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u-MAPPINGS ON TREES

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David Bellamy has shown that there exist tree-like continua which do not have the fixed point property. We give sufficient conditions for a tree-like continuum to have the fixed point property. In order to establish this result, we define u-mappings on trees and show that each u-mapping is universal. Our results generalize similar theorems of C. A. Eberhart and J. B. Fugate in [3].

In 1969 R. H. Bing [2] asked if each tree-like continuum has the fixed point property. In 1979 David Bellamy [1] answered Bing's question in the negative; i.e., he gave an example of a tree-like continuum which admits a fixed point free map to itself. In this paper, we give sufficient conditions for a tree-like continuum to have the fixed point property. Our result generalizes a similar theorem of C. A. Eberhart and J. B. Fugate [3, Theorem 7]. Other papers concerned with fixed point theorems for tree-like continua are [6] and [7].

W. Holsztynski [5, Corollary 1] has shown that whenever a continuum is the inverse limit of absolute retracts with universal bonding maps, then the continuum has the fixed point property. In [3], Eberhart and Fugate showed that if a mapping of trees is weakly arc-preserving, then it is also universal. Their fixed point result for tree-like continua follows from Holsztynski's theorem. We define a u-mapping of trees and prove that each u-mapping is universal. We also show that u-mappings are more general than weakly arc-preserving mappings.

By a *continuum* we will mean a compact, connected metric space. A *tree* is a finite, connected, simply connected graph. Each continuous function will be referred to as a *map* or *mapping*.

A mapping $f\colon X \to Y$ of trees is *arc-preserving* provided that f is a surjection and if A is an arc in X, then $f(A)$ is an arc or a point. The mapping f is *weakly arc-preserving* provided that there is a subtree X' of X so that the restriction of f to X' is arc-preserving. A mapping $f\colon X \to Y$ of topological spaces is said to be *universal* provided that whenever $g\colon X \to Y$ is a mapping, there is a point $x \in X$ such that $f(x) = g(x)$.

Suppose that X is a tree. We define the sets $E(X)$ and $B(X)$ of *endpoints* and *branchpoints* of X, respectively, by

$$E(X) = \{x \in X \mid X - \{x\} \text{ is connected}\} \quad \text{and}$$
$$B(X) = \{x \in X \mid X - \{x\} \text{ has at least three components}\}.$$

For each pair of points x_1, x_2 in X, the unique arc in X linearly ordered from x_1 to x_2 will be denoted by $[x_1, x_2]$. The arc $[v_1, v_2]$ in X will be called *an edge of* X provided that v_1, $v_2 \in B(X) \cup E(X)$ and if $x \in [v_1, v_2]$ and $v_1 \neq x \neq v_2$, then $x \notin B(X) \cup E(X)$. If $[v_1, v_2]$ is an edge of X and one of v_1 or v_2 is in $E(X)$, then $[v_1, v_2]$ is said to be *a terminal edge of* X. Otherwise, $[v_1, v_2]$ is an *interior edge of* X.

If H is a subcontinuum or a point of a tree X, we define $st(H)$ to be the union of all edges of X which intersect H. The arc s is said to be *a leg of* $st(H)$ provided that s is the closure of some component of $\mathrm{st}(H) - H$. Notice that each leg of $\mathrm{st}(H)$ contains an endpoint of $\mathrm{st}(H)$ and is a subarc of some edge of X.

Suppose that f: $X \to Y$ is a mapping of trees, $w \in B(Y)$, $\{t_i\}_{i=1}^{n}$ are the legs of $\mathrm{st}(w)$, and $[u, v]$ is an arc in X so that $f(u) = w$, but $f([u, v]) \neq \{w\}$. We will say that $[u, v]$ *has an initial image under* f provided that there is an integer $j \in \{1, 2, \ldots, n\}$ and a point $x \in [u, v]$ such that $f(x) \in T_j - \{w\}$ and, if $x' \in [u, x]$, then $f(x') \in t_j$. In this case, we will also say that t_j *is the initial image of* $[u, v]$ under f. The reference to f will be omitted if such reference is clear. If C is the closure of the component of $X - \{u\}$ that contains $[u, v]$ and D is the closure of the component of $Y - \{w\}$ that contains t_j, we will also say that D is the initial image of C. Finally, if M is any component of $f^{-1}(w)$, we will say that the legs of $\mathrm{st}(M)$ *initially cover* the legs of $\mathrm{st}(w)$ provided that, for each leg t_i of $\mathrm{st}(w)$, there is a leg s of $\mathrm{st}(M)$ whose initial image is t_i.

For each tree X in this paper, we will assume that we have a metric d defined on $X \times X$ so that each edge of X has length one. Since each mapping from a connected metric space onto an arc is universal **[5]**, and O. H. Hamilton **[4]** has shown that arc-like continua have the fixed point property, we will further assume, throughout this paper, that all trees have non-empty branchpoint sets.

DEFINITION. Suppose that f: $X \to Y$ is a mapping of a tree X onto a tree Y. We will say that f is a *u-mapping* (*u-map*) provided that f satisfies the following properties.

(1) $f(B(X)) \subset B(Y)$,

(2) if s is a terminal edge of X, then $f(s)$ is a terminal edge of Y,

(3) if $w \in B(Y)$ and M is a component of $f^{-1}(w)$ which contains a branchpoint of X, then the legs of $\mathrm{st}(M)$ initially cover the legs of $\mathrm{st}(w)$,

(4) if $w \in B(Y)$ and $[v_1, v_2]$ is an interior edge of X such that $f(v_1) = w = f(v_2)$ and $f([v_1, v_2]) \neq \{w\}$, then there is a component N of $f^{-1}(w)$ and two legs t_1 and t_2 of $\mathrm{st}(w)$ such that $N = [z_1, z_2] \subset [v_1, v_2]$,

and $f([v_i, z_i])$ is a nondegenerate subarc of t_i for $i = 1, 2$, and

(5) if $v \in B(X)$, then $f(\mathrm{st}(v)) \subset \mathrm{st}(f(v))$.

