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THE ETA INVARIANT, Pin^c BORDISM, AND EQUIVARIANT Spin^c BORDISM FOR CYCLIC 2-GROUPS

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The eta invariant and equivariant Stiefel-Whitney numbers completely detect Z_n equivariant Spin^c bordism and Pin^c bordism. The additive structure of Pin^c bordism and of equivariant Spin^c bordism for cyclic 2-groups is determined using these invariants in terms of K-theory. The analysis is used to embed the K-theory in the bordism.

0. Introduction. The eta invariant of Atiyah-Patodi-Singer [**4**] is an R/Z valued measure of the spectral asymmetry of self-adjoint partial differential operators. It defines both equivariant bordism and locally flat K-theory invariants in suitable categories. Let n be a power of 2, let $Z_n = \{\lambda \in C\colon \lambda^n = 1\}$ act by scalar multiplication on the unit sphere S^{2k-1} of C^k, and let $L^k(n) = S^{2k-1}/Z_n$ be a generalized lens space.

THEOREM 0.1. (a) *The eta invariant and equivariant Stiefel-Whitney numbers completely detect the Z_n-Spin^c reduced bordism groups $\tilde\Omega_*^{\mathrm{Spin}^c}(BZ_n)$.*

(b) *Let $A_{2k}(n) = 0$, and let $A_{2k-1}(n)$ be the subgroup of $\tilde\Omega_*^{\mathrm{Spin}^c}(BZ_n)$ generated by all possible Spin^c structures on $L^k(n)$. Let $\ker_*(\eta, n)$ be the kernel of all eta invariants. Then as additive groups,*

$$\tilde\Omega_*^{\mathrm{Spin}^c}(BZ_n) \approx \{A_*(n) \otimes Z[CP^2, CP^4, \dots]\} \oplus \ker_*(\eta, n),$$

$$\ker_m(\eta, n) \approx \bigoplus_{j<m} \mathrm{Tor}\left(\Omega_j^{\mathrm{Spin}^c}\right), \quad \textit{and} \quad A_{2k-1}(n) \approx \tilde K\left(S^{2k+1}/Z_n\right).$$

REMARK. Equivariant bordism decomposes as a direct sum for different primes so it suffices to study prime powers. Gilkey [**13**] showed the eta invariant completely detects $\tilde\Omega_*^{\mathrm{U}}(BZ_v)$ for all v and $\tilde\Omega_*^{\mathrm{SO}}(BZ_v)$ for v odd. The arguments given there generalize to show the eta invariant completely detects $\tilde\Omega_*^{\mathrm{Spin}^c}(BZ_v)$ if v is odd. See also Wilson [**20**]. The coefficient ring Ω_*^{U} is torsion free; all the torsion in Ω_*^{SO} and $\Omega_*^{\mathrm{Spin}^c}$ is of order 2. The torsion in $\Omega_*^{\mathrm{Spin}^c}$ enters in an essential fashion when n is a power of 2 as we shall see. The K-theory of the lens spaces is well known. Let $n = 2^v$. If $1 \le i < n$ choose s so $2^s \le i < 2^{s+1}$. If $i > k$, let $t(i,k,n) = 0$. If $i \le k$, let $t(i,k,n) = v - s + [(k-i)/2^s]$. Then $A_{2k-1}(n) = \oplus_{i=1}^{n-1} Z_{2^{t(i,k,n)}}$ by Fujii et al. [**9**]. One can also show $A_*(n) = bu_*(BZ_n)$.

The Smith homomorphism defines an isomorphism between $\tilde\Omega_m^{\mathrm{Spin}^c}(BZ_2)$ and $\tilde\Omega_{m-1}^{\mathrm{Pin}^c}$; under this isomorphism, the eta invariant for odd dimensional Z_2-Spin^c manifolds corresponds to the eta invariant of even dimensional Pin^c manifolds.

THEOREM 0.2. (a) *The eta invariant and Stiefel-Whitney numbers completely detect the* Pin^c *bordism groups* $\Omega_*^{\mathrm{Pin}^c}$.

(b) *Let* $B_{2k+1} = 0$ *and* $B_{2k} = Z_{2^{k+1}}$ *be the subgroup of* $\Omega_{2k}^{\mathrm{Pin}^c}$ *generated by real projective space* RP^{2k}. *Let* $\ker_*(\eta)$ *be the kernel of all eta invariants. Then as additive groups*

$$\Omega_*^{\mathrm{Pin}^c} \approx \{B_* \otimes Z[CP^2, CP^4, \dots]\} \oplus \ker_*(\eta) \quad \textit{and}$$

$$\ker_m(\eta) \approx \bigoplus_{j \le m} \mathrm{Tor}\left(\Omega_j^{\mathrm{Spin}^c}\right).$$

In [5] we studied $\tilde\Omega_*^{\mathrm{Spin}^c}(BZ_n)$ and $\Omega_*^{\mathrm{Pin}^c}$ using the Anderson, Brown, and Peterson splitting of the spectrum $\underline{M\mathrm{Spin}^c}$. The methods of that paper were homotopy theoretic. In this paper, we use geometry as a bridge between the analysis and the topology and obtain explicit representatives of the generators. In particular, the splitting $\tilde\Omega_*^{\mathrm{Spin}^c}(BZ_n) = \{A_* \otimes Q_*\} \oplus \ker_*(\eta, n)$ is an analytic splitting whereas in [5] it was a purely algebraic splitting. This paper rests heavily upon the results of Anderson-Brown-Peterson [1] and of Stong [18, 19]. We refer to Giambalvo [10] for analogous results regarding (S)Pin bordism. The results of Stong show that for unitary and Spin^c bordism, all relations among characteristic numbers follow from the Atiyah-Singer index theorem. Theorem 0.1 is a generalization of these results to equivariant bordism.

Here is a brief guide to the paper. In the first section, we discuss the results concerning $\Omega_*^{\mathrm{Spin}^c}$ and the equivariant Stiefel-Whitney numbers we shall need. In the second section, we define the eta invariant and recall its properties. In the third section we discuss the Smith homomorphism and in the fourth section we complete the proofs of Theorems 0.1 and 0.2. We acknowledge with pleasure helpful correspondence and conversations with Professors Giambalvo, Landweber, and Peterson.

