# CONVOLUTION SEMIGROUPS ON HYPERGROUPS

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*Dedicated to Professor Elmar Thoma on the occasion of his 60th birthday*

The purpose of this paper is to establish a unified treatment of many disparate theorems of Levy-Hinčin type. The appropriate framework to do this is the theory of commutative hypergroups. In this way we not only generalize the results mentioned above but also settle some asymmetries indicated above. Roughly speaking a hypergroup $K$ is a space in which the product of two elements is a probability measure on this space satisfying certain conditions. If $K$ is commutative and if the space $\hat{K}$ of characters is a hypergroup under pointwise operations a Levy-Hinčin formula for convolution semigroups is obtained. Before setting up some notation we show how the examples fit in.

**Introduction.** Continuous convolution semigroups $(\mu_t)_{t \geq 0}$ on locally compact abelian groups $G$ are described completely by the so-called Levy-Hinčin formula, see [1] and [6]. This is done by a canonical representation of the negative definite function $\psi$ on $\hat{G}$, which is in one-to-one correspondence to $(\mu_t)_{t \geq 0}$. Beyond that there exist theorems of Levy-Hinčin type for convolution semigroups built up in various ways. The following subsumes some approaches to this subject.

For certain Gelfand pairs $(G, H)$ convolution semigroups consisting of $H$-biinvariant probability measures $\mu_t$ on $G$ allow a Levy-Hinčin representation. In this situation one studies negative definite functions $\psi$ defined on the set $S^p(G, H)$ of all positive definite spherical functions. Negative definiteness on $S^p(G, H)$ has to be defined in an appropriate way. A nice summary on this topic containing a lot of references is [11]. As a dual concept a Levy-Hinčin formula for $H$-biinvariant negative definite functions on $G$ is established in [5]. Of course here one misses a corresponding convolution semigroup on $S^P(G, H)$.

In [4] Bochner characterized homogeneous stochastic processes associated with ultraspherical polynomials via a Levy-Hinčin formula, see also [7]. The part of the convolution semigroup is played by a semigroup of bounded sequences $(c_n(t))_{n=0}^{\infty}$ bearing a certain positive definite property. In [13] Kennedy studied homogeneous stochastic processes which may be viewed as dual to the class considered by Bochner. Here the part of the convolution semigroup is played by a semigroup of bounded functions $f_t(x)$, $-1 \leq x \leq 1$, bearing a certain positive definite property.

For a Gelfand pair $(G, H)$ the double coset space $K = G//H$ is a commutative hypergroup, see [12, 8.2B]. Now in many cases $\hat{K}$ is a hypergroup and each member of $\hat{K}$ may be identified with a function of $S^p(G, H)$ as well as each $H$-biinvariant measure on $G$ may be identified with a measure on $K$. Moreover in many cases the dual of $S^p(G, H) = \hat{K}$ is $G//H = K$ again. Then there is also a connection between $H$-biinvariant negative definite functions on $G$ and convolution semigroups defined on $S^p(G, H) = \hat{K}$ via a Bochner theorem. In this manner convolution semigroups with respect to $(G, H)$ or with respect to $S^p(G, H)$ have a Levy-Hinčin representation in a unified way.

The stochastic processes studied by Bochner are corresponding via Fourier transformation with a convolution semigroup defined on a hypergroup $K = [-1, 1]$. The "dual" stochastic process in [13] is the Fourier transform of a convolution semigroup on a hypergroup $K = \mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Hypergroup structures on $\mathbf{N}_0$ connected with orthogonal polynomials are studied in [14]. In the case of ultraspherical polynomials or more general Jacobi polynomials the dual space $\hat{K}$ of $K = \mathbf{N}_0$ is equal to $[-1, 1]$, and $\hat{K}$ is a hypergroup with Pontrjagin duality $\hat{\hat{K}} = K$, see [14]. In this manner the Levy-Hinčin formula of the convolution semigroup corresponding to the stochastic process of Bochner resp. Kennedy is established by the Levy-Hinčin representation based on hypergroups.

In our exposition we follow that of the relevant chapters in [1]. We note that some methods of proof used in the group case are not available for hypergroups. Let $K$ be a locally compact Hausdorff space. $M(K)$ denotes the space of all bounded Radon measures, $M^1(K)$ the subset of all probability measures and $p_x$ the point measure of $x \in K$. The support of a measure $\mu$ is denoted by $\mathrm{supp}\,\mu$. $C(K)$ denotes the space of continuous functions on $K$. The space $K$ is called a hypergroup if the following conditions are satisfied:

(H1) There exists a map: $K \times K \to M^1(K)$, $(x, y) \mapsto p_x * p_y$, called convolution, which is continuous, where $M^1(K)$ bears the vague topology. The linear extension to $M(K)$, see [12, Lemma 2.4B], satisfies $p_x * (p_y * p_z) = (p_x * p_y) * p_z$.

(H2) $\mathrm{supp}\,p_x * p_y$ is compact.

(H3) There exists a homeomorphism $K \to K$, $x \mapsto \bar{x}$, called involution, such that $x = \bar{\bar{x}}$ and $(p_x * p_y)^- = p_{\bar{y}} * p_{\bar{x}}$.

(H4) There exists an element $e \in K$, called unit element, such that $p_e * p_x = p_x * p_e = p_x$.

(H5) $e \in \mathrm{supp}\,p_x * p_{\bar{y}}$ if and only if $x = y$.

(H6) The map $(x, y) \mapsto \mathrm{supp}\, p_x * p_y$ of $K \times K$ into the space of nonvoid compact subsets of $K$ is continuous, the latter space with the topology as given in [12, 2.5].

Many examples are contained in [3], [12], [14], [19].

In this paper we only deal with commutative hypergroups, i.e. $p_x * p_y = p_y * p_x$. Then $K$ always has a Haar measure, [22]. Denote

$$\hat{K} = \big\{ \alpha \in C(K): \alpha \neq 0, \text{ bounded, } p_x * p_y(\alpha) = \alpha(x)\alpha(y),$$
$$\alpha(\bar{x}) = \overline{\alpha(x)} \big\}.$$

Equipped with the topology of uniform convergence on compacta $\hat{K}$ is a locally compact Hausdorff space. For basic results of Fourier analysis we refer to [12], where hypergroups are called "convos". We shall say that $\hat{K}$ is a hypergroup with respect to pointwise multiplication, if for $\alpha, \gamma \in \hat{K}$ there exists a measure $p_\alpha * p_\gamma \in M^1(\hat{K})$ such that $\alpha(x)\gamma(x) = \int_{\hat{K}} \tau(x)\, dp_\alpha * p_\gamma(\tau)$ for each $x \in K$, and $\hat{K}$ is a hypergroup with this convoltuion and complex conjugation as involution and the constant function 1 as unit. In general the dual $\hat{K}$ is not a hypergroup. If $\hat{K}$ is a hypergroup with respect to pointwise multiplication, then $K \subseteq K^{\wedge\wedge}$ in a natural manner, [12, Theorem 12.4B]. If in addition $K = K^{\wedge\wedge}$ holds, we shall call $K$ a *strong hypergroup*.