We will show that whenever a mapping f of trees has a restriction to a subtree that is a u-mapping, then f must be universal. Although the properties of a u-map are technical and many in number, each property is necessary in the sense that its omission yields an example of a non-universal mapping of trees. We give examples later in the paper. Also, it is generally easy to check if a mapping $f: X \to Y$ of trees has properties (1) through (5), while it is not easy to check if f has a coincidence point with each mapping $g: X \to Y$.

We begin with a lemma concerning u-mappings.

LEMMA 1. *Suppose that* $f: X \to Y$ *is a u-mapping,* $[w_1, w_2]$ *is an edge of* Y *with* $w_1 \in B(Y)$, $v_1 \in B(X)$, *and* $f(v_1) = w_1$. *Then there is an arc* $[v_1, v_2]$ *in* X *such that the initial image of* $[v_1, v_2]$ *is* $[w_1, w_2]$, *and* $[w_1, w_2] \subset f([v_1, v_2]) \subset \mathrm{st}(w_1)$. *Moreover, if* $[w_1, w_2]$ *is a terminal edge of* Y, *then* v_2 *can be chosen from* $E(X)$, *and if* $[w_1, w_2]$ *is an interior edge of* Y, *then* v_2 *can be chosen from* $B(X)$ *with* $f(v_2) = w_2$.

Proof. Let M_1 be the component of $f^{-1}(w_1)$ which contains v_1. Since f is a u-mapping, we may choose a leg s_1 of $\mathrm{st}(M_1)$ whose initial image is $[w_1, w_2]$. Let α_1 be the edge of X such that $s_1 \subset \alpha_1$, and let u_1 be the endpoint of α_1 which is not in M_1.

If α_1 is a terminal edge of X, then $f(\alpha_1)$ is a terminal edge of Y. Hence, $[w_1, w_2]$ must be a terminal edge of Y and we have that $[w_1, w_2] = f([v_1, u_1]) \subset \mathrm{st}(w_1)$. So, $[v_1, u_1]$ satisfies the conclusion of the lemma.

We assume that α_1 is an interior edge of X. By property (5) of a u-mapping, we have that $f([v_1, u_1]) \subset \mathrm{st}(w_1)$. Since the initial image of α_1 is $[w_1, w_2]$, by properties (1) and (5), $f(u_1)$ must be either w_1 or w_2. If $f(u_1) = w_2$, then again we have that $[v_1, u_1]$ satisfies the conclusion of the lemma. So, we assume that $f(u_1) = w_1$. Let M_2 be the component of $f^{-1}(f(u_1))$ which contains u_1. Let s_2 be a leg of $\mathrm{st}(M_2)$ whose initial image is $[w_1, w_2]$, α_2 be the edge of X such that $s_2 \subset \alpha_2$, and u_2 be the endpoint of α_2 which is not in M_2. Now, by property (4) of a u-mapping, $\alpha_2 \neq \alpha_1$.

Again, if either α_2 is a terminal edge of X or $f(u_2) = w_2$, then the result follows for the arc $[v_1, u_2]$. So, we assume that α_2 is an interior edge of X and that $f(u_2) = w_1$. Now, since X has finitely many interior edges, a continuation of this procedure eventually gives us a positive integer n

for which α_n is either a terminal edge of X or $f(u_n) = w_2$. We then have the desired result for the arc $[v_1, u_n]$.

THEOREM 1. *Suppose that F: $X' \to Y$ is a mapping from a tree X' onto a tree Y such that there is a subcontinuum X of X' for which $F|_X$ is a u-mapping from X onto Y. Then F is universal.*

Proof. Let $f = F|_X$. We will show that f: $X \to Y$ is universal. It follows that F is universal.

Suppose there is a point $w \in B(Y)$ such that $f^{-1}(w) \cap B(X) = \varnothing$. Let $v \in B(X)$. By property (1), $f(v) \in B(Y)$. Let z be a point of Y such that $w \in [f(v), z]$ and $w \neq z$. Let $x \in f^{-1}(z)$. Then $[f(v), z]$ is a subset of $f([v, x])$. Let u be the last branchpoint in $[v, x]$ with the property that w does not separate $f(v)$ from $f(u)$ and if u' is in $B(X) \cap (u, x]$, then w separates $f(v)$ from $f(u')$. It follows that $f(\mathrm{st}(u)) \not\subset \mathrm{st}(f(u))$, which is a contradiction. Hence, for each branchpoint $w \in Y$, there is a branchpoint v in X such that $f(v) = w$.

If $[u_1, u_2]$ is an interior edge of X with the property that $f(u_1) = f(u_2)$ but $f([u_1, u_2]) \neq \{f(u_2)\}$, then we will say that $[u_1, u_2]$ *is folded by* f. We will use induction on the number of interior edges of X which are folded by f.

Suppose there is no interior edge of X which is folded by f. We claim that

(∗) if $[w, b]$ is an edge of Y with $w \in B(Y)$ and v is a branchpoint of X such that $f(v) = w$, then there is an arc $[v, a]$ in X such that $f([v, a]) = [w, b]$. Moreover, if $b \in B(Y)$, then a can be chosen from $B(X)$ with $f(a) = b$.

So, let $[w, b]$ be an edge of Y with $w \in B(Y)$ and let $v \in B(X) \cap f^{-1}(w)$. By Lemma 1, there is an arc $[v, u]$ in X such that u is a vertex of X, $[w, b]$ is the initial image of $[v, u]$, and $[w, b] \subset f([v, u]) \subset \mathrm{st}(w)$. Let a be the first vertex of X in $[v, u]$ such that $f([v, a]) \neq \{w\}$. Now, by properties (1), (2), and (5), and the assumption that no edge of X is folded by f, it follows that $f([v, a]) = [w, b]$.

We will now show that f is universal, in this case, using an induction argument on the number of branchpoints in Y.