1. The Spin^c **bordism ring and equivariant Stiefel-Whitney classes.** Let Spin(m) be the universal cover group of the special orthogonal group

SO(m) for $m > 2$. Since $\pi_1(\mathrm{SO}(m)) = Z_2$, there is a non-trivial short exact sequence $0 \to Z_2 \to \mathrm{Spin}(m) \xrightarrow{\pi} \mathrm{SO}(m) \to 0$; define Spin($m$) in terms of Clifford algebras if $m = 1, 2$ (see Atiyah-Bott-Shapiro [3]). Let $\mathrm{Spin}^c(m) = \mathrm{Spin}(m) \times \mathrm{U}(1)/Z_2$ by identifying $(g, \lambda) = (-g, -\lambda)$. Let $\tau(g, \lambda) = \lambda^2$ define a representation τ: $\mathrm{Spin}^c(m) \to \mathrm{U}(1)$, then $0 \to Z_2 \to \mathrm{Spin}^c(m) \xrightarrow{\pi \times \tau} \mathrm{SO}(m) \times \mathrm{U}(1) \to 0$. Similarly if O($m$) is the orthogonal group, let Pin(m) be the non-trivial double cover for $m > 2$, and let $\mathrm{Pin}^c(m) = \mathrm{Pin}(m) \times \mathrm{U}(1)/Z_2$. $\mathrm{Spin}^c(m)$ is the connected component of the identity in $\mathrm{Pin}^c(m)$. The forgetful homomorphism from U(m) to SO($2m$) lifts to $\mathrm{Spin}^c(2m)$ and the determinant representation lifts to τ (see Hitchin [**15**]).

Let $W^* = H^*(\mathrm{BO}; Z_2) = Z_2[w_j]$ be the algebra of Stiefel-Whitney classes. Let V be a real vector bundle over a compact manifold. V is orientable if $w_1(V) = 0$. V admits a Pin^c structure if $w_2(V)$ is the mod 2 reduction of an integral class. V admits a Spin^c structure if $w_1 = 0$ and if w_2 is integral—i.e. V is both oriented and Pin^c.

LEMMA 1.1. *Let* $0 \to V_1 \to V_2 \to V_3 \to 0$ *be a short exact sequence of vector bundles over a compact manifold* M.

(a) *If two of the* V_i *admit* Spin^c *structures, then there is a natural* Spin^c *structure induced on the third.*

(b) *If one of the* V_i *admits a* Spin^c *structure and another admits a* Pin^c *structure, then there is a natural* Pin^c *structure induced on the third.*

(c) *If two of the* V_i *admit* Pin^c *structures and if the third is orientable, then there is a natural* Spin^c *structure on the third.*

Proof. As this is an elementary calculation in characteristic classes, we omit details in the interests of brevity. This fails if we replace (S)Pin^c by (S)Pin.

A (S)Pin^c structure on a compact manifold N with boundary M is a (S)Pin^c structure on the tangent bundle $T(N)$. Since $T(N)|_M = T(M) \oplus 1$, we use Lemma 1.1 to induce a (S)Pin^c structure on M. Let $\Omega_m^{(\mathrm{S})\mathrm{Pin}^c}$ denote the bordism group of compact smooth m-dimensional (S)Pin^c manifolds modulo the subgroup which bound and let $\Omega_*^{(\mathrm{S})\mathrm{Pin}^c}$ denote the corresponding graded group. We define $\Omega_*^{(\mathrm{S})\mathrm{O}}$ similarly. If M admits a stable unitary structure, then there is a natural Spin^c structure induced on M. Let CP^k be complex projective space with the Spin^c structure induced from the holomorphic structure. The direct sum of two Spin^c bundles is a Spin^c bundle and the direct sum of a Spin^c bundle and a Pin^c bundle is a Pin^c bundle. Consequently, Cartesian product makes $\Omega_*^{\mathrm{Spin}^c}$ into a graded ring and makes $\Omega_*^{\mathrm{Pin}^c}$ into an $\Omega_*^{\mathrm{Spin}^c}$ module. Since the direct sum of two Pin^c bundles is not a Pin^c bundle in general, there is no natural ring structure on $\Omega_*^{\mathrm{Pin}^c}$. The Stiefel-Whitney numbers are Z_2 valued Ω_*^0 bordism invariants completely detecting Ω_*^0. The forgetful functor maps $\Omega_*^{(\mathrm{S})\mathrm{Pin}^c} \to \Omega_*^0$ and lets us regard the Stiefel-Whitney numbers as $(\mathrm{S})\mathrm{Pin}^c$ bordism invariants.

The representation τ: $\mathrm{Spin}^c(m) \to U(1)$ defines a complex line bundle over any Spin^c manifold M; let $c_1(M) \in H^2(M; Z)$ be the Chern class of this bundle. Let $p_k(M) \in H^{4k}(M; Z)$ be the Pontrjagin classes. We form the ring generated by $\{c_1, p_*\}$. The integer and rational characteristic numbers obtained from this ring are called the Chern/Pontrjagin numbers. Let Λ^k be the complexified exterior representations of $\mathrm{SO}(m)$; extend the Λ^k to $\mathrm{Spin}^c(m)$ by composing with the projection π. Let $R(\mathrm{Spin}^c)$ be the free polynomial algebra $R(\mathrm{Spin}^c) = Z[\Lambda^k, \tau]$; this is not the full representation ring of Spin^c. If $\theta \in R(\mathrm{Spin}^c)$ and if $M \in \Omega_{\mathrm{even}}^{\mathrm{Spin}^c}$, let index $(\theta, M) \in Z$ be the index of the Spin^c complex over M with coefficients in the virtual bundle defined by θ. Since index (θ, M) is expressible in terms of rational Chern-Pontrjagin numbers, it is a bordism invariant. Since $\mathrm{index}(\theta, M) \in Z$, $\mathrm{index}(\theta, M) = 0$ if $M \in \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})$. We set $\mathrm{index}(\theta, M) = 0$ for notational convenience if $M \in \Omega_{\mathrm{odd}}^{\mathrm{Spin}^c}$. We summarize below the results we shall need concerning $\Omega_*^{\mathrm{Spin}^c}$ and refer to Anderson-Brown-Peterson **[1]** and Stong **[18, 19]** for details.

THEOREM 1.2. *Let* $P_* = Z[CP^1, CP^2, CP^4, \dots, CP^{2k}, \dots]$.

(a) $\Omega_*^{\mathrm{Spin}^c}$ *is a commutative ring. All the torsion in* $\Omega_*^{\mathrm{Spin}^c}$ *is of order* 2. *If* $M \in \Omega_*^{\mathrm{Spin}^c}$, *then* $M = 0$ *iff all the rational Chern/Pontrjagin numbers of* M *vanish and all the Stiefel-Whitney numbers of* M *vanish.*

(b) P_* *embeds in* $\Omega_*^{\mathrm{Spin}^c}$ *and* $\Omega_*^{\mathrm{Spin}^c} \otimes_Z Z_2 = (P_* \otimes_Z Z_2) \oplus \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})$. *If* $M \in P_*$, *then* $M \in 2^v P_*$ *iff* $\mathrm{index}(\theta, M)$ *is divisible by* 2^v *for all* $\theta \in R(\mathrm{Spin}^c)$.