**1. Negative definite functions.** Let $K$ be a commutative hypergroup. We note that for many results in this section commutativity is not really used. A continuous function $\psi: K \to \mathbf{C}$ is called *negative definite* if for any $x_1, \ldots, x_n \in K$, $c_1, \ldots, c_n \in \mathbf{C}$

(N) $$\sum_{i,\, j=1}^{n} \big( \psi(x_i) + \overline{\psi(x_j)} - p_{x_i} * p_{\bar{x}_j}(\psi) \big) c_i \bar{c}_j \geq 0$$

For example each constant function $c1$, $c \geq 0$, is negative definite. Obviously the following holds for a negative definite function $\psi$:

$$\psi(e) \geq 0, \quad p_x * p_{\bar{x}}(\psi) \in \mathbf{R}, \quad \overline{\psi(x)} = \psi(\bar{x}),$$
$$\psi(x) + \psi(\bar{x}) \geq p_x * p_{\bar{x}}(\psi).$$

We note that $\mathrm{Re}\,\psi$ is in general not a nonnegative function, see the remark below. The following statement and its corollary can be proved exactly as in [1, Proposition 7.5, Corollary 7.6 and 7.7].

PROPOSITION 1.1. *A function $\psi: K \to \mathbf{C}$ is negative definite if and only if the following conditions are satisfied*:
 (a) $\psi$ *is continuous*, $\psi(e) \geq 0, \overline{\psi(x)} = \psi(\bar{x})$ *for each $x \in K$ and*

(b) *if* $x_1, \ldots, x_n \in K$, $c_1, \ldots, c_{\underset{\neq}{\supseteq}} \in \mathbf{C}$ *with* $\sum_{i=1}^{n} c_i = 0$, *then*

$$\sum_{i,\,j=1}^{n} p_{x_i} * p_{\overline{x_j}}(\psi) c_i \overline{c_j} \leq 0$$

*holds*.

COROLLARY 1.2. *Let* $\psi$ *be a function on K.*

(a) *If* $\psi$ *is negative definite, then* $x \mapsto \psi(x) - \psi(e)$ *is negative definite.*

(b) *If* $\psi$ *is positive definite, then* $x \mapsto \psi(e) - \psi(x)$ *is negative definite.*

REMARK. Consider $K = \mathbf{N}_0$ equipped with a hypergroup structure as defined in [**14**, §2]. This hypergroup $\mathbf{N}_0$ is intimately connected with a certain orthogonal polynomial sequence $\{P_n(x)\}$. In fact given $x \in \mathbf{R}$ every function $\varphi_x \colon \mathbf{N}_0 \to \mathbf{R}$, $\varphi_x(n) = P_n(x)$ has the property that $p_n * p_m(\varphi_x) = \varphi_x(n)\varphi_x(m)$ for each $n \in \mathbf{N}_0$. Thus $\varphi_x$ is a positive definite function on the hypergroup $\mathbf{N}_0$. By Corollary 1.2(b) $\psi_x(n) = 1 - P_n(x)$ is negative definite. If $x > 1$ we have $P_1(x) > 1$. Hence $\psi_x(1) < 0$.

PROPOSITION 1.3. *Let* $\psi \colon K \to \mathbf{C}$ *be a negative definite function on K.*

(a) *If* $\operatorname{Re} \psi \geq 0$ *holds, then* $\operatorname{Re} \psi \geq \psi(e) \geq 0$.

(b) *If* $\psi$ *is bounded, then* $\operatorname{Re} \psi \geq \psi(e) \geq 0$.

*Proof*. Given $x \in K$ choose $x_1 = e$, $x_2 = x$ and $c_1 = 1$, $c_2 = -1$. By Proposition 1.1 we know that $2 \cdot \operatorname{Re} \psi(x) \geq \psi(e) + p_x * p_{\overline{x}}(\psi)$. Now $p_x * p_{\overline{x}}(\psi) \in \mathbf{R}$ and $x \mapsto p_x * p_{\overline{x}}(\psi)$ is a continuous function on $K$. Thus we obtain

$$\operatorname{Re} \psi(x) \geq (1/2)\psi(e) + (1/2^2)\psi(e) + (1/2^2) \int_K p_y * p_{\overline{y}}(\psi)\, dp_x * p_{\overline{x}}(y).$$

Continuing in this fashion we have

$$\operatorname{Re} \psi(x) \geq \left( \sum_{k=1}^{n} 1/2^k \right) \psi(e)$$

$$+ (1/2^n) \int_K \cdots \int_K p_{x_{n-1}} * p_{\overline{x_{n-1}}}(\psi) \cdots dp_{x_1}$$

$$* p_{\overline{x_1}}(x_2)\, dp_x * p_{\overline{x}}(x_1).$$

If $\operatorname{Re} \psi \geq 0$, we know that $p_x * p_{\overline{x}}(\psi) \geq 0$ for each $x \in K$. Therefore $\operatorname{Re} \psi \geq (\sum_{k=1}^{n} 1/2^k)\psi(e)$ for each $n \in \mathbf{N}$ and then $\operatorname{Re} \psi \geq \psi(e)$. If $\psi$ is bounded, the absolute value of the $(n - 1)$-fold integral above is less than $\|\psi\|_u$. This implies again that $\operatorname{Re} \psi \geq \psi(e)$.

PROPOSITION 1.4. *Let* $\psi\colon K \to \mathbf{C}$ *be a positive definite function with* $\mathrm{Re}\,\psi \geq 0$. *Then* $\sqrt{|p_x * p_y(\psi)|} \leq \sqrt{|\psi(x)|} + \sqrt{|\psi(y)|}$ *for* $x, y \in K$.

*Proof.* A small modification of the arguments in [**1**, Proposition 7.15] yields that

$$\left| \psi(x) + \overline{\psi(y)} - p_x * p_y(\psi) \right|^2$$

$$\leq \left( 2\,\mathrm{Re}\,\psi(x) - p_x * p_{\bar{x}}(\psi) \right)\left( 2\,\mathrm{Re}\,\psi(y) - p_y * p_{\bar{y}}(\psi) \right).$$

Since $p_x * p_{\bar{x}}(\psi) \in \mathbf{R}$, we know that

$$p_x * p_{\bar{x}}(\psi) = \int_K \mathrm{Re}\,\psi(y)\,dp_x * p_{\bar{x}}(y) \geq 0.$$

Therefore $|\psi(x) + \overline{\psi(y)} - p_x * p_{\bar{y}}(\psi)|^2 \leq 4|\psi(x)||\psi(y)|$, and the assertion follows.

THEOREM 1.5. *Let* $\psi\colon K \to \mathbf{C}$ *be a function on* $K$. *Assume that*
(a) $\psi$ *is continuous and* $\psi(e) \geq 0$,
(b) $\varphi_t\colon x \mapsto \exp(-t\psi(x))$ *are positive definite for each* $t > 0$. *Then* $\psi$ *is negative definite.*

*Proof.* By (a) the functions $\varphi_t$ are continuous and $\varphi_t(e) \leq 1$. Therefore Corollary 1.2(b) implies that $x \mapsto (1/t)(1 - \varphi_t(x))$ is negative definite for any $t > 0$. Since

$$\left| \psi(x) - (1/t)(1 - \varphi_t(x)) \right| \leq t\exp|\psi(x)| \quad \text{for } 0 < t < 1,$$

we obtain that $\lim_{t \to 0}(1/t)(1 - \varphi_t) = \psi$ uniformly on compact subsets of $K$. Now one can easily prove that $\psi$ satisfies $(N)$.

We do not know whether the inverse multiplication of this theorem does hold in general. Now some general examples of negative definite functions are given. A continuous function $h\colon K \to \mathbf{R}$ is called a *homomorphism*, if $h(\bar{x}) = -h(x)$ and $p_x * p_y(h) = h(x) + h(y)$ for $x, y \in K$. The following assertion is immediate.