Suppose that Y has only one branchpoint w. Then $Y = \mathrm{st}(w)$. Let $\{t_i\}_{i=1}^n$ be the legs of $\mathrm{st}(w)$. Let v be a branchpoint of X such that $f(v) = w$. By (∗), for each $i \in \{1, 2, \ldots, n\}$, we can choose an arc $[v, a_i]$ in

X such that $f([v, a_i]) = t_i$. Let $T = \bigcup_{i=1}^n [v, a_i]$. Since each t_i is an arc, $f|_{[v, a_i]}$ is universal. Also, for $i \neq j$, $f([v, a_i]) \cap f([v, a_j]) = \{w\}$. A theorem of Holsztynski [5, Prop. 7] gives us that $f|_T$ is universal. Hence, f is universal.

Suppose that Y has exactly m branchpoints. Let w be a branchpoint of Y and let $v \in B(X) \cap f^{-1}(w)$. Let $\{Y_i\}_{i=1}^n$ be the collection of closures of components of $Y - \{w\}$. We intend to produce, for each i in $\{1, \ldots, n\}$, a subtree X_i of X such that $f(X_i) = Y_i$ and $f|_{X_i}$ is universal.

For $i \in \{1, \ldots, n\}$, let $[w, b_i]$ be the terminal edge of Y_i with endpoint w. Applying (*), for each $i \in \{1, \ldots, n\}$, we choose an arc $[v, a_i]$ in X such that $f([v, a_i]) = [w, b_i]$.

Now, if for some $j \in \{1, \ldots, n\}$, $b_j \in E(Y)$, then $[w, b_j] = Y_j$. So, in this case, we let $X_j = [v, a_j]$, and thus, $f|_{X_j}$ is universal.

Suppose for some $j \in \{1, \ldots, n\}$, $b_j \in B(Y)$. Now, $f([v, a_j]) = [w, b_j]$, and, by (*), we may assume that a_j was chosen from $B(X)$. We now will apply (*) to each leg of $\operatorname{st}(b_j)$ except for $[b_j, w]$. Let $\{d_i\}_{i=1}^k$ be the vertices of Y that are adjacent to b_j. Assume $d_1 = w$. For each i in $\{2, 3, \ldots, k\}$, choose an arc $[a_j, c_i]$ in X such that $f([a_j, c_i]) = [b_j, d_i]$. Let $X_{j1} = \bigcup_{i=2}^k [a_j, c_i] \cup [a_j, v]$. Now, X_{j1} is a simple k-od and $f|_{X_{j1}}$ maps the edges of X_{j1} onto the edges of $\operatorname{st}(b_j)$. So, clearly $f|_{X_{j1}}: X_{j1} \to \operatorname{st}(b_j)$ is a u-mapping. For each d_i that is a branchpoint of Y_j, we repeat the above procedure on $\operatorname{st}(d_i)$. This gives us another k-od, say X_{ji}, (not necessarily the same k as above) in X which shares the edge $[c_i, a_j]$ with X_{j1} and whose edges are mapped by $f|_{X_{ji}}$ onto the edges of $\operatorname{st}(d_i)$. We repeat the process again for each vertex of Y_j that is both a branchpoint of Y_j and adjacent to some d_i that was a branchpoint of Y_j. In this manner, since Y_j has but finitely many branchpoints, in fact fewer than m, we will generate a subtree X_j of X (X_j will be the union of all the X_{ji}'s produced by this procedure) that is a homeomorph of Y_j and whose edges are mapped by $f|_{X_j}$ onto the corresponding edges of Y_j. Thus it is clear that $f|_{X_j}$ is a u-mapping. Since X_j has fewer than m branchpoints, we have by the inductive assumption that $f|_{X_j}$ is universal.

For each $i \in \{1, \ldots, n\}$, we have constructed a subtree X_i of X such that $f(X_i) = Y_i$ and $f|_{X_i}$ is universal. Thus, by Holsztynski's theorem [5, Prop. 7], it follows that f is universal.

Suppose that X has exactly m interior edges which are folded by f. We assume that whenever $f': Z \to Y$ is a u-mapping of a tree Z onto Y such that Z has fewer than m interior edges which are folded by f', then f' is universal.

By way of contradiction, we assume that f is not universal. Let $g: X \to Y$ be a mapping such that $f(x) \neq g(x)$ for each $x \in X$.

Let $[v_1, v_2]$ be an interior edge of X which is folded by f. Let $w = f(v_1)$. We let $[z_1, z_2]$ be the component of $f^{-1}(w)$ as indicated in property (4) of a u-mapping. Also, we let t_1 and t_2 be the legs of $\mathrm{st}(w)$ as indicated in property (4). Let b_1 and b_2 be the vertices of Y which are adjacent to w and belong to t_1 and t_2 respectively.

For $i = 1, 2$, let a_i be the endpoint of $f([v_i, z_i])$ which lies in t_i. We have that, for $i = 1, 2$, $w < a_i \leq b_i$ in the order on $[w, b_i]$. Since we are assuming that each leg of Y has length one, we let $|a_i|$ denote the distance from a_i to w. For $i = 1, 2$, let $\varepsilon_i = 1 - 1/|a_i|$; we notice that $\varepsilon_i \leq 0$. Assuming that each of X and Y is a subset of E^2, we define the mapping $\hat{f}\colon X \to Y$ by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \notin [v_1, v_2], \\ \varepsilon_1 w + (1 - \varepsilon_1) f(x) & \text{if } x \in [v_1, z_2], \\ \varepsilon_2 w + (1 - \varepsilon_2) f(x) & \text{if } x \in [z_1, v_2]. \end{cases}$$

Now, it is clear that, for each point $f(x)$ in t_i ($i = 1, 2$), $\hat{f}(x)$ is on the line containing w and $f(x)$. We claim, in fact, that $\hat{f}(x)$ is in the arc $[w, b_i]$. The calculations which follow will make this clear.