REMARK. (b) is a scholium to the theorem of Stong **[18, 19]** that all relations among characteristic numbers in $\Omega_*^{\mathrm{Spin}^c}/\text{torsion}$ are given by the index theorem. Let $\pi_Z(m) = \mathrm{Rank}_Z(\Omega_m^{\mathrm{Spin}^c}) = \mathrm{Rank}_Z(P_m)$ and let $\pi_2(m) = \mathrm{Rank}_2(\mathrm{Tor}(\Omega_m^{\mathrm{Spin}^c}))$; since all torsion in $\Omega_m^{\mathrm{Spin}^c}$ is of order 2, these numbers determine the additive structure. We computed them for $m \leq 59$ on a computer using the Anderson-Brown-Peterson **[1]** algorithm and list them below as follows:

m	$\pi_2(m)$	$\pi_Z(m)$	m	$\pi_2(m)$	$\pi_Z(m)$	m	$\pi_2(m)$	$\pi_Z(m)$
0	0	1	20	1	19	40	26	139
1	0	0	21	0	0	41	8	0
2	0	1	22	5	19	42	59	139
3	0	0	23	0	0	43	10	0
4	0	2	24	2	30	44	44	195
5	0	0	25	0	0	45	16	0
6	0	2	26	9	30	46	90	195
7	0	0	27	0	0	47	20	0
8	0	4	28	4	45	48	72	272
9	0	0	29	1	0	49	29	0
10	1	4	30	14	45	50	138	272
11	0	0	31	1	0	51	36	0
12	0	7	32	8	67	52	116	373
13	0	0	33	2	0	53	51	0
14	1	7	34	24	67	54	207	373
15	0	0	35	2	0	55	64	0
16	0	12	36	15	97	56	183	508
17	0	0	37	4	0	57	88	0
18	3	12	38	37	97	58	311	508
19	0	0	39	5	0	59	110	0

We shall need the following technical lemma about Stiefel-Whitney numbers later in the paper. We acknowledge with gratitude helpful suggestions by Peter Landweber about the proof. Let $\ker_*(SW) = \{M \in \Omega_*^{\mathrm{Spin}^c}: x(M) = 0\, \forall x \in W^*\}$.

LEMMA 1.3. *Let* $Q_* = Z[CP^2, CP^4, \ldots, CP^{2k}, \ldots]$ *so* $P_* = Q_*[CP^1]$. *Let* $M = M_1 + M_2$ *for* $M_1 \in Q_*$ *and* $M_2 \in \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})$. *If* $M \in \ker_*(SW)$, *then* $M_1 \in 2 \cdot Q_*$ *and* $M_2 = 0$.

Proof. We assemble the appropriate results from Stong's book (see **[19]** pages 42 and 352). Let $\overline{M}$ and $\overline{M}_i$ denote the corresponding elements reduced mod 2 in $\Omega_*^{\mathrm{Spin}^c} \otimes Z_2$. Let F: $\Omega_*^{\mathrm{Spin}^c} \to \Omega_*^{\mathrm{SO}}$, G: $\Omega_*^{\mathrm{SO}} \to \Omega_*^{\mathrm{O}}$, and p: $\Omega_*^{\mathrm{SO}} \to \Omega_*^{\mathrm{SO}}/$ torsion be the natural maps. The Stiefel-Whitney numbers completely detect unoriented bordism so $GF(M) = 0$. Since CP^1 bounds in Ω_*^{SO} and since pF is surjective, pF: $Q_* \otimes F_2 \to \{\Omega_*^{\mathrm{SO}}/$ torsion$\} \otimes Z_2$ is surjective by Theorem 1.2. Since $\Omega_*^{\mathrm{SO}}/$ torsion $= Z[y_4, y_8, \ldots]$ is a polynomial algebra, we see pF is an isomorphism by counting dimensions. With Z_2 coefficients, G: $\Omega_*^{\mathrm{SO}} \otimes Z_2 = \Omega_*^{\mathrm{SO}}/2\Omega_*^{\mathrm{SO}} \to \Omega_*^{\mathrm{O}}$ is injective. Since $GF(\overline{M}) = 0$, $F(\overline{M}) = 0$ so $pF(\overline{M}) = pF(\overline{M}_1) = 0$. Since pF is an isomorphism with coefficients in Z_2, this implies $\overline{M}_1 = 0$ so $M_1 \in 2 \cdot \Omega_m^{\mathrm{Spin}^c}$. Consequently $\mathrm{index}(\theta, M_1) \equiv 0(2) \forall \theta \in R(\mathrm{Spin}^c)$ so $M_1 \in 2 \cdot P_*$

and hence $M_1 \in 2Q_*$ by Theorem 1.2. Consequently $M_1 \in \ker_*(SW)$ so $M_2 \in \ker_*(SW)$ so $M_2 = 0$ which completes the proof.

Let BZ_n be the classifying space of Z_n. A Z_n-structure on M is a homotopy class of a map $M \to BZ_n$. This is equivalent to either a principal Z_n bundle over M or to a representation of $\pi_1(M)$ in Z_n. If N is a compact Spin^c manifold with boundary M and if the classifying map extends over N, then M bounds. Let $\Omega_m^{\mathrm{Spin}^c}(BZ_n)$ be the resulting bordism group and let $\Omega_*^{\mathrm{Spin}^c}(BZ_n)$ be the graded direct sum. Cartesian product makes $\Omega_*^{\mathrm{Spin}^c}(BZ_n)$ into an $\Omega_*^{\mathrm{Spin}^c}$ module. The forgetful functor induces an $\Omega_*^{\mathrm{Spin}^c}$ module morphism $\Omega_*^{\mathrm{Spin}^c}(BZ_n) \to \Omega_*^{\mathrm{Spin}^c}$. Since any Spin^c manifold admits a trivial Z_n structure, this map is surjective and splits. Let $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$ be the kernel of the forgetful functor; $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$ is an $\Omega_*^{\mathrm{Spin}^c}$ module and we split $\Omega_*^{\mathrm{Spin}^c}(BZ_n) = \tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n) \oplus \Omega_*^{\mathrm{Spin}^c}$.