PROPOSITION 1.6. *If* $h\colon K \to \mathbf{R}$ *is a homomorphism, then* $\psi = ih$ *is negative definite.*

A continuous function $q\colon K \to \mathbf{R}$ is called a *quadratic form*, if

$$p_x * p_y(q) + p_x * p_{\bar{y}}(q) = 2(q(x) + q(y)), \qquad x, y \in K.$$

Obviously a quadratic form $q$ satisfies:

(Q)    $$q(e) = 0, \quad q(\bar{x}) = q(x) \quad \text{and}$$
$$\mu * \nu(q) + \mu * \bar{\nu}(q) = 2(\nu(q)\mu(K) + \mu(q)\nu(K)),$$
$$\mu, \nu \in M(K).$$

LEMMA 1.7. *Let $q$ be a quadratic form, and let $\mu \in M(K)$. Then*

$$\mu^n(q) = n^2 \cdot \mu(K)^{n-1}\mu(q) - (n(n-1)/2)\mu(K)^{n-2}\mu * \bar{\mu}(q)$$

*for each $n \in \mathbf{N}$, $n \geq 2$, holds, where $\mu^n = \mu * \cdots * \mu$, the n-fold convolution of $\mu$.*

*Proof.* We prove the statement by induction on $n$. By (Q) we obtain

$$\mu^{n+1}(q) = 2\big(\mu^n(q)\mu(K) + \mu(K)^n\mu(q)\big) - \mu^n * \bar{\mu}(q) \quad \text{and}$$

$$\mu^n * \bar{\mu}(q) = (1/2)\big[\mu^{n-1} * (\mu * \bar{\mu})(q) + \mu^{n-1} * (\mu * \bar{\mu})(q)\big]$$

$$= \mu^{n-1}(q)\mu(K)^2 + \mu(K)^{n-1}\mu * \bar{\mu}(q).$$

Thus by the induction assumption

$$\mu^{n+1}(q) = 2\mu(K)^n\mu(q)$$

$$+ 2\mu(K)\big[n^2\mu(K)^{n-1}\mu(q) - (n(n-1)/2)\mu(K)^{n-2}\mu * \bar{\mu}(q)\big]$$

$$- \mu(K)^2\big[(n-1)^2\mu(K)^{n-2}\mu(q)$$

$$- ((n-1)(n-2)/2)\mu(K)^{n-3}\mu * \bar{\mu}(q)\big]$$

$$- \mu(K)^{n-1}\mu * \bar{\mu}(q)$$

$$= \mu(K)^n(n+1)^2\mu(q) - \mu(K)^{n-1}((n+1)n/2)\mu * \bar{\mu}(q).$$

COROLLARY 1.8.. *Let $q$ be a quadratic form. Then*

$$\lim_{n \to \infty} \frac{p_x^n(q)}{n^2} = q(x) - (1/2)p_x * p_{\bar{x}}(q).$$

*In particular if $q$ is nonnegative (i.e. $q(x) \geq 0$) then*

$$p_x * p_{\bar{x}}(q) \leq 2q(x) \leq p_x * p_x(q).$$

COROLLARY 1.9. *The only bounded quadratic form $q$ on $K$ is $q = 0$.*

*Proof.* Since $q$ is a quadratic form, $p_x^n * p_{\bar{x}}^n(q) = n \cdot p_x * p_{\bar{x}}(q)$ is satisfied for each $n \in \mathbf{N}$, $x \in K$. In fact $p_x^{n+1} * p_{\bar{x}}^{n+1}(q) = p_x^n * p_{\bar{x}}^n(q) + p_x * p_{\bar{x}}(q)$ holds by means of (Q). Now a simple induction argument applies. The boundedness of $q$ implies that $p_x * p_{\bar{x}}(q) = 0$ for each $x \in K$. Therefore $\lim_{n \to \infty} = (1/n^2)p_x^n(q) = q(x)$. Again by the boundedness of $q$ wee see that $q = 0$.

PROPOSITION 1.10. *Each nonnegative quadratic form on $K$ is a negative definite function.*

*Proof.* By means of Proposition 1.1 we have to show: if $x_1, \ldots, x_m \in K$, $c_1, \ldots, c_m \in \mathbf{R}$ with $\sum_{i=1}^m c_i = 0$ ($c_i \neq 0$) and $\lambda = \sum_{i=1}^m c_i p_{x_i} \in M(K)$, then $\lambda * \bar{\lambda}(q) \leq 0$. Let $\lambda = \mu - \nu$, $\mu = \lambda^+ \in M^+(K)$, $\nu = \lambda^- \in M^+(K)$, the decomposition of $\lambda$ into its positive parts. In particular $\mu(K) = \sum_{c_i > 0} c_i > 0$, $\nu(K) = -\sum_{c_i < 0} c_i > 0$, and consequently $\mu(K) = \nu(K)$. By Lemma 1.7 we know

$$n^2 \mu * \nu(K)^{n-1} \mu * \bar{\nu}(q)$$

$$= \mu^n * \bar{\nu}^n(q) + (n(n-1)/2)\mu * \bar{\nu}(K)^{n-2}\mu * \bar{\mu} * \nu * \bar{\nu}(q)$$

for each $n \in \mathbf{N}$. Further by (Q)

$$\mu * \bar{\mu} * \nu * \bar{\nu}(q) = \mu * \bar{\mu}(q)\nu * \bar{\nu}(K) + \nu * \bar{\nu}(q)\mu * \bar{\mu}(K).$$

Therefore

$$\mu * \bar{\nu}(q) = \frac{\mu^n * \bar{\nu}^n(q)}{n^2 \mu(K)^{n-1}\nu(K)^{n-1}}$$

$$+ \frac{\mu * \bar{\mu}(q)\nu(K)^2 + \nu * \bar{\nu}(q)\mu(K)^2}{\mu(K)\nu(K)}(1/2 - 1/(2n))$$

for each $n \in \mathbf{N}$. Consequently

$$\mu * \bar{\nu}(q) - (1/2)\mu * \bar{\mu}(q) - (1/2)\nu * \bar{\nu}(q)$$

$$= \lim_{n \to \infty} \frac{\mu^n * \bar{\nu}^n(q)}{n^2 \mu(K)^{n-1}\nu(K)^{n-1}} \geq 0.$$

Thus we have established that

$$\lambda * \bar{\lambda}(q) = \mu * \bar{\mu}(q) - 2\mu * \bar{\nu}(q) + \nu * \bar{\nu}(q) \leq 0.$$

EXAMPLES. We conclude this section by calculating nonnegative quadratic forms on $K = \mathbf{N}_0$ with a hypergroup structure as defined in [14, §2] and yet mentioned in the remark above. Since in this situation the

identity map is the involution on $K = \mathbf{N}_0$ the defining property of quadratic forms reduces to $p_m * p_n(q) = q(m) + q(n)$ for each $m$, $n \in \mathbf{N}_0$. The only homomorphism is the null-function. Let $(a_n)$, $(b_n)$, $(c_n)$ be the sequences, which define the convolution on $\mathbf{N}_0$, [14, §2]. Define recursively the following sequence $(s_n)$:

(i)    $s_0 = 0$,   $s_1 = 1$,   $s_{n+1} = (1/a_n)(1 + (1 - b_n)s_n - c_n s_{n-1})$

for $n = 2, 3, \ldots$. We note that $(s_n)$ is an increasing sequence. In fact $s_{n+1} - s_n = (1/a_n)(1 + c_n(s_n - {}_{n-1}))$, since $a_n + b_n + c_n = 1$. By induction we see that $s_{n+1} \geq s_n$.