For each $x \in [v_1, v_2]$ such that $f(x) = w$, we notice that $\hat{f}(x) = w$. Thus, $\hat{f}$ is continuous. Suppose that $x \in [v_i, z_i]$, for either $i = 1$ or $i = 2$, and $f(x) = a_i$. Then

$$\begin{aligned} \hat{f}(x) &= \varepsilon_i w + (1 - \varepsilon_i) a_i \\ &= \varepsilon_i w + (1 - \varepsilon_i)\big(|a_i| b_i + (1 - |a_i|) w\big) \\ &= \varepsilon_i w + (1 - \varepsilon_i)|a_i| b_i + (1 - \varepsilon_i)(1 - |a_i|) w \\ &= \big[\varepsilon_i + (1 - \varepsilon_i)(1 - |a_i|)\big] w + (1 - \varepsilon_i)|a_i| b_i \\ &= \big[1 - |a_i| + \varepsilon_i |a_i|\big] w + (1 - \varepsilon_i)|a_i| b_i \\ &= \big[1 - |a_i| + (1 - 1/|a_i|)|a_i|\big] w + \big(1 - (1 - 1/|a_i|)\big)|a_i| b_i \\ &= 0 \cdot w + 1 \cdot b_i = b_i. \end{aligned}$$

Thus, $\hat{f}([v_i, z_i]) \subset t_i$ for $i = 1, 2$. We also notice that if $a_i = b_i$ for either $i = 1$ or $i = 2$, then $\varepsilon_i = 0$, and $\hat{f}(x) = f(x)$ for each $x \in [v_i, z_i]$. In addition, it is easy to see that $\hat{f}$ is a u-mapping and exactly m interior edges of X are folded by $\hat{f}$.

For $i = 1, 2$, let u_i be a point of $[v_i, z_i]$ such that $\hat{f}(u_i) = b_i$.

We now would like to modify the mapping $\hat{f}$ and perhaps we will also need to modify the continuum X by adjoining a homeomorphic copy of a subcontinuum of Y to X. However, our procedure is dependent upon whether b_i is an endpoint or a branchpoint of Y, for each $i = 1, 2$. We see that there are actually four cases to consider. We will consider only one case. It will be clear that the proof of the other cases can be carried out in a similar manner.

We suppose that b_2 is a branchpoint of Y and b_1 is an endpoint of Y. Let X_1 be the closure of the component of $X - \{z_1\}$ which contains v_1. We claim that $Y \subset \hat{f}(X_1)$. We only need to show that each endpoint of Y is in $\hat{f}(X_1)$. Now, $b_1 = \hat{f}(u_1)$ and $u_1 \in [v_1, z_1] \subset X_1$. Let $e \neq b_1$ be an endpoint of Y. Let $\{e_i\}_{i=1}^{k}$ be the vertices (in order) of Y that lie in the arc $[w, e]$, with $e_1 = w$ and $e_k = e$. Notice that $[w, e] \cap [w, b_1] = \{w\}$. By Lemma 1, there is an arc $[v_1, d_2]$ in X such that the initial image of $[v_1, d_2]$ under $\hat{f}$ is $[w, e_2]$ and $[w, e_2] \subset \hat{f}([v_1, d_2]) \subset \text{st}(w)$. Since the initial image of $[v_1, d_2]$ is $[w, e_2]$, it follows that $[v_1, d_2] \cap [v_1, z_1] = \{v_1\}$. So, $[v_1, d_2] \subset X_1$.

If $k = 2$, then $e_2 = e$, $[w, e] \subset \hat{f}([v_1, d_2])$ and we are done. Otherwise, e_2 is a branchpoint and we may assume, by Lemma 1, that $d_2 \in B(X)$ and $\hat{f}(d_2) = e_2$. We repeat the process above. By Lemma 1, there is an arc $[d_2, d_3]$ in X such that the initial image of $[d_2, d_3]$ is $[e_2, e_3]$ and $[e_2, e_3] \subset \hat{f}([d_2, d_3]) \subset \text{st}(e_2)$. It follows that $[d_2, d_3] \subset X_1$. If $k = 3$, then $e_3 = e$, $[e_2, e] \subset \hat{f}([d_2, d_3])$, and we are done. Otherwise, $e_3 \in B(Y)$ and we continue the process. After finitely many steps, we get that $e \in \hat{f}(X_1)$. Thus, $Y \subset \hat{f}(X_1)$.

Let U be the component of $Y - \{b_2\}$ which contains w. Let $Y_2 = Y - U$. Let h be a homeomorphism from Y_2 into E^2 such that $h(b_2) = u_2$ and $h(Y_2) \cap X = \{u_2\}$. Also, let X_2 be the union of $h(Y_2)$ and the closure of the component of $X - \{z_1\}$ which contains v_2. Let $Z = X_1 \cup X_2$.

We now wish to define mappings f': $Z \to Y$ and g': $Z \to Y$. Let f' be defined by

$$f'(x) = \begin{cases} \hat{f}(x) & \text{if } x \in X, \\ h^{-1}(x) & \text{if } x \in Z - X. \end{cases}$$

Let g' be defined by

$$g'(x) = \begin{cases} g(x) & \text{if } x \in X, \\ g(u_2) & \text{if } x \in Z - X. \end{cases}$$

Let $f_i' = f' \mid_{X_i}$, for $i = 1, 2$. Now, it is clear that each of f_1' and $f_2' \mid_{X_2 - (u_2, z_1]}$ is a u-mapping. Since $Y \subset \hat{f}(X_1)$ it follows that the image of f_1' is Y. We will also show that the image of $f_2' \mid_{X_2 - (u_2, z_1]}$ is Y. Again, we show that each endpoint of Y is in the desired image. If e is an endpoint of Y_2, then $h(e) \in X_2 - (u_2, z_1]$. Furthermore, $f_2'(h(e)) = f'(h(e)) = h^{-1}(h(e)) = e$. Suppose e is an endpoint of Y and e is not in Y_2. Then $e \in U$. Let $\{w_i\}_{i=1}^{k}$ be the vertices (in order) of Y that lie in the arc $[w, e]$, where $w_1 = w$ and $w_k = e$. By Lemma 1, there is an arc $[v_2, c_2]$ in X such that the initial image of $[v_2, c_2]$ is $[w, w_2]$ and $[w, w_2] \subset \hat{f}([v_2, c_2]) \subset \mathrm{st}(w)$. Now, since the initial image of $[v_2, c_2]$ is $[w, w_2]$, it follows that $[v_2, c_2] \cap [v_2, u_2] = \{v_2\}$. So, $[v_2, c_2] \subset X_2 - (u_2, z_1]$. Also, since $[v_2, c_2] \subset X$, it follows that $f_2'([v_2, c_2]) = f'([v_2, c_2]) = \hat{f}([v_2, c_2])$.