Let $W^*(BZ_n) = W^* \otimes H^*(BZ_n;\ Z_2)$ be the algebra of Z_n equivariant Stiefel Whitney classes. If $x \in W^m(BZ_n)$ and if $M \in \tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)$, then there is a natural evaluation $x(M) \in Z_2$ obtained by cupping the Stiefel-Whitney classes of the tangent bundle with the cohomology classes of $H^*(BZ_n;\ Z_2)$ of the principal bundle and then evaluating on the fundamental class of M. This defines a pairing $W^m(BZ_n) \otimes \Omega_m^{\mathrm{Spin}^c}(BZ_n) \to Z_2$. We refer to Conner-Floyd [7] for further details.

Let $\rho_s(\lambda) = \lambda^s$ be the irreducible representations of Z_n where s is defined modulo n and let V_s be the complex line bundle corresponding to ρ_s. $V_{n/2}$ is a real bundle. Let $x_1 = w_1(V_{n/2}) \in H^1(BZ_n;\ Z_2)$ and let $x_2 \in H^2(BZ_n;\ Z_2)$ be the mod 2 reduction of $c_1(V_1)$. Let $a(m,n) = |\oplus_{j \geq 0} P_{m-2j-1} \otimes Z_n|$, let $b(m) = |\oplus_{j \geq 0} \mathrm{Tor}(\Omega_{m-2j-1}^{\mathrm{Spin}^c})|$, and let $c(m) = b(m-1) = |\oplus_{j \geq 0} \mathrm{Tor}(\Omega_{m-2j-2}^{\mathrm{Spin}^c})|$.

LEMMA 1.4. (a) $\tilde{H}_{2j}(BZ_n;\ Z) = 0$, $\tilde{H}_{2j-1}(BZ_n;\ Z) = Z_n$, *and* $\tilde{H}_j(BZ_n;\ Z_2) = Z_2$ *for* $j > 0$.

(b) $H^*(BZ_2;\ Z_2) = Z_2[x_1]$. *If* $n > 2$, *then* $H^*(BZ_n;\ Z_2) = Z_2[x_1, x_2]/\{x_1^2 = 0\}$.

(c) $|\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)| \leq a(m,n)b(m)c(m)$ *and* $\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)$ *is a finite* 2-*group*.

(d) *If* $M \in \tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$, *then*

$$M \times \Omega_*^{\mathrm{Spin}^c} = M \times P_* + M \times \mathrm{Tor}\left(\Omega_*^{\mathrm{Spin}^c}\right).$$

Proof. We refer to [6] for (a, b). $|\oplus_j \tilde{H}_j(BZ_n;\ \Omega_{m-j}^{\mathrm{Spin}^c})| = a(m,n)b(m)c(m)$ by (a) and Theorem 1.2. Since the $E_{p,q}^2$ term of the bordism spectral sequence is $\tilde{H}_p(BZ_m;\ \Omega_q^{\mathrm{Spin}^c})$ (see [7]), $\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)$ is a finite 2-group and $|\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)| \leq a(m,n)b(m)c(m)$ which proves (c); (d) now follows from Theorem 1.2 which completes the proof.

REMARK. In Lemma 4.1, we will show

$$|\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)| = a(m,n)b(m)c(m)$$

so the bordism spectral sequence degenerates.

2. The eta invariant. Let M be a smooth compact Riemannian manifold of dimension m without boundary and let D be a self-adjoint elliptic differential operator on M. If $\lambda \in R$, let $E(D,\lambda)$ denote the eigenspace of D corresponding to the eigenvalue λ. Let

$$\eta(s,D) = \frac{1}{2}\left\{\dim E(D,0) + \sum_{\lambda \neq 0} \dim E(D,\lambda) \cdot \mathrm{sign}(\lambda) \cdot |\lambda|^{-s}\right\}$$

be an analytic measure of the spectral asymmetry of D. The series converges to define a holomorphic function of s for $\mathrm{Re}(s) >> 0$; $\eta(s,D)$ has a meromorphic extension to C with isolated simple poles on the real axis. The value at $s = 0$ is regular and we let $\eta(D) = \eta(0,D) \in R/Z$. If D_t is a smooth 1-parameter family of such operators, although $\eta(0,D_t)$ has integer jumps as spectral values cross the origin, the mod Z reduction is smooth in t.

There is a general procedure for constructing such operators. Let N be a posibly non-compact manifold with boundary M. Choose the metric to be product near M. Let Q: $C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic first order complex over N. Use the geodesic flow to identify a neighborhood of M in N with $M \times [0,\varepsilon)$. Let $n \in [0,\varepsilon)$ be the distance to the boundary M. Using the symbol of dn, identify V_1 with V_2 on the collar and decompose $Q = \partial/\partial n + D$ where D is a first order elliptic tangential differential operator on $C^\infty(V_1|_M)$. For the classical elliptic complexes, D is selfadjoint and does not depend on the particular N chosen. If N is compact, the generalized index theorem of Atiyah et al **[4]** is

THEOREM 2.1. *With the notation above,*

$$\mathrm{index}(Q) = \int_N a_0(x,Q)\,dx - \{\eta(0,D) + \dim\ker(D)\}/2.$$

$a_0(x,Q)$ is a local invariant of Q and of the formal adjoint Q^* which vanishes if M is even dimensional. Index(Q) *is computed with respect to suitable non-local elliptic boundary conditions.*

Let M be an odd dimensional Z_n-Spin^c manifold. Choose a not necessarily compact even dimensional Spin^c manifold N so $\partial N = M$; for example we could take $N = M \times [0, \infty)$. Since N is even dimensional, the Spin^c complex is defined over N. Let Q be the operator of the Spin^c complex over N and let D be the tangential operator of the Spin^c complex over M. If $\theta \in R_0(Z_n) \otimes R(\text{Spin}^c)$, let $\theta(M)$ be the virtual bundle defined by θ over M and let D_θ be D with coefficients in $\theta(M)$. Let $\eta(\theta, M) = \eta(D_\theta)$ and set $\eta(\theta, M) = 0$ if M is even dimensional. Similarly, if M is an even dimensional Pin^c manifold and if $\theta \in R(\text{Spin}^c)$, let $\overline{D}_\theta$ be the tangential operator of the Pin^c complex over M with coefficients in $\theta(M)$ and let $\eta(\theta, M) = \eta(\overline{D}_\theta)$. Set $\eta(\theta, M) = 0$ if M is odd dimensional.

LEMMA 2.2. (a) *Let $M = M_1 \times M_2$, let Q be a first order elliptic complex over M_1, let R be a first order self-adjoint elliptic operator over M_2, and let*

$$P = \begin{pmatrix} R \otimes 1 & 1 \otimes Q^* \\ 1 \otimes Q & -R \otimes 1 \end{pmatrix} \quad \text{on } M.$$

Then $\eta(P)(M) = \text{index}(Q)(M_1) \cdot \eta(R)(M_2)$.