PROPOSITION 1.11. *Let* $K = \mathbf{N}_0$ *be a hypergroup defined by the sequences* $(a_n)$, $(b_n)$, $(c_n)$ *as in* [14, §2]. *Denote by* $(s_n)$ *the sequence from* (i). *Then the nonnegative quadratic forms on* $K = \mathbf{N}_0$ *are exactly the functions* $q(n) = s_n \cdot a$, *where* $a \geq 0$.

*Proof.* Assume that $q: \mathbf{N}_0 \to \mathbf{R}$ is a nonnegative quadratic form. Then $q(0) = 0$. Let $a = q(1) \geq 0$. Since $p_1 * p_n = a_n p_{n+1} + b_n p_n + c_n p_{n-1}$, $n \in \mathbf{N}$ see [14], we have $q(n + 1) = (1/a_n)(q(1) + (1 - b_n)q(n) - c_n q(n - 1)$. Now by induction it follows that $q(n) = s_n \cdot a$. Conversely we show that for $a \geq 0$ the function $q(n) = s_n \cdot a$ is a nonnegative quadratic form. Since $(s_n)$ is increasing $q$ is nonnegative. We use induction on $m$ to prove that $p_m * p_n(q) = q(m) + q(n)$ for each $m$, $n \in \mathbf{N}_0$, $m \leq n$. For $m = 1$ this equation follows by the definition of $s_n$. Let $m + 1 \leq n$. By the induction assumption we see that

$$p_{m+1} * p_n(q) = (1/a_m)p_1 * (p_m * p_n)(q) - (b_m/a_m)p_m * p_n(q)$$

$$- (c_m/a_m)p_{m-1} * p_n(q)$$

$$= (1/a_m)(q(1) + q(m) + q(n))$$

$$- (b_m/a_m)(q(m) + q(n)) - (c_m/a_m)(q(m - 1) + q(n))$$

$$= q(n) + q(m + 1).$$

For the prominent case, where $K = \mathbf{N}_0$ bears the convolution structure which is "realized" by the ultraspherical polynomials, i.e. $a_n = (n + 1 + 2\alpha)/(2n + 1 + 2\alpha)$, $b_n = 0$, $c_n = n/(2n + 1 + 2\alpha)$, $\alpha \geq -1/2$, see [14, §3(a)], we can calculate $s_n$ explicitly:

(ii)                              $$s_n = \frac{n(n + 1 + 2\alpha)}{2 + 2\alpha}.$$

This follows by a direct induction argument.

REMARK. Let $K = G//H$, a double coset space, where $G$ is a locally compact group, $H$ a compact subgroup of $G$. An $H$-biinvariant function on $G$ may be viewed as a function on $K$. It is immediate that every (continuous) $H$-biinvariant negative definite function on the group $G$ is a negative definite function on the hypergroup $G//H$. In [10] Harzallah has noted that there exist a double coset space $G//H$ and a $H$-biinvariant function on $G$, which is a nonnegative quadratic form on the hypergroup $K = G//H$, but is not a negative definite function on the group $G$. By Proposition 1.10 we see that a $H$-biinvariant function on $G$, which is negative definite on the hypergroup $K = G//H$, is in general not negative definite on the group $G$.

## 2. Convolution semigroups.

Let $K$ again be a commutative hypergroup. A family $(\mu_t)_{t > 0}$, $\mu_t \in M^+(K)$ is called a *convolution semigroup* on $K$, if

(a) $\mu_t(K) \leq 1$ for each $t > 0$,

(b) $\mu_t * \mu_s = \mu_{t+s}$ for $t, s > 0$,

(c) $\lim_{t \to 0} \mu_t = p_e$ with respect to the vague topology on $M(K)$.

LEMMA 2.1. *Let $(\mu_t)$ be a convolution semigroup on $K$. Then for $\alpha \in \hat{K}$ the function $t \mapsto \hat{\mu}_t(\alpha)$, $\mathbf{R}^+ \to \mathbf{C}$ is continuous.*

*Proof.* Obviously it is sufficient to show that $\lim_{t \to 0} \hat{\mu}_t(\alpha) = 1$. By [1, Proposition 1.4] we have to show that $\lim_{t \to 0} \mu_t(K) = 1$. This follows by the argument of [1, Proposition 8.2].

As one might expect one can study convolution semigroups on $K$ by means of negative definite functions on $\hat{K}$. Of course we have to assume that $\hat{K}$ is a hypergroup with respect to pointwise multiplication.

THEOREM 2.2. *Assume that $\hat{K}$ is a hypergroup with respect to pointwise multiplication. If $(\mu_t)_{t > 0}$ is a convolution semigroup on $K$, then there exists exactly one negative definite function $\psi$: $\hat{K} \to \mathbf{C}$ with $\mathrm{Re}\, \psi \geq 0$ such that*

$$\hat{\mu}_t(\alpha) = \exp(-t\psi(\alpha)) \text{ for each } \alpha \in \hat{K}, \qquad t > 0.$$

*Proof.* One has to give some obvious modifications to the proof of [1, Theorem 8.3].

The negative definite function $\psi$: $\hat{K} \to \mathbf{C}$ with $\mathrm{Re}\, \psi \geq 0$ defined in the above theorem is called associated to $(\mu_t)$. Unfortunately we can only prove a rather weak converse implication of Theorem 2.2.

THEOREM 2.3. *Let $K$ be a strong hypergroup. Let $\psi$: $\hat{K} \to \mathbf{C}$ be a negative definite function with $\operatorname{Re}\psi \geq 0$, such that $\psi_t(\alpha) = \exp(-t\psi(\alpha))$ is positive definite for $t > 0$. There exists a unique convolution semigroup $(\mu_t)$ on $K$ such that $\psi$ is associated to $(\mu_t)$.*

*Proof.* Since $\operatorname{Re}\psi \geq 0$ we know that $|\exp(-t\psi(\alpha)| \leq 1$. Thus using $K = \hat{\hat{K}}$ by [**12**, Theorem 12.3B] there are unique determined measures $\mu_t \in M^+(K)$, $t > 0$, such that $\hat{\mu}_t(\alpha) = \exp(-t\psi(\alpha))$. Obviously $(\mu_t)$ satisfies the properties (a), (b); use [**21**, Theorem II.5.5]. Further using the boundedness of $\psi$ on compact subsets of $\hat{K}$, we know

$$\lim_{t \to 0} \hat{\mu}_t(\alpha) = \lim_{t \to 0} \exp(-t\psi(\alpha)) = 1$$

on compact subsets. Let $f \in C_{00}(K)$, $\varepsilon > 0$. By [**15**, Proposition 1.1], or using $K = \hat{\hat{K}}$ and [**12**, Theorem 7.3H], there exists a function $g \in C_{00}(\hat{K})$ such that $\|f - \check{g}\|_u < \varepsilon$. Now we obtain

$$|\mu_t(f) - p_e(f)| \leq 2\varepsilon + \int_{\hat{K}} |g(\alpha)| |\hat{\mu}_t(\bar{\alpha}) - 1| \, d\pi(\alpha).$$

Therefore $\lim_{t \to 0} \mu_t = p_e$ in the vague topology on $M(K)$.

