D_{pq}; M)$
P^i, $0 \le i \le q-1$	R_q	$\hat{P}_i$	0
$\mathbf{Z}$	$\mathbf{Z}_q$	$\hat{\mathbf{Z}}_0$	$\mathbf{Z}/q$
S	S_q	$\sum_{i=1}^{q-1} \hat{\mathbf{Z}}_i$	$\mathbf{Z}/p$
H	$\mathbf{Z}_q H$	$\sum_{i=0}^{q-1} \hat{\mathbf{Z}}_i$	$\mathbf{Z}/p$
X_T	$S_q + R_q^{\lvert T\rvert}$	$\sum_{i \in T} \hat{\mathbf{Z}}_{i-1}^G + \sum_{i \notin T \cup \{1\}} \hat{\mathbf{Z}}_{i-1}$	$\begin{cases} \mathbf{Z}/p & \text{if } 2 \notin T \pmod q \\ 0 & \text{otherwise} \end{cases}$
Y_T	$\mathbf{Z}_q H + R_q^{\lvert T\rvert}$	$\sum_{i \in T} \hat{\mathbf{Z}}_{i-1}^G + \sum_{i \notin T} \hat{\mathbf{Z}}_{i-1}$	$\begin{cases} \mathbf{Z}/p & \text{if } 2 \notin T \pmod q \\ 0 & \text{otherwise} \end{cases}$
V	$R_q + \mathbf{Z}_q$	$\hat{\mathbf{Z}}_0^G$	$\mathbf{Z}/q$

where T is any non-empty subset of $\{0, 1, \dots, q-1\}$ except that to form X_T we cannot have $1 \in T$.

The cohomology groups are as given since $H^*(G; M)_p \cong H^*(G; M_p) \cong H^*(G; \hat{M}_p)$ for any prime p.

$H^2(D_{pq}; M)$ has a special point if and only if it has both p-torsion and q-torsion since $\operatorname{cor}_H^G \operatorname{res}_H^G = |G:H|x$. In any case the comparability class depends only on the genus since all the p-torsion in $H^2(D_{pq}; M)$ comes from $\hat{\mathbf{Z}}_p$ and all the q-torsion from the irreducible $\mathbf{Z}_q$ and it is easy to construct an automorphism of these which takes any non-zero element of H^2 to any other.

However the automorphisms of the group permute the $\hat{\mathbf{Z}}_i$, $i \neq 0$, and hence also the P^i, $i \neq 1$, in order to preserve sequence (†). This permutes the sets T by changing the elements not equal to 1 according to $\alpha \mapsto (\alpha - 1)^r + 1 \bmod q$ for some r depending on the automorphism, and any r prime to q is possible. Note that it acts in the same way on each copy of T.

THEOREM C. *The comparability classes of flat manifolds with holonomy group isomorphic to D_{pq} are in one-to-one correspondence with the equivalence classes of the genera of faithful special D_{pq}-lattices under the relation on T described above.*

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