(b) $\eta\colon R_0(Z_n) \otimes R(\text{Spin}^c) \otimes \Omega_*^{\text{Spin}^c}(BZ_n) \to Q/Z$.

(c) $\eta\colon R(\text{Spin}^c) \otimes \Omega_*^{\text{Pin}^c} \to Q/Z$.

Proof. We refer to Gilkey [**11, 14**] for the proof of (a). To prove (b), we use Theorem 2.1. Suppose N is a compact Z_n-Spin^c manifold with boundary M and let $\theta \in R_0(Z_n) \otimes R(\text{Spin}^c)$. We extend the bundle $\theta(M)$ over N as follows. Let $r(\Lambda^k) = \Lambda^k + \Lambda^{k-1}$ and $r(\tau) = \tau$ define an $R(Z_n)$ module ring iromorphism of $R_0(Z_n) \otimes R(\text{Spin}^c)$. Choose Θ so $r(\Theta) = \theta$. Since $T(N)|_M = T(M) \oplus 1$, $\Theta(N)|_M = \theta(M)$ so this provides the desired extension. Let Q_Θ be the operator of the Spin^c complex over N with coefficients in $\Theta(N)$. The tangential operator of Q_Θ is D_θ. If $\Theta = \rho \otimes \psi$ for $\rho \in R_0(Z_n)$ and $\psi \in R(\text{Spin}^c)$, then Q_Θ is locally isomorphic to $\dim(\rho)$ copies of Q_ψ. Since a_0 is a local invariant, $a_0(x, Q_\Theta) = \dim(\rho) a_0(x, Q_\psi) = 0$ since $\dim(\rho) = 0$ for ρ in the augmentation ideal. In general Q_Θ is a sum of such operators. Since $a_0(x, -)$ is additive with respect to direct sums, the local term vanishes and Theorem 2.1 implies index $(Q_\Theta) = -\{\eta(0, D_\theta) + \dim\ker(D_\theta)\}/2 \in Z$. Therefore $\eta(\theta, M) = 0$ in R/Z so eta is a bordism invariant. If the Z_n structure on M is trivial, then $\theta(M) = 0$ so $\eta(\theta, M) = 0$; consequently we will often restrict eta to $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$ with no loss of information. Since $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$ is a torsion group by Lemma 1.4, eta takes values in Q/Z which proves (b).

The proof of (c) is similar. The representations of $R(\text{Spin}^c)$ extend to Pin^c so the operators in question are well defined. Let $\overline{Q}_\Theta$ be the operator of the Pin^c complex with coefficients in the bundle defined by Θ. Since M

is *even* dimensional, $a(x, \overline{Q}_\Theta) = 0$. The remainder of the argument is the same; we refer to Gilkey [**11**] for a proof $\eta(\theta, M) \in Z[1/2]/Z \subseteq Q/Z$ which proves (c).

We introduce the following notation for certain Dedekind sums which arise in evaluating the eta invariant. Let $C(Z_n)$ denote the space of complex class functions on Z_n. The map $\rho \to \mathrm{Tr}(\rho)$ embeds $R(Z_n)$ in $C(Z_n)$ with $R(Z_n) \otimes C = C(Z_n)$. If $f,\ g \in C(Z_n)$, let $(f, g) = \sum_{\lambda \in Z_n} f(\lambda) g(\lambda)/n$. If $\rho,\ \hat{\rho} \in R(Z_n)$, then $(\rho, \hat{\rho}) \in Z$ by the orthogonality relations. Let $\alpha(1) = 0$ and $\alpha(\lambda) = \lambda/(\lambda - 1)$ for $\lambda \neq 1$.

LEMMA 2.3. (a) $L^k(n) \in \tilde{\Omega}^{\mathrm{Spin}^c}_{2k-1}(BZ_n)$. *If* $\theta \in R_0(Z_n)$, *then* $\eta(\theta, L^k(n)) = (\theta, \alpha^k)$.

(b) $\tilde{\Omega}^{\mathrm{Spin}^c}_0(BZ_n) = 0$. $\eta(\rho_s - \rho_0, L^1(n)) = -s/n$. $L^1(n)$ *generates* $\tilde{\Omega}^{\mathrm{Spin}^c}_1(BZ_n) = Z_n$.

Proof. $T(L^k(n)) \oplus 1$ inherits a natural unitary structure. Since $L^k(n)$ is odd dimensional, it bounds in Ω^U_* and hence in $\Omega^{\mathrm{Spin}^c}_*$. Let $Z_n \to S^{2k-1} \to L^k(n)$ define a Z_n structure on $L^k(n)$ so $L^k(n) \in \tilde{\Omega}^{\mathrm{Spin}^c}_{2k-1}(BZ_n)$. Atiyah-Patodi-Singer [**4**, see II-(2.9)] calculate the eta invariant of the tangential operator of the Signature complex in terms of Dedekind sums; the same argument proves (a). $\tilde{\Omega}^{\mathrm{Spin}^c}_0(BZ_n) = 0$ by Lemma 1.4. If $\chi(1) = 0$ and $\chi(\lambda) = 1$ for $\lambda \neq 1$, then

$$\begin{aligned}\eta\big(\rho_s - \rho_0, L^1(n)\big) &= \big(\rho_1(\rho_s - \rho_0)/(\rho_1 - \rho_0), \chi\big) = \big(\rho_s + \cdots + \rho_1, \chi\big) \\ &= \big(\rho_s + \cdots + \rho_1, \rho_0\big) - s/n = -s/n \bmod Z\end{aligned}$$

by (a) and the orthogonality relations. Thus $L^1(n)$ is an element of order at least n in $\tilde{\Omega}^{\mathrm{Spin}^c}_1(BZ_n)$. Since $|\tilde{\Omega}^{\mathrm{Spin}^c}_1(BZ_n)| \leq n$ by Lemma 1.4, $L^1(n)$ generates $Z_n = \tilde{\Omega}^{\mathrm{Spin}^c}_1(BZ_n)$ which completes the proof.

Define

$$\begin{aligned}\ker_*(\eta, n) &= \big\{M \in \tilde{\Omega}^{\mathrm{Spin}^c}_*(BZ_n)\colon \eta(\theta, M) = 0\ \forall\theta \in R_0(Z_n) \otimes R(\mathrm{Spin}^c)\big\},\\ \ker_*(\eta) &= \big\{M \in \Omega^{\mathrm{Pin}^c}_*\colon \eta(\theta, M) = 0\ \forall\theta \in R(\mathrm{Spin}^c)\big\},\\ \ker_*(SW, n) &= \big\{M \in \Omega^{\mathrm{Spin}^c}_*(BZ_n)\colon x(M) = 0\ \forall x \in W^*(BZ_n)\big\}.\end{aligned}$$

LEMMA 2.4. *Let* $M \in L^1(n) \times \Omega^{\mathrm{Spin}^c}_{m-1}$.