If $k = 2$, then $w_2 = e$, $[w, e] \subset f_2'([v_2, c_2])$ and we are done. Otherwise, w_2 is a branchpoint and we may assume, by Lemma 1, that $c_2 \in B(X)$ and $\hat{f}(c_2) = w_2$. We repeat this process, as we have done before, finally getting that e is in the image of $f_2' |_{X_2 - (u_2, z_1]}$.

Hence, we have that each of f_1' and $f_2' |_{X_2 - (u_2, z_1]}$ is a u-mapping whose image is Y. Also, each X_i has fewer than m interior edges which are folded by f_i'. Hence, for $i = 1, 2$, there is a point x_i in X_i such that $f_i'(x_i) = g'(x_i)$.

We consider the point x_1 in X_1. Since $X_1 \subset X$, $g(x_1) = g'(x_1) = f_1'(x_1) = \hat{f}(x_1)$. So, $f(x_1) \neq g(x_1)$ implies that $\hat{f}(x_1) \neq f(x_1)$. It follows that $x_1 \in [v_1, z_1]$ and $f(x_1) \neq w$. Hence, $w < f(x_1) < \hat{f}(x_1) = g(x_1) \leq b_1$ in the ordering on $[w, b_1]$.

We will now find a point z in $[v_1, v_2]$ so that $f(z) \in [w, b_2]$ and $f(z)$ separates w from $g(z)$ in Y. We indicate this separation by writing $w < f(z) < g(z)$.

If $x_2 \in X$, we may apply the same argument to x_2 that we applied to x_1 to get that $x_2 \in [v_2, z_2]$ and $w < f(x_2) < \hat{f}(x_2) = g(x_2) \leq b_2$. In this case, we let $z = x_2$.

If $x_2 \in Z - X$, we get that $g(u_2) = g'(x_2) = f_2'(x_2) = h^{-1}(x_2)$. Thus, $g(u_2)$ is in Y_2. We have that $w < a_2 = f(u_2) < \hat{f}(u_2) = b_2$ and either $g(u_2)$ is separated from $f(u_2)$ by b_2 or $g(u_2) = b_2$. We indicate this separation by writing $b_2 \leq g(u_2)$. In this case, we let $z = u_2$ and we again have that $z \in [v_1, v_2]$ and $w < f(z) < g(z)$.

We consider the arc $[x_1, z]$. Since $[x_1, z] \subset [v_1, v_2]$ and $f([v_1, v_2]) = [a_1, a_2] \subset t_1 \cup t_2$, it follows that $f|_{[x_1, z]}$ and $g|_{[x_1, z]}$ have a coincidence point, which is a contradiction. Hence, $f\colon X \to Y$ is universal. It follows that $F\colon X' \to Y$ is universal.

In [**8**], Nadler showed that each universal mapping from a compact Hausdorff space onto a locally connected metric continuum is weakly confluent. Hence, by Theorem 1, a u-mapping of trees must be weakly confluent.

We also have the following fixed point result as a corollary to Theorem 1.

THEOREM 2. *Suppose that D is a directed set and* $X = \varprojlim \{X_i, f_i^j, D\}$, *where, for each* $i \leq j$, X_i *is a tree and there is a subtree* X_j' *of* X_j *such that* $f_i^j \mid_{X_j'}$ *is a u-mapping onto* X_i. *Then X has the fixed point property.*

Proof. The theorem follows immediately from Holsztynski's [**5**, Corollary 1] result and Theorem 1.

We will now show that Theorem 2 is a generalization of Eberhart and Fugate's theorem [**3**, Theorem 7]. We first show that each weakly arc-preserving mapping of trees can be restricted to a *u*-mapping of trees. Then we give an example of a *u*-mapping of trees which is not weakly arc-preserving.

Suppose hereafter that $f\colon X \to Y$ is an arc-preserving mapping of a tree X onto a tree Y and that X is minimal with respect to mapping onto Y; i.e., if X' is a proper subcontinuum of X, then $f(X') \neq Y$.

LEMMA 2. *Let w be a point of Y and M a component of* $f^{-1}(w)$. *If* z_1 *and* z_2 *are endpoints of Y which belong to a component D of* $Y - \{w\}$, *and each of* C_1 *and* C_2 *is a component of* $X - M$ *such that* $z_1 \in f(C_1)$ *and* $z_2 \in f(C_2)$, *then* $C_1 = C_2$.

Proof. Suppose that $C_1 \neq C_2$. For $i = 1, 2$, let a_i be a point of C_i such that $f(a_i) = z_i$. Let $\alpha = [a_1, a_2]$. Now, α is an arc and $f(\alpha)$ contains each of w, z_1, and z_2. But w, z_1, and z_2 are distinct endpoints of $\overline{D}$ which implies that $f(\alpha)$ is not an arc, a contradiction.

LEMMA 3. *Let w be a point of Y and M a component of* $f^{-1}(w)$ *which contains a branchpoint v of X. If s is a leg of* $\operatorname{st}(M)$, *then* $f(s)$ *intersects exactly one component of* $Y - \{w\}$.

Proof. Suppose that $f(s)$ intersects the components D_1 and D_2 of $Y - \{w\}$. Let $\{C_i\}_{i=1}^m$ be the set of components of $X - M$. Assume, without loss of generality, that $s \subset \overline{C}_1$. We notice that $m \geq 3$, for otherwise, some component C of $X - \{v\}$ is a subset of M, in which case X is not minimal with respect to mapping onto Y. For $j = 1, 2, 3$, let z_j be in $E(Y) - f(\bigcup_{i=1/i \neq j}^m \overline{C}_i)$.