EXAMPLE. Fix $\alpha \geq -1/2$. We consider the quadratic forms $q(n) = s_n \cdot a$, $a \geq 0$, on the "ultraspherical hypergroup" $\mathbf{N}_0$ to $\alpha$, where $s_n$ is given by the formula (ii) in §1. This hypergroup is a strong one, see [**14**, §4]. We give a convolution semigroup $(\mu_t)$ on the dual hypergroup $\widehat{\mathbf{N}_0} = [-1, 1]$ such that $q$ is associated to $(\mu_t)$. We call this $(\mu_t)$ the *Brownian semigroup* on $[-1, 1] = \widehat{\mathbf{N}_0}$. Our arguments are simple modifications of those of [**5**]. First we note for arbitrary $\mu \in M^+(K)$ that $\exp(t\hat{\mu})$, $t > 0$, is positive definite on $\hat{K}$. In fact $(\hat{\mu})^n = (\mu^n\hat{)}$ and $\mu^n \in M^+(K)$. Using the exponential power series we see that $\exp(t\hat{\mu})$ is positive definite. Therefore $\exp(-t(1 - \hat{\mu}))$ is positive definite. Let $\{P_n^\alpha(x)\}$ be the ultraspherical polynomial sequence. In particular we know, that

$$n \mapsto \exp\left(-t\left(1 - P_n^\alpha(x)\right)\right)$$

is positive definite for each $x \in [-1, 1]$. Let $x \in [-1, 1[$. Replace $t$ by $t(1 - x)^{-1}$ obtaining that $n \mapsto \exp(-t(1 - P_n^\alpha(x))(1 - x)^{-1})$ is positive definite. Now

$$\lim_{x \to 1} \left(1 - P_n^\alpha(x)\right)(1 - x)^{-1} = n(n + 2\alpha + 1)/(2 + 2\alpha),$$

see [**4**, 2.2.19]. It follows that $n \mapsto \exp(-tq(n))$ is positive definite. Now Theorem 2.3 yields the Brownian semigroup on $[-1, 1]$.

3. **The Lévy-Hinčin representation.** We assume throughout this section that $K$ is a commutative hypergroup such that $\hat{K}$ is a hypergroup with respect to pointwise multiplication. Let $S = \{\mu \in M^1(K): \mu = \bar{\mu}, \operatorname{supp}\mu \text{ compact}\}$.

LEMMA 3.1. *Let $V$ be a compact neighbourhood of $e \in K$. Then there exists a $\sigma \in S$ such that $-1/2 \leq \check{\sigma}(x) \leq 1/2$ for each $x \in K \setminus V$.*

*Proof.* There exists a compact neighbourhood $U$ of $e$ such that $\overline{U} = U$, $U * U \subseteq V$. Let $\varphi = 1/m(U)\chi_U * \chi_U$, where $\chi_U$ is the characteristic function of $U$, and $m$ the Haar measure on $K$. We know that $\varphi$ is a positive definite function on $K$ such that $0 \leq \varphi \leq 1$, $\varphi(e) = 1$, $\varphi(x) = 0$ for $x \in K \setminus V$, see [**12**, Lemma 6.2E and Lemma 11.3A]. By [**12**, Theorem 12.3B] there exists a $\mu \in M^+(\hat{K})$ such that $\check{\mu} = \varphi$. One easily obtains that $\mu \in M^1(\hat{K})$ and $\mu = \bar{\mu}$. Choose a compact subset $C \subseteq \hat{K}$ such that $\overline{C} = C$ and $\mu(C) \geq 3/4$. Then for $\sigma = (\mu|C)/\mu(C)$ $\quad \|\varphi - \check{\sigma}\|\|_u \leq \|\mu - \sigma\| \leq 1/2$ and thus $-1/2 \leq \check{\sigma}(x) \leq 1/2$ for $x \in K \setminus V$.

The following statement is contained in the proof of Levy's continuity theorem in [**2**, Theorem 4.6].

PROPOSITION 3.2. *Let $f$ be a bounded continuous function on $\hat{K}$. Let $(\mu_\alpha)$ be a net of positive measures in $M(K)$ such that $\lim \widehat{\mu_\alpha \pi} = f\pi$ vaguely ($\pi$ the Haar measure on $\hat{K}$) and $\lim \widehat{\mu_\alpha}(1) = f(1)$. Then there exists a $\mu \in M^+(K)$ such that $\hat{\mu} = f$ and $\lim \mu_\alpha = \mu$ weakly.*

PROPOSITION 3.3. *Let $(\mu_t)$ be a convolution semigroup on $K$ and $\psi: \hat{K} \to \mathbf{C}$ the negative definite function associated to $(\mu_t)$. The net $((1/t) \cdot \mu_t | K \setminus \{e\})_{t > 0}$ converges vaguely as $t \to 0$ to a positive measure $\mu$ on $K \setminus \{e\}$. For each $\sigma \in S$ the function $\psi * \sigma - \psi$ is positive definite and bounded. There exists a measure $\mu_\sigma \in M^+(K)$ such that $\widehat{\mu_\sigma} = \psi * \sigma - \psi$. These measures satisfy*

(L) $\qquad\qquad (1 - \check{\sigma})\mu = \mu_\sigma | K \setminus \{e\} \quad$ for $\sigma \in S$.

*The equation* (L) *determines $\mu$ uniquely.*

*Proof.* Let $\sigma \in S$. Slightly modifying the proof of [**1**, Proposition 18.2] we obtain for $\alpha \in \hat{K}$:

$$(1/t)[(1 - \check{\sigma})\mu_t]\check{}(\alpha) = (1/t)\left[\widehat{\mu_t}(\alpha) - \int_{\hat{K}} \int_{\hat{K}} \widehat{\mu_t}(\gamma)\, dp_\alpha * p_{\bar{\tau}}(\gamma)\, d\sigma(\tau)\right]$$

$$= (1/t)\left[\widehat{\mu_t}(\alpha) - \widehat{\mu_t} * \sigma(\alpha)\right] = (1/t)[1 - \exp(-t\psi)] * (\sigma - p_1)(\alpha).$$

We know that $\lim_{t \to 0}(1/t)(1 - \exp(-t\psi)) = \psi$ uniformly on compact subsets of $\hat{K}$; see the proof of Theorem 1.5. Thus

$$\lim_{t \to 0}(1/t)[(1 - \check{\sigma})\mu_t]\check{} = \psi * \sigma - \psi$$

uniformly on compact subsets of $\hat{K}$ and $\psi * \sigma - \psi$ is positive definite and bounded. Therefore $\lim_{t \to 0} (1/t)[(1 - \check{\sigma})\mu_t]\hat{}\pi = (\psi * \sigma - \psi)\pi$ vaguely in the space of all Radon measures on $\hat{K}$. By Proposition 3.2 there exists a measure $\mu_\sigma \in M^+(K)$ such that $\widehat{\mu_\sigma} = \psi * \sigma - \psi$ and

$$\lim_{t \to 0} (1/t)(1 - \check{\sigma})\mu_t = \mu_\sigma$$

in the weak topology. The same arguments as in [1, Proposition 18.2] using Lemma 3.1 now show that there exists a positive Radon measure $\mu$ on $K \setminus \{e\}$ such that $\mu = \lim_{t \to 0}(1/t)\mu_t | K \setminus \{e\}$ vaguely and $(1 - \check{\sigma})\mu = \mu_\sigma | K \setminus \{e\}$ for any $\sigma \in S$. Let $\nu$ be another positive Radon measure on $K \setminus \{e\}$ such that $(1 - \check{\sigma})\mu = (1 - \check{\sigma})\nu$. For $x \in K \setminus \{e\}$ let $\alpha \in \hat{K}$ such that $\operatorname{Re} \alpha(x) < 1$. Denote $\sigma = 1/2(p_\alpha + p_{\bar{\alpha}}) \in S$ and let $V_x$ be a compact neighbourhood of $x$ such that $e \notin V_x$ and $\check{\sigma} | V_x < 1$. Now it is obvious that $\mu | V_x = \nu | V_x$. Thus $\mu = \nu$.