(a) $M = L^1(n) \times (N_1 + N_2)$ *for* $N_1 \in P_{m-1}$ *and* $N_2 \in \mathrm{Tor}(\Omega^{\mathrm{Spin}^c}_{m-1})$.

(b) $M \in \ker_m(\eta, n)$ *iff* $N_1 \in n \cdot P_{m-1}$.

(c) $M \in \ker_m(SW, n)$ *iff* $N_2 = 0$ *and* $N_1 \in 2 \cdot Q_{m-1} + CP^1 \cdot P_{m-3}$.

(d) $|L^1(n)\times\Omega^{\text{Spin}^c}_{m-1}|=|\Omega^{\text{Spin}^c}_{m-1}\otimes Z_n|=a(m,n)b(m)/a(m-2,n)b(m-2)$.
(e) $L^1(n)\times\Omega^{\text{Spin}^c}_{m-1}\cap\ker_m(SW,n)\cap\ker_m(\eta,n)=0$.

Proof. If $M\in L^1(n)\times\Omega^{\text{Spin}^c}_{m-1}$, then M has this form by Lemma 1.4 which proves (a). Suppose $M\in\ker_m(\eta,n)$. If $\dim(M)$ is even, then $N_1=0$ so we suppose $\dim(M)$ odd. If $\psi\in R(\text{Spin}^c)$, choose Ψ so $r(\Psi)=\psi$. Since $T(L^1(n))=1$, $\Psi(L^1(n)\times N)=\Psi(1\oplus T(N))=\psi(N)$. Let Q be the operator of the Spin^c complex on N with coefficients in ψ, let D be the tangential operator of the Spin^c complex on $L^1(n)$ with coefficients in $\rho_1-\rho_0$, and let $\hat{D}$ be the tangential operator of the Spin^c complex on M with coefficients in $(\rho_1-\rho_0)\otimes\Psi$. Then $\hat{D}=\begin{pmatrix} D\otimes 1 & 1\otimes Q^* \\ 1\otimes Q & -D\otimes 1\end{pmatrix}$ so $\eta((\rho_1-\rho_0)\otimes\Psi,M)=\eta(\rho_1-\rho_0,L^1(n))\cdot\text{index}(\psi,N_1+N_2)=-\text{index}(\psi,N_1)/n$ by Lemmas 2.2 and 2.3 since $\text{index}(\psi,N_2)=0$. Consequently, $M\in\ker_m(\eta,n)$ iff index $(\psi,N_1)\equiv 0 \bmod n$ iff $N_1\in n\cdot P_{m-1}$ by Theorem 1.2 which proves (b). The only possibly non-zero equivariant Stiefel-Whitney numbers of $L^1(n)\times N$ are of the form $x_1\cdot y$ for $y\in W_{m-1}$. Since $x_1(L^1(n))=1$, $x_1\cdot y(L^1(n)\times N)=y(N)$. Decompose $N_1=X_1+CP^1\times X_2$ for $X_1\in Q_{m-1}$ and $X_2\in P_{m-3}$. Since CP^1 bounds a 3 ball in Ω_2^{SO}, $CP^1\times X_2\in\ker_{m-1}(SW)$ and $y(N)=y(X_1)+y(N_2)$. $M\in\ker_m(SW,n)$ iff $X_1+N_2\in\ker_{m-1}(SW)$ iff $X_1\in 2\cdot Q_*$ and $N_2=0$ by Lemma 1.3 which proves (c). Since $L^1(n)$ is an element of order n, $L^1(n)\times(N_1+N_2)=0$ iff $N_1\in n\cdot P_{m-1}$ and $N_2=0$ so $|L^1(n)\times\Omega^{\text{Spin}^c}_{m-1}|=n^{\pi_z(m-1)}\cdot 2^{\pi_2(m-1)}=|\Omega^{\text{Spin}^c}_{m-1}\otimes Z_n|$ which proves (d). (e) is a direct consequence of (b,c) which completes the proof.

Let $s(\tau)=\tau\otimes\tau$ and $s(\Lambda^k)=\Sigma_{i+j=k}\Lambda^i\otimes\Lambda^j$ define an $R(Z_n)$ module coproduct s: $R_0(Z_n)\otimes R(\text{Spin}^c)\to\{R_0(Z_n)\otimes R(\text{Spin}^c)\}\otimes R(\text{Spin}^c)$. Let $M\in\Omega^{\text{Spin}^c}_{\text{odd}}(BZ_n)$ and let $N\in\Omega^{\text{Spin}^c}_{\text{even}}$. If $s(\theta)=\Sigma_i a_i\otimes b_i$, then $\theta(M\times N)=\Sigma_i a_i(M)\otimes b_i(N)$. By Lemma 2.2, $\eta(\theta,M\times N)=\Sigma_i\eta(a_i,M)\cdot\text{index}(b_i,N)$. If N is a torsion class, then $\text{index}(b_i,N)=0$. This proves

LEMMA 2.5. $\tilde{\Omega}^{\text{Spin}^c}_{\text{odd}}(BZ_n)\times\text{Tor}(\Omega^{\text{Spin}^c}_{\text{even}})\subseteq\ker_*(\eta,n)$.

Embed $L^{k-1}(n)$ into $L^k(n)$ using the first $k-1$ coordinates. The complex normal bundle of the embedding is given by the representation ρ_1 so that complexification of the real normal bundle corresponds to $\rho_1+\rho_{-1}$. Let $t(\tau)=\rho_1\otimes\tau$ and $t(\Lambda^k)=\Lambda^{k-2}+(\rho_1+\rho_{-1})\otimes\Lambda^{k-1}+\Lambda^k$. Extend t to an $R(Z_n)$ module algebra isomorphism of $R_0(Z_n)\otimes R(\text{Spin}^c)$. If s is as above, then $(t\otimes 1)\cdot s=s\cdot t$. If $\theta\in R_0(Z_n)\otimes$

$R(\text{Spin}^c)$ and $N \in \Omega_*^{\text{Spin}^c}$, then

$$\theta(L^k(n)) \times N|_{L^{k-1}(n)\times N} = t(\theta)(L^{k-1}(n) \times N).$$

Let $\beta = (\rho_1 - \rho_0)/\rho_1 \in R_0(Z_n)$. We will use the following Lemma to discuss the Smith homomorphism later.