Suppose that for some $j \in \{1, 2, 3\}$, $z_j \notin D_1 \cup D_2$. Assume that z_j belongs to the component D_3 of $Y - \{w\}$. Let a_j be a point of C_j such that $f(a_j) = z_j$. Now, let α be the minimal arc in X such that $a_j \in \alpha$ and

$s \subset \alpha$. We have that $f(\alpha)$ intersects each of D_1, D_2, and D_3. So, f is not arc-preserving which is a contradiction.

Hence, for $j = 1, 2, 3$, $z_j \in D_1 \cup D_2$. Thus, two of z_1, z_2, and z_3 are either in D_1 or D_2. But this contradicts Lemma 2.

We are now ready to see that f has the properties of a u-mapping.

(1) Suppose that $v \in B(X)$ but $f(v) \notin B(Y)$. Let $w = f(v)$ and let M be the component of $f^{-1}(w)$ that contains v. Let $\{C_i\}_{i=1}^m$ be the set of components of $X - M$. As in Lemma 3, $m \geq 3$. Since $w \notin B(Y)$, $Y - \{w\}$ has at most two components, say D_1 and D_2. For $j = 1, 2, 3$, let z_j be in $E(Y) - f(\bigcup_{i=1/i \neq j}^m \overline{C}_i)$. So, two of z_1, z_2, and z_3 must either be in D_1 or D_2. But this contradicts Lemma 2. Hence, $f(v)$ must be a branchpoint of Y.

(2) Let $[v, a]$ be a terminal edge of X with $v \in B(X)$. Let $w = f(v)$; by (1), we know that $w \in B(Y)$. Let M be the component of $f^{-1}(w)$ that contains v. Now, $[v, a] \not\subset M$, for otherwise, X in not minimal with respect to mapping onto Y. By Lemma 3, $f([v, a])$ intersects exactly one component D of $Y - \{w\}$. Suppose that $\overline{D}$ is not an arc. Then D contains at least two endpoints z_1 and z_2 of Y. Since $f([v, a])$ is an arc, only one of z_1 and z_2 is in $f([v, a])$, say z_1. Hence, there is a component C of $X - M$ such that $C \cap [v, a] = \varnothing$ and $z_2 \in f(C)$. This contradicts Lemma 2. Thus, $\overline{D}$ is an arc which implies that $f([v, a])$ is a terminal edge of Y.

(3) Let $w \in B(Y)$ and let M be a component of $f^{-1}(w)$ which contains a branchpoint v of X. Let $\{t_i\}_{i=1}^n$ be the legs of st(w). Suppose that the leg t_1 of st(w) is not initially covered by a leg of st(M). Let D_1 be the component of $Y - \{w\}$ such that $t_1 \subset \overline{D}_1$ and let z_1 be in $E(Y) \cap D_1$. Let C_1 be a component of $X - M$ such that $z_1 \in f(C_1)$. Choose a point $a_1 \in C_1$ such that $f(a_1) = z_1$. By Lemma 3, each leg of st(M) has an initial image, so C_1 has an initial image. Assume that the initial image of C_1 is t_2. Let D_2 be the component of $Y - \{w\}$ such that $t_2 \subset \overline{D}_2$. For $j = 2, 3$, let $z_j \in E(Y) - f(\bigcup_{i=1/i \neq j}^m C_i)$, where $C_2, C_3, \ldots, C_m$ are the remaining components of $X - M$. So, neither z_2 nor z_3 is in $f(C_1)$. By Lemma 2, neither z_2 nor z_3 is in D_1. Also, z_2 and z_3 are not in the same component of $Y - \{w\}$. Hence, one of z_2 and z_3 is not in $D_1 \cup D_2$. Assume that z_3 is in the component D_3 of $Y - \{w\}$. Let a_3 be a point of C_3 such that $f(a_3) = z_3$. Let $\alpha = [a_1, a_3]$. Now α is an arc, but $f(\alpha)$ intersects each of D_1, D_2, and D_3, a contradiction. Hence, the legs of st(M) initially cover the legs of st(w).

(4) Let $w \in B(Y)$. We will show that there is no interior edge $[v_1, v_2]$ of X with the property that $f(v_1) = w = f(v_2)$ and $f([v_1, v_2]) \neq \{w\}$. Suppose otherwise. Let $\{t_i\}_{i=1}^n$ be the legs of st(w). Let M_1 and M_2 be

the components of $f^{-1}(w)$ that contain v_1 and v_2, respectively. Assume that $f([v_1, v_2])$ intersects $t_1 - \{w\}$. By property (3), there is a leg r of $\mathrm{st}(M_1)$ whose initial image is t_2 and there is a leg s of $\mathrm{st}(M_2)$ whose initial image is t_3. Let α be the unique minimal arc in X such that $r \cup s \subset \alpha$. Notice that $[v_1, v_2] \subset \alpha$ also. Hence, $f(\alpha)$ intersects each of t_1, t_2, and t_3, a contradiction. Thus, property (4) holds by default.

(5) Let $v \in B(X)$. By (1), $f(v) \in B(Y)$. Let $[v, u]$ be an edge of X. We need to show that $f([v, u]) \subset \mathrm{st}(f(v))$.

If $[v, u]$ is a terminal edge of X, then by (2), $f([v, u])$ is a terminal edge of Y. So, $f([v, u]) \subset \mathrm{st}(f(v))$.

Suppose that $[v, u]$ is an interior edge of X and $f([v, u])$ is not a subset of $\mathrm{st}(f(v))$. Then $f([v, u]) \neq \{f(v)\}$ and, by the proof of (4), $f(v) \neq f(u)$.