The positive Radon measure $\mu$ on $K \setminus \{e\}$ is called the *Levy measure* of $(\mu_t)$.

PROPOSITION 3.4. *Let $\mu$ denote the Levy measure of a given convolution semigroup $(\mu_t)$. Then*
   (a) $\int_{K \setminus \{e\}} (1 - \operatorname{Re} \gamma(x)) \, d\mu(x) < \infty$ *for each $\gamma \in \hat{K}$,*
   (b) *if $V$ is a compact neighbourhood of $e$ in $K$, then $\mu | K \setminus V \in M^+(K)$.*

*Proof.* (a) For $\gamma \in \hat{K}$ let $\sigma = 1/2(p_\gamma + p_{\bar{\gamma}}) \in S$. Then by (L)

$$\int_{K \setminus \{e\}} (1 - \operatorname{Re} \gamma(x)) \, d\mu(x) = \mu_\sigma(K \setminus \{e\}) < \infty.$$

The statement of (b) follows as in [1, Proposition 18.4] using Lemma 3.1.

The following assertions can be proved by an argument as in [1, Lemma 18.13 and Lemma 18.16].

LEMMA 3.5. *Let $h$: $\hat{K} \to \mathbf{R}$ be continuous and $h(1) = 0$. $h$ is a homomorphism if and only if $h * \sigma - h = 0$ for each $\sigma \in S$.*

LEMMA 3.6. *Let $q$: $\hat{K} \to \mathbf{R}$ be continuous with $q(\alpha) = q(\bar{\alpha})$, $q(1) = 0$. $q$ is a quadratic form if and only if $q * \sigma - q$ is a constant function for each $\sigma \in S$. Moreover, in the affirmative case $q$ is nonnegative if and only if $q * \sigma - q \geq 0$ for all $\sigma \in S$.*

COROLLARY 3.7. *Let $(\mu_t)$ be a convolution semigroup on $K, \psi$ the associated negative definite function. Assume that the Levy measure $\mu$ of $(\mu_t)$ is symmetric. Then $\operatorname{Im} \psi$ is a homomorphism. In particular $i \operatorname{Im} \psi$ is*

*negative definite. Further* $\mu$ *is also the Levy measure of* $(\nu_t)$, *where* $\nu_t = \mu_{t/2} * \overline{\mu_{t/2}}$.

*Proof.* $\overline{\mu} = \mu$ is equivalent to $\overline{\mu}_\sigma = \mu_\sigma$ for each $\sigma \in S$. This is equivalent to $\psi * \sigma - \psi$ being real-valued for each $\sigma \in S$. Thus $(\operatorname{Im}\psi) * \sigma - \operatorname{Im}\psi = 0$ for each $\sigma \in S$, and by Lemma 3.5 we know that $\operatorname{Im}\psi$ is a homomorphism. Thus $i \operatorname{Im}\psi$ is negative definite. Further we have seen $(\operatorname{Re}\psi) * \sigma - \operatorname{Re}\psi = \psi * \sigma - \psi$. The proof of Proposition 3.3 yields that $(\mu_t)$ and $(\nu_t)$ define the same class of measures $\mu_\sigma$, $\sigma \in S$. Therefore the uniqueness of the measure satisfying (L) implies the second assertion.

We shall say that $K$ satisfies property (F) if the following holds: If $C \subseteq \hat{K}$ is compact then there exist a constant $M_C \geq 0$, a neighbourhood $U_C$ of $e$ in $K$ and a finite subset $N_C$ of $C$ such that for each $x \in U_C$

(F)   $\sup\{1 - \operatorname{Re}\alpha(x): \alpha \in C\} \leq M_C \cdot \sup\{1 - \operatorname{Re}\alpha(x): \alpha \in N_C\}$.

Obviously each compact or discrete hypergroup satisfies the property (F). It is known that each locally compact abelian group satisfies property (F), see [18, Lemma 5.1]. At the end of this section we shall give another important class of hypergroups for which property (F) is valid.

LEMMA 3.8. *Let $K$ satisfy property* (F). *Assume that $\mu$ is a positive symmetric Radon measure on $K \setminus \{e\}$ such that*

$$\int_{K \setminus \{e\}} (1 - \operatorname{Re}\alpha(x)) \, d\mu(x) < \infty \text{ for each } \alpha \in \hat{K} \text{ and}$$

$\mu \,|\, K \setminus V \in M^+(K)$ *for each compact neighbourhood $V$ of $e$.*

*Then the function* $\psi_\mu: \hat{K} \to \mathbf{R}$, $\psi_\mu(\alpha) = \int_{K \setminus \{e\}} (1 - \operatorname{Re}\alpha(x)) \, d\mu(x)$ *is continuous. Further $\psi_\mu$ is negative definite.*

*Proof.* Let $\alpha_0 \in \hat{K}$, $\varepsilon > 0$ and $C$ a compact neighbourhood of $\alpha_0$. Property (F) of $K$ yields $M_C \geq 0$, $N_C = \{\alpha_1, \ldots, \alpha_n\} \subseteq \hat{K}$ and a neighbourhood $U_C$ of $e$ in $K$ such that

$$\int_{U_C \setminus \{e\}} \sup_{\alpha \in C} (1 - \operatorname{Re}\alpha(x)) \, d\mu(x)$$

$$\leq M_C \cdot \int_{U_C \setminus \{e\}} \sup_{\alpha \in N_C} (1 - \operatorname{Re}\alpha(x)) \, d\mu(x)$$

$$\leq M_C \cdot \sum_{i=1}^{n} \int_{U_C \setminus \{e\}} (1 - \operatorname{Re}\alpha_i(x)) \, d\mu(x) \leq M_C \cdot \sum_{i=1}^{n} \psi_\mu(\alpha_i).$$

Thus there exists a neighbourhood $V$ of $e$ such that

$$\int_{V\setminus\{e\}} (1 - \operatorname{Re}\alpha(x))\, d\mu(x) < \varepsilon/4$$

for each $\alpha \in C$. Since $\mu \mid K\setminus V$ is bounded there exists a neighbourhood $W_{\alpha_0} \subseteq C$ of $\alpha_0$ in $\hat{K}$ such that $|\int_{K\setminus V} (\alpha(x) - \alpha_0(x))\, d\mu(x)| < \varepsilon/2$ for each $\alpha \in W_{\alpha_0}$. Thus

$$\left|\psi_\mu(\alpha) - \psi_\mu(\alpha_0)\right| \le \int_{V\setminus\{e\}} (1 - \operatorname{Re}\alpha(x))\, d\mu(x)$$

$$+ \int_{V\setminus\{e\}} (1 - \operatorname{Re}\alpha_0(x))\, d\mu(x)$$

$$+ \left|\int_{K\setminus V} (\operatorname{Re}\alpha_0(x) - \operatorname{Re}\alpha(x))\, d\mu(x)\right| < \varepsilon$$

for each $\alpha \in W_{\alpha_0}$. In order to show that $\psi_\mu$ is negative definite we note that for $x \in K$ the function $\tau \mapsto \tau(x)$ is positive definite. Thus $\tau \mapsto 1 - \operatorname{Re}\tau(x)$ is negative definite. Given $\alpha, \beta \in \hat{K}$ apply again property (F) to $C = \operatorname{supp} p_\alpha * p_\beta$ to show that

$$\int_{K\setminus\{e\}} \sup_{\alpha\in C} (1 - \operatorname{Re}\alpha(x))\, d\mu(x) < \infty.$$

Thus we may apply Fubini's theorem leading to the following equation:

$$\psi_\mu(\alpha) + \psi_\mu(\beta) - p_\alpha * p_{\bar{\beta}}(\psi_\mu)$$

$$= \int_{K\setminus\{e\}} \left[(1 - \operatorname{Re}\alpha(x)) + (1 - \operatorname{Re}\beta(x))\right.$$

$$\left. - \int_{\hat{K}} (1 - \operatorname{Re}\tau(x))\, dp_\alpha * p_{\bar{\beta}}(\tau)\right] d\mu(x).$$

Therefore $\psi_\mu$ is negative definite.