LEMMA 2.6. (a) *If* $\theta \in R_0(Z_n) \otimes R(\text{Spin}^c)$ *and if* $N \in \Omega_*^{\text{Spin}^c}$, *then* $\eta(\theta \cdot \beta, L^k(n) \times N) = \eta(t(\theta), L^{k-1}(n) \times N)$.

(b) *If* $\Sigma_k L^k(n) \times N_k \in \ker_*(\eta, n)$, *then* $\Sigma_k L^{k-1}(n) \times N_k \in \ker_*(\eta, n)$.

Proof. If $s(\theta) = \Sigma_i a_i \otimes b_i$, then $s(t(\theta)) = (t \otimes 1)s(\theta) = \Sigma_i t(a_i) \otimes b_i$ so

$$\eta(\theta \cdot \beta, L^k(n) \times N) = \sum_i \eta(a_i \cdot \beta, L^k(n)) \cdot \text{index}(b_i, N),$$

$$\eta(t(\theta), L^{k-1}(n) \times N) = \sum_i \eta(t(a_i), L^{k-1}(n)) \cdot \text{index}(b_i, N).$$

$\tau(L^k(n)) = \rho_k(L^k(n))$ is given by the representation theory. The Λ^k are the complexified exterior representations. Since $(T(L^k(n)) \oplus 1) \otimes C$ corresponds to $k \cdot \rho_1 + k \cdot \rho_{-1}$, $\Lambda^k + \Lambda^{k-1}$ corresponds to the $2k$th elementary symmetric function in the $k \cdot \rho_1$ and $k \cdot \rho_{-1}$. We solve this relation to express Λ^k in terms of the (virtual) representation theory. Choose $\phi_i \in R_0(Z_n)$ so that $\phi_i(L^k(n)) = a_i(L^k(n))$. Since $a_i(L^k(n))|_{L^{i-1}(n)} = t(a_i)(L^{k-1}(n))$, $\phi_i(L^{k-1}(n)) = t(a_i)(L^{i-1}(n))$. Since $\beta \cdot \alpha^k = \alpha^{k-1}$,

$$\eta(a_i \cdot \beta, L^k(n)) = \eta(\phi_i \cdot \beta, L^k(n)) = (\phi_i \cdot \beta, \alpha^k) = (\phi_i, \alpha^{k-1})$$
$$= \eta(\phi_i, L^{k-1}(n)) = \eta(t(a_i), L^{k-1}(n))$$

by Lemmas 2.2 and 2.3 which proves (a); (b) follows from (a) since t is surjective.

3. The Smith homomorphism. The classifying space BZ_n is the limit $\text{Lim}_{k\to\infty} L^k(n)$ with respect to the inclusions defined previously. If $M \in \tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$, let $f\colon M \to L^k(n)$ be the classifying map for k large. Make f transverse to $L^{k-1}(n)$ and let $\Delta(M) = f^{-1}(L^{k-1}(n))$. The normal bundle of $L^{k-1}(n)$ in $L^k(n)$ is given by the representation ρ_1 and has a natural Spin^c structure. We use Lemma 1.1 to give $\Delta(M)$ a Spin^c structure. Δ defines an $\Omega_*^{\text{Spin}^c}$ module morphism in bordism.

If $n = 2$, we split $\Delta = \delta_2 \cdot \delta_1$. Let $RP^k = S^k/Z_2$ be real projective space and let L be the non-trivial real line bundle over RP^k. We also use the notation L_{RP^k} to emphasize the base space is RP^k. Identify the

0-section with RP^k. If $x = 1 = w_1(L)$ generates $H^1(RP^k; Z_2) = Z_2$, then

$$w(k \cdot L) = (1 + x_1)^k = 1 + k \cdot x_1 + k(k-1)/2 \cdot x_1^2 + \cdots .$$

x_1^2 is the reduction mod 2 of $c_1(L \otimes C)$. The stable tangent bundle $T(RP^k) \oplus 1 = (k+1) \cdot L$ so by Lemma 1.1

$$RP^k \text{ admits a} \begin{Bmatrix} \text{Pin}^c \text{ structure and } \tau(RP^k) = L \otimes C & \text{if } k \equiv 0(4) \\ \text{Spin}^c \text{ structure and } \tau(RP^k) = L \otimes C & \text{if } k \equiv 1(4) \\ \text{Pin structure and } \tau(RP^k) = 1 & \text{if } k \equiv 2(4) \\ \text{Spin structure and } \tau(RP^k) = 1 & \text{if } k \equiv 3(4) \end{Bmatrix}$$

Embed S^{k-1} equivariantly in S^k using the first k coordinates to induce an embedding of RP^{k-1} in RP^k. The classifying space $BZ_2 = \text{Lim}_{k \to \infty} RP^k$. Let ∞ be the image of the north pole $(0, 0, \ldots, 0, 1)$ of S^k in RP^k. There is a diffeomorphism between $(RP^k - \infty, RP^{k-1})$ and $(L_{RP^{k-1}}, RP^{k-1})$ so the normal bundle of RP^{k-1} in RP^k is $L_{RP^{k-1}}$. $RP^k - RP^{k-1}$ is a contractable neighborhood of ∞ where L is trivial.

If $M \in \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2)$, let $f\colon M \to RP^k$ be the classifying map for k large. Make f transverse to RP^{k-1} and let $\delta_1(M) = f^{-1}(RP^{k-1})$. The normal bundle of $\delta_1(M)$ in M is $L_M = f^*(L)$. L_M has a Pin^c structure so $\delta_1(M)$ inherits a natural Pin^c structure by Lemma 1.1. $\delta_1\colon \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2) \to \Omega_{m-1}^{\text{Pin}^c}$ and $\delta_1(RP^{2k+1}) = RP^{2k}$. $L_M|_{\delta_1(M)}$ is the orientation line bundle of $\delta_1(M)$. Similarly if $M \in \Omega_{m-1}^{\text{Pin}^c}$, let $f\colon M \to RP^{k-1}$ classify the orientation line bundle L_M. Make f transverse to RP^{k-2} and let $\delta_2(M) = f^{-1}(RP^{k-2})$. Since the normal bundle of $\delta_2(M)$ in M is $L_M|_{\delta_2(M)}$, $\delta_2(M)$ has a natural Spin^c structure by Lemma 1.1. $\delta_2\colon \Omega_{m-1}^{\text{Pin}^c} \to \Omega_{m-2}^{\text{Spin}^c}(BZ_2)$ and $\delta_2(RP^{2k}) = RP^{2k-1}$. δ_1 and δ_2 are $\Omega_*^{\text{Spin}^c}$ morphisms and $\Delta = \delta_2 \cdot \delta_1$ if $n = 2$.