Suppose that $f(v)$ and $f(u)$ are not adjacent branchpoints of Y. Let $w \in B(Y) \cap (f(v), f(u))$. Let b be a branchpoint of X such that $f(b) = w$. Now, either the arc $[b, v]$ contains u or the arc $[b, u]$ contains v. Assume, without loss of generality, that $u \in [b, v]$. Since $f([v, u])$ is an arc and w is a branchpoint of Y, there is a leg t of $\mathrm{st}(w)$ such that $f([v, u]) \cap t = \{w\}$. Let M be the component of $f^{-1}(w)$ that contains b and let s be a leg of $\mathrm{st}(M)$ whose initial image is t. Let a be a point of s such that $f(a) \in t - \{w\}$. Finally, let $\alpha = [a, v]$. Now, $[v, u] \subset \alpha$; so, $f(\alpha)$ intersects each of t, $[w, f(v)]$, and $[w, f(u)]$. Thus, $f(\alpha)$ intersects three distinct components of $Y - \{w\}$. This is a contradiction.

Suppose that $f(v)$ and $f(u)$ are adjacent branchpoints of Y; i.e., $[f(v), f(u)]$ is an interior edge of Y. Since $f([v, u])$ is not a subset of $\mathrm{st}(f(v))$, then either $f([v, u])$ intersects a leg r of $\mathrm{st}(f(v))$ different from $[f(v), f(u)]$ or $f([v, u])$ intersects a leg s of $\mathrm{st}(f(u))$ different from $[f(v), f(u)]$. We assume that the latter is the case. Let t be a leg of $\mathrm{st}(f(u))$ different from both $[f(v), f(u)]$ and s. Let M be the component of $f^{-1}(f(u))$ that contains u and let h be a leg of $\mathrm{st}(M)$ whose initial image is t. Let c be a point in h such that $f(c)$ is in $t - \{f(u)\}$. Let $\alpha = [c, v]$. Then α is an arc but $f(\alpha)$ intersects three legs of $\mathrm{st}(f(u))$, namely t, s, and $[f(u), f(v)]$. This is a contradiction.

Having established properties (1) through (5), f must be a u-mapping. We have the following theorem.

THEOREM 3. *If $f\colon X \to Y$ is a weakly arc-preserving mapping of trees, then there is a subcontinuum X' of X such that $f(X') = Y$ and $f|_{X'}$ is a u-mapping.*

Proof. Since f is weakly arc-preserving, there is a subcontinuum X'' of X such that $f(X'') = Y$ and $f|_{X''}$ is arc-preserving. Let X' be a subcontinuum of X'' which is minimal with respect to mapping onto Y. Clearly, $f|_{X'}$ is arc-preserving. We have shown that the mapping $f|_{X'}$: $X' \to Y$ must satisfy properties (1) through (5). Hence, $f|_{X'}$ is a u-mapping.

We now wish to look at a few examples. The first example together with Theorem 3 shows that u-mappings are more general than weakly arc-preserving mappings. The other examples show the necessity of properties (1) through (5) in Theorem 1.

In each example the maps will be piecewise linear with respect to some triangulation of the domain. Hence, we will only indicate what the mappings do to the vertices of these triangulations.

EXAMPLE 1. A u-mapping of trees which is not weakly arc-preserving.

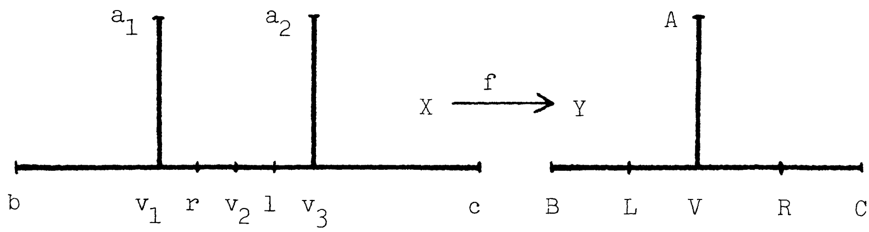

FIGURE 1

Let f be given by $f(a_i) = A$ for $i = 1, 2$, $f(b) = B$, $f(c) = C$, $f(v_i) = V$ for $i = 1, 2, 3$, $f(r) = R$, and $f(l) = L$.

Figure 2 below is a schematic indication of how f maps X onto Y.

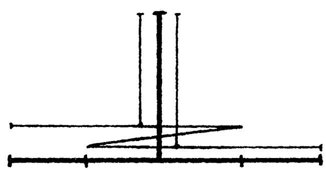

FIGURE 2

It is easy to check that f is a u-mapping. The image of each of the arcs $[a_1, l]$ and $[a_2, r]$ in X is the simple triod with endpoints A, L, and R in Y. Thus, f is not arc-preserving. Since any subcontinuum of X that maps onto Y must contain either $[a_1, v_1]$ or $[a_2, v_3]$, it follows that f is not weakly arc-preserving.

Since each subcontinuum of a given tree is characterized by its endpoints, we will refer to a continuum in X or in Y by listing its endpoints; e.g., Y may be denoted by $\langle A, B, C \rangle$.

EXAMPLE 2. A non-universal mapping of trees which satisfies properties (1), (3), (4), and (5), but does not satisfy (2).

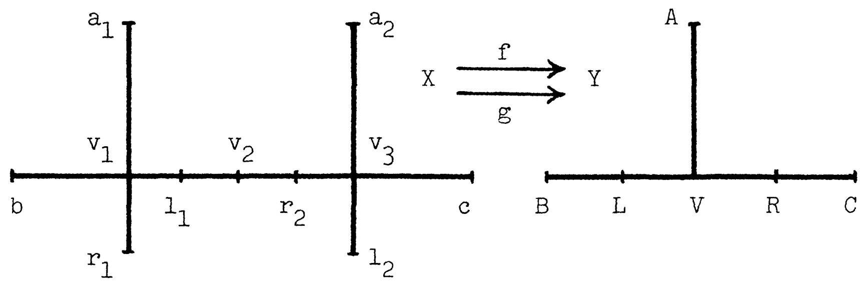

FIGURE 3

Let f be given by $f(a_i) = A$ for $i = 1,2$, $f(b) = B$, $f(c) = C$, $f(v_i) = V$ for $i = 1,2,3$, $f(r_i) = R$ for $i = 1,2$, and $f(l_i) = L$ for $i = 1,2$.