Now we can prove a Levy-Hinčin formula.

THEOREM 3.9. *Suppose that $K$ satisfies property* (F). *Let* $(\mu_t)$ *be a convolution semigroup on $K$ with associated negative definite function* $\psi$: $\hat{K} \to \mathbf{C}$, *and Levy measure $\mu$. Assume that $\mu$ is symmetric.*

(a) *Then $\psi$ can be written*

$$(*) \quad \psi(\alpha) = c + ih(\alpha) + q(\alpha) + \int_{K\setminus\{e\}} (1 - \operatorname{Re}\alpha(x))\, d\mu(x),$$

*for $\alpha \in \hat{K}$, where $c$ is a nonnegative constant, $h$: $\hat{K} \to \mathbf{R}$ a homomorphism and $q$: $\hat{K} \to \mathbf{R}$ a nonnegative quadratic form.*

(b) *Moreover $c, h, q$ in $(*)$ are determined uniquely by $(\mu_t)$*: $c = \psi(1)$, $h = \text{Im} \psi$ *and*

$$q(\alpha) = \lim_{n \to \infty} \left[ \frac{p_\alpha^n(\psi)}{n^2} + \frac{p_\alpha^n * p_{\bar{\alpha}}^n(\psi)}{2n} \right].$$

*Proof.* (a) Let $c = \psi(1)$ and $\psi' = \psi - c1$. Then $\psi'$ is a negative definite function associated to a convolution semigroup having the same Levy measure $\mu$ as $(\mu_t)$. Since $\mu$ is symmetric, $h = \text{Im} \psi$ is a homomorphism by Corollary 3.7. Further the function $\psi'' = \psi - c1 - ih$ is negative definite and associated to a convolution semigroup with Levy measure $\mu$. Define $\psi_\mu$ as in Lemma 3.8 using Proposition 3.4. Consider $q = \psi'' - \psi_\mu$. The function $q$ is continuous, real-valued, symmetric and $q(1) = 0$. Given $\sigma \in S$ we know that $h * \sigma - h = 0$ by Lemma 3.5. Thus $\psi'' * \sigma - \psi'' = \psi * \sigma - \psi = \widehat{\mu_\sigma}$, where the measure $\mu_\sigma$ is defined as in Proposition 3.3. Consequently $q * \sigma - q = \widehat{\mu_\sigma} - (\psi_\mu * \sigma - \psi_\mu)$. One easily obtains that

$$\psi_\mu * \sigma - \psi_\mu = \int_{K \backslash \{e\}} \text{Re}\, \alpha(x)(1 - \check{\sigma}(x))\, d\mu(x)$$

$$= \int_{K \backslash \{e\}} \overline{\alpha(x)}(1 - \check{\sigma}(x))\, d\mu(x).$$

Since $\mu_\sigma | K \backslash \{e\} = (1 - \check{\sigma})\mu$, we see that $q * \sigma - q = \mu_\sigma(\{e\}) \geq 0$. By Lemma 3.6 the function $q$ is a nonnegative quadratic form. Now the statement of (a) is proved.

(b) Of course $c = \psi(1)$ and $h = \text{Im}\, \psi$. Denote again $\psi_\mu(\alpha) = \int_{K \backslash \{e\}} (1 - \text{Re}\, \alpha(x))\, d\mu(x)$. We know that $\psi_\mu$ is negative definite. By Corollary 1.8 we obtain

$$q(\alpha) - (1/2)p_\alpha * p_{\bar{\alpha}}(q) = \lim_{n \to \infty} p_\alpha^n(q)/n^2$$

$$= \lim_{n \to \infty} p_\alpha^n(\psi)/n^2 - \lim_{n \to \infty} (1/n^2) \int_{K \backslash \{e\}} \left(1 - \text{Re}(\alpha(x)^n)\right) d\mu(x).$$

Note that Fubini's theorem is available by property (F). Obviously $\lim_{n \to \infty} (1/n^2)(1 - \text{Re}(\alpha(x)^n)) = 0$ for each $x \in K$. If $\alpha(x) \neq 0$, let $0 < r \leq 1$, $\theta \in [-\pi, \pi]$ such that $\alpha(x) = r \exp(i\theta)$. Then for $n \in \mathbf{N}$

$$(1/n^2)\left(1 - \text{Re}(\alpha(x)^n)\right) = (1/n^2)(1 - r^n) + (r^n/n^2)(1 - \cos n\theta)$$

$$\leq (1/n^2)(1 - r^n) + r^n C(1 - \cos \theta),$$

where $C > 0$ can be established as in [1, p. 183]. Thus

$$\left(1/n^2\right)\left(1 - \mathrm{Re}\left(\alpha(x)^n\right)\right) \le (1 - r) + r^{n-1}C(r - r\cos\theta)$$

$$\le (1 - r^2) + C(1 - r\cos\theta) = \left(1 - |\alpha(x)|^2\right) + C(1 - \mathrm{Re}\,\alpha(x)).$$

For $\alpha(x) = 0$ this inequality holds obviously. Since

$$\int_{K\backslash\{e\}} \left(1 - |\alpha(x)|^2\right) d\mu(x) + C \cdot \int_{K\backslash\{e\}} (1 - \mathrm{Re}\,\alpha(x))\, d\mu(x)$$

$$= p_\alpha * p_{\bar{\alpha}}(\psi_\mu) + C\psi_\mu(\alpha) < \infty,$$

the dominated convergence theorem yields that $q(\alpha) = \lim_{n\to\infty} p_\alpha^n(\psi)/n^2 + p_\alpha * p_{\bar{\alpha}}(q)/2$. In the proof of Corollary 1.9 we have noted that $p_\alpha^n * p_{\bar{\alpha}}^n(q) = n \cdot p_\alpha * p_{\bar{\alpha}}(q)$ for each $n \in \mathbf{N}$. Thus

$$p_\alpha * p_{\bar{\alpha}}(q)/2 = \lim_{n\to\infty} p_\alpha^n * p_{\bar{\alpha}}^n(q)/(2n)$$

$$= \lim_{n\to\infty} p_\alpha^n * p_{\bar{\alpha}}^n(\psi)/(2n) - \lim_{n\to\infty} (1/2n)\int_{K\backslash\{e\}} \left(1 - |\alpha(x)|^{2n}\right) d\mu(x).$$

Since $(1/2n)(1 - |\alpha(x)|^{2n}) \le (1 - |\alpha(x)|) \le (1 - |\alpha(x)|^2)$, the dominated convergence theorem applies again. Consequently $p_\alpha * p_{\bar{\alpha}}(q)/2 = \lim_{n\to\infty}(1/2n)p_\alpha^n * p_{\bar{\alpha}}^n(\psi)$.