Now, g is also piecewise linear with respect to the triangulation of X shown in Figure 3. Thus, we will indicate, for example, that g maps the arc $[v_1, l_1]$ linearly onto the arc $[c, v]$ by the notation $[v_1, l_1] \to [c, v]$. If g is constant on some subtree of X, say $g(\langle a_1, b, r_1\rangle) = \{C\}$, we use the notation $\langle a_1, b, r_1\rangle \to \{C\}$. According to this convention, we define g as follows.

$$g\colon \langle a_1, b, r_1\rangle \to \{C\} \qquad [l_1, v_2] \to [V, A] \qquad [r_2, v_3] \to [V, B]$$
$$[v_1, l_1] \to [C, V] \qquad [v_2, r_2] \to [A, V] \qquad \langle a_2, c, l_2\rangle \to \{B\}$$

Figure 4 below gives a schematic representation of the mappings f and g.

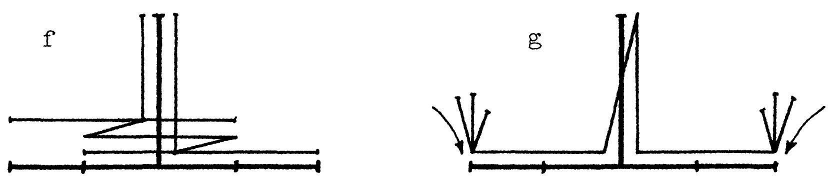

FIGURE 4

It is easy to check that f satisfies properties (1), (3), (4), and (5). Property (2) is not satisfied since the image under f of the terminal edge $[v_1, r_1]$ in X is the arc $[V, R]$ in Y which is not a terminal edge.

We will now show that f and g have no coincidence point. Referring to the definition of g, our notation makes it easy to see the behavior of both g and f over a given arc. On the triod $\langle a_1, b, r_1\rangle$, we see that the images under g and f are disjoint. On the arc $[v_1, l_1]$, the image under f goes from V to L as the image under g goes from C to V; thus, no coincidence occurs. On $[l_1, v_2]$, f goes from L to V as g goes from V to A. The action of each of f and g is symmetric on the subtrees $\langle a_1, b, r_1, v_2\rangle$ and $\langle a_2, c, l_2, v_2\rangle$ of X. Hence, f and g have no coincidence point.

EXAMPLE 3. A non-universal mapping of trees which satisfies properties (1), (2), (4), and (5), but does not satisfy (3).

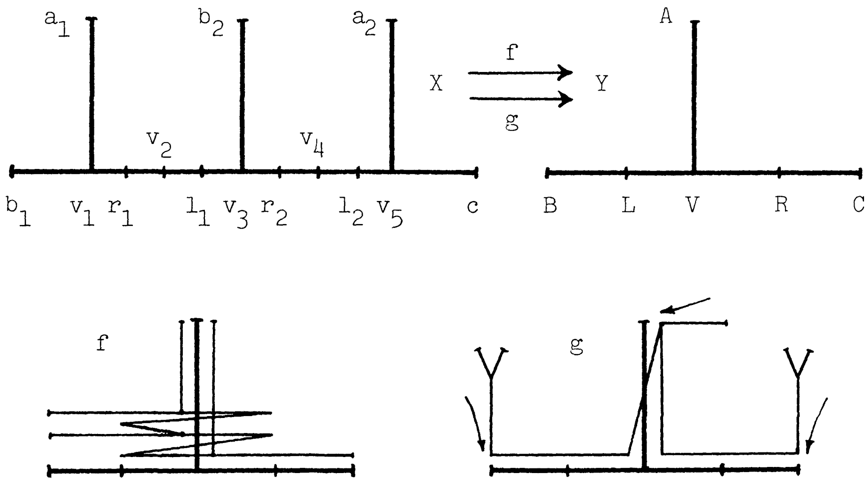

FIGURE 5

Let f be given by $f(a_i) = A$ for $i = 1, 2$, $f(b_i) = B$ for $i = 1, 2$, $f(c) = C$, $f(v_i) = V$ for $i = 1, 2, 3, 4, 5$, $f(r_i) = R$ for $i = 1, 2$, and $f(l_i) = L$ for $i = 1, 2$.

Let g be given by

$$g\colon \langle a_1, b_1, v_2\rangle \to \{C\} \qquad [l_1, v_3] \to [V, A] \qquad [r_2, v_4] \to [V, B]$$
$$[v_2, l_1] \to [C, V] \qquad [v_3, b_2] \to \{A\} \qquad \langle v_4, c, a_2\rangle \to \{B\}$$
$$[v_3, r_2] \to [A, V]$$

In a manner similar to that outlined in Example 2, it is easy to check that f has the desired properties. We also notice that a restriction of the mapping f would yield an example of a non-universal mapping which satisfies properties (1), (2), (3), and (5), but not (4). Let $X' = X - (v_3, b_2]$. Then the mapping $f|_{X'}\colon X' \to Y$ has the desired properties.

Examples of non-universal mappings which do not satisfy property (1) or do not satisfy property (5) can also be given.

REFERENCES

[1] D. Bellamy, *A tree-like continuum without the fixed point property*, Houston J. Math., **6** (1979), 1–14.

[2] R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly, **76** (1969), 119–132.

[3] C. A. Eberhart and J. B. Fugate, *Weakly confluent maps on trees*, General Topology and Modern Analysis, Academic Press, New York (1981), 209–215.
[4] O. H. Hamilton, *A fixed point theorem for pseudo arcs and certain other metric continua*, Proc. Amer. Math. Soc., **2** (1951), 173–174.
[5] W. Holsztynski, *Universal mappings and fixed point theorems*, Bull. Acad. Polon. Sci., **XV** (1967), 433–438.
[6] M. M. Marsh, *Fixed Point Theorems for Certain Tree-Like Continua*, Dissertation, Unversity of Houston (1981).
[7] _____, *A fixed point theorem for inverse limits of fans*, Proc. Amer. Math. Soc., **91** (1984), 139–142.
[8] S. B. Nadler, Jr., *Universal mappings and weakly confluent mappings*, CX (1980), #3, 221–235.

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