EXAMPLES. (a) Consider the hypergroup $K = \mathbf{N}_0$ with the structure which corresponds to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha \ge \beta > -1$, $\alpha + \beta + 1 \ge 0$. If we assume that in addition that $\beta \ge -1/2$ or $\alpha + \beta \ge 0$ then the dual $\hat{K}$ is a hypergroup and may be identified with $[-1, 1]$, see [14, §§3 and 4]. Since $\hat{K}$ is compact the homomorphism and the quadratic form in the Levy-Hinčin formula of Theorem 3.9 are zero. Further given a convolution semigroup $(\mu_t)$ on $\mathbf{N}_0$ the Levy measure $\mu$ of $(\mu_t)$ is symmetric and bounded. Hence

$$\widehat{\mu_t}(x) = \exp(-t\psi(x)) \text{ for } x \in [-1, 1], \quad \text{where}$$

$$\psi(x) = c + \sum_{n=1}^{\infty} \left(1 - P_n^{(\alpha,\beta)}(x)\right)\mu(n).$$

Compare [13] for the ultraspherical case $\alpha = \beta$.

(b) Consider the dual hypergroup $K = [-1, 1]$ corresponding to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, where $(\alpha, \beta)$ belongs to the same region as in (a). The dual of $K$ is the hypergroup $\mathbf{N}_0$ of (a), see [14, §4]. Given a convolution semigroup $(\mu_t)$ on $[-1, 1]$ the Levy measure $\mu$ is symmetric.

Further the homomorphism is zero. Thus

$$\widehat{\mu_t}(n) = \exp(-t\psi(n)) \text{ for } n \in \mathbf{N}_0, \quad \text{where}$$

$$\psi(n) = c + q(n) + \int_{-1}^{1-0} \left(1 - P_n^{(\alpha,\beta)}(x)\right) d\mu(x) \quad \text{and}$$

$$q(n) = as_n(a \geq 0), \text{ where } s_n = \frac{n(n + \alpha + \beta + 1)}{\alpha + \beta + 2}.$$

The quadratic form $s_n$ can be computed by means of formula (i) in Chapter 1. Compare [4] and [7].

(c) We consider now commutative hypergroups which are obtained from certain groups. Let $G$ denote a locally compact group and let $B$ denote a subgroup of the automorphism group $\text{Aut}(G)$ that contains the group $I(G)$ of inner automorphisms. One calls $G$ an $[FIA]_B^-$ group provided the closure $\overline{B}$ of $B$ in $\text{Aut}(G)$ is compact, where $\text{Aut}(G)$ bears the Birkhoff topology. It is easily established that the $\overline{B}$-orbit space $G_B$ of $G$ is commutative hypergroup with natural operations, see [19, §1]. There is an obvious relation between $\overline{B}$-invariant measures or functions on $G$ and measures or functions on $G_B$. We note that $G_B$ may be identified with the set of all continuous nonzero positive definite $B$-invariant functions $\varphi$ on $G$ which satisfy $\varphi(x)\varphi(y) = \int_{\overline{B}} \varphi(x\beta(y)) \, d\beta$, where $d\beta$ is the normalized Haar measure on $\overline{B}$, see [16] and [17]. In [9] it is shown that $\widehat{G_B}$ is a hypergroup with respect to pointwise multiplication. We prove that $G_B$ satisfies property (F).

LEMMA 3.10. *Let $K_1$ and $K_2$ be two hypergroups satisfying property* (F). *Then $K_1 \oplus K_2$ satisfies property* (F).

*Proof.* Let $0 \leq r_1, r_2 \leq 1$ and $0 \leq x, y \leq 2\pi$. Then

$$1 - r_1 r_2 \cos(x + y) \leq 2[(1 - r_1 \cos x) + (1 - r_2 \cos y)]$$

is valid. In fact the inequalities $1 - r_1 r_2 \leq (1 - r_1) + (1 - r_2)$ and $1 - \cos(x + y) \leq 2[(1 - \cos x) + (1 - \cos y)]$, compare [18, p. 82], imply that

$$1 - r_1 r_2 \cos(x + y) = (1 - r_1 r_2) + r_1 r_2 (1 - \cos(x + y))$$

$$\leq (1 - r_1) + (1 - r_2) + 2r_1(1 - \cos x) + 2r_2(1 - \cos y)$$

$$\leq 2[(1 - r_1 \cos x) + (1 - r_2 \cos y)].$$

Since each $\alpha \in (K_1 \oplus K_2)^\frown$ is written $\alpha((z_1, z_2)) = \beta(z_1)\gamma(z_2)$, where $z_1 \in K_1$, $z_2 \in K_2$, $\alpha \in \widehat{K_1}$, $\gamma \in \widehat{K_2}$, the assertion can be easily proved.

The structure theorem for $[FIA]_B^-$ groups $G$, see [23], says that $G = V \oplus L$, $L$ containing an open, compact subgroup $H$, and $V$ being a vector group. By Lemma 3.10 we have to show that $V_B$ and $L_B$ satisfy property (F). Using [17, Theorem 5.8] we obtain that $V_B$ satisfies (F), since $V$ satisfies (F) as an abelian group. The methods of [17, Proposition 2.9] show that the map $r_H : \widehat{L_B} \to \widehat{H_B}$, $r_H(\alpha) = \alpha | H$ is continuous. $\widehat{H_B}$ is discrete. Thus for $\alpha \in \widehat{L_B}$ the set $C_\alpha = \{ \beta \in \widehat{L_B} : \beta | H = \alpha | H \}$ is open. This fact implies property (F) for $L_B$. Hence (F) holds for $G_B$.

Thus Theorem 3.9 characterizes convolution semigroups $(\mu_t)$ on the $[FIA]_B^-$ group $G$ consisting of $B$-invariant measures $\mu_t$. The class of $[FIA]_B^-$ groups covers for instance orbit spaces of compact groups or, for $B = I(G)$ of locally compact groups having relatively compact conjugacy classes and having small invariant neighbourhoods of the identity. In particular if $G = \mathbf{R}^n$, $B = \mathrm{SO}(n)$, the special orthogonal group, we obtain the Bessel functions. In fact $G_B$ and $\widehat{G_B}$ may be identified with $[0, \infty[$. Each non-constant character $\alpha \in \widehat{G_B}$ is given by $\alpha = \alpha_y$, $y \in ]0, \infty[$, where

$$\alpha_y(x) = \Gamma(\nu + 1) 2^\nu \frac{J_\nu(yx)}{(yx)^\nu} \quad \text{for } x \in ]0, \infty[ \text{ and } \alpha_y(0) = 1,$$

and $J_\nu$ is the Bessel function of the first kind of order $\nu$ and $\nu = n/2 - 1$. We refer to [20], [5, p. 201] where a Levy-Hinčin formula is established, and [17, Theorem 5.8].

Finally we note that similar arguments yield that $\widehat{G_B}$ satisfies property (F), too. The dual of $\widehat{G_B}$ is $G_B$, see [8], and Theorem 3.9 applies for $K = \widehat{G_B}$, $G$ an arbitrary $[FIA]_B^-$ group.

<h1 style="text-align:center">REFERENCES</h1>

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