LEMMA 3.1 (*The Smith homomorphism*). (a) $\delta_1\colon \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2) \to \Omega_{m-1}^{\text{Pin}^c}$ *is an isomorphism.*

(b) *Let* $\nu\colon \Omega_*^{\text{Spin}^c} \to \Omega_*^{\text{Pin}^c}$ *be the forgetful functor. Then* δ_2 *defines a short exact sequence*

$$0 \to \nu\left(\Omega_{m-1}^{\text{Spin}^c}\right) \to \Omega_{m-1}^{\text{Pin}^c} \xrightarrow{\delta_2} \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2) \oplus \text{Tor}\left(\Omega_{m-2}^{\text{Spin}^c}\right) \to 0.$$

(c) *If* $\Delta = \delta_2 \cdot \delta_1$, *then* Δ *defines a short exact sequence*

$$0 \to RP^1 \times \Omega_{m-1}^{\text{Spin}^c} \to \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2) \xrightarrow{\Delta} \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2) \oplus \text{Tor}\left(\Omega_{m-2}^{\text{Spin}^c}\right) \to 0.$$

Proof. We adopt the following notational conventions. If V is a vector bundle, let $S(V)$ and $D(V)$ be the unit sphere and disk bundles. Let L_M be a real line bundle over some manifold M. Let O_a be a simple cover of M and let w_a be sections to the Z_2 bundle $S(L_M)$ over O_a. Let $w_a = g_{ab}w_b$ for $g_{ab} = \pm 1$ over $O_a \cap O_b$. Let $S(L_M \oplus 1)$ be a circle bundle over M. Let $\theta_a \in [0, 2\pi]$ be local angular parameters where we identify θ_a with $(\sin(\theta_a)w_a, \cos(\theta_a))$ in $S(L_M \oplus 1)$; $\theta_a = g_{ab}\theta_b$ on $O_a \cap O_b$. Parametrize the associated projective bundle $RP(L_M \oplus 1)$ by letting $\theta_a \in [0, \pi]$. We construct the inverse to δ_1 using some ideas of Korschorke and Stong **[16, 19]**. Let $M \in \Omega^{\mathrm{Pin}^c}_{m-1}$ and let L_M be the orientation line bundle. Let $Z_2 \to S(L_M \oplus 1) \to RP(L_M \oplus 1)$ define a Z_2 structure on $RP(L_M \oplus 1)$. Since $T(L_M \oplus 1) = T(M) \oplus L_M \oplus 1$, the orientation bundle of the manifold $L_M \oplus 1$ is $L_M \otimes L_M = 1$ so $L_M \oplus 1$ is orientable. Consequently $L_M \oplus 1$ has a Spin^c structure by Lemma 1.1. Give

$$S(L_M \oplus 1) = \partial D(L_M \oplus 1)$$

the bounding Spin^c structure. The map $\theta_a \to 2\theta_a$ is a diffeomorphism from $RP(L_M \oplus 1)$ to $S(L_M \oplus 1)$ which we use to define a Spin^c structure on $RP(L_M \oplus 1)$. Let $\alpha_1(M) = RP(L_M \oplus 1)$ define an $\Omega^{\mathrm{Spin}^c}_*$ module morphism from $\Omega^{\mathrm{Pin}^c}_{m-1}$ to $\tilde{\Omega}^{\mathrm{Spin}^c}_m(BZ_2)$. We compute $\delta_1 \cdot \alpha_1$ as follows. Let $f\colon M \to RP^{k-1}$ be the classifying map for the line bundle L_M. f induces a Z_2 equivariant map $f\colon S(L_M) \to S^{k-1}$. Extend $f\colon L_M \to R^k$ to be fiber linear. Let $F(x, t) = (f(x), t)$ be a Z_2 equivariant map from $S(L_M \oplus 1)$ to S^k. This descends to a map $F\colon RP(L_M \oplus 1) \to RP^k$ which is the classifying map for the Z_2 structure. F is transverse to RP^{k-1} and $F^{-1}(RP^{k-1}) = M$ corresponds to the embedding of M as $(0, -1)$ in $S(L_M \oplus 1) \subseteq L_M \oplus 1$. Since this is homotopic to the embedding of M as the zero section of $L_M \oplus 1$, the induced Pin^c structure agrees with the original Pin^c structure on M so $\delta_1 \cdot \alpha_1(M) = M$. This shows δ_1 is surjective. We will show $\alpha_1 \cdot \delta_1 = \mathrm{id}$ by showing δ_1 is an isomorphism presently.

Since $\delta_1(RP^1) = RP^0$, $\delta_1(RP^1 \times \Omega^{\mathrm{Spin}^c}_{m-1}) = \nu(\Omega^{\mathrm{Spin}^c}_{m-1})$. Conversely, if $N \in \Omega^{\mathrm{Spin}^c}_{m-1}$, then the orientation line bundle L_N is trivial so $\alpha_1(N) = RP^1 \times N$. Thus δ_1 and α_1 provide isomorphisms between $RP^1 \times \Omega^{\mathrm{Spin}^c}_{m-1}$ and $\nu(\Omega^{\mathrm{Spin}^c}_{m-1})$. Since δ_1 is surjective and since $\tilde{\Omega}^{\mathrm{Spin}^c}_*(BZ_2)$ is a 2-group by Lemma 1.4, $\Omega^{\mathrm{Pin}^c}_*$ is a torsion group so

$$\delta_2\left(\Omega^{\mathrm{Pin}^c}_{m-1}\right) \subseteq \mathrm{Tor}\left(\Omega^{\mathrm{Spin}^c}_{m-2}(BZ_2)\right) = \tilde{\Omega}^{\mathrm{Spin}^c}_{m-2}(BZ_2) \oplus \mathrm{Tor}\left(\Omega^{\mathrm{Spin}^c}_{m-2}\right).$$

Conversely, let $N \in \tilde{\Omega}^{\mathrm{Spin}^c}_{m-2}(BZ_2) \oplus \mathrm{Tor}(\Omega^{\mathrm{Spin}^c}_{m-2})$. Then $2 \cdot N = 0$ in $\Omega^{\mathrm{Spin}^c}_{m-2}$. Let L_N be the real line bundle over N corresponding to the Z_2 structure so $Z_2 \to S(L_N) \to N$. The Stiefel-Whitney numbers and Chern/Pontrjagin numbers are multiplicative under finite coverings so $S(L_N) = 2 \cdot N = 0$ in $\Omega^{\mathrm{Spin}^c}_{m-2}$ by Theorem 1.2. Let U be a $m - 1$ dimensional compact Spin^c

manifold with boundary $S(L_N)$. Use the outward normal to orient $D(L_N) - N$. Let $M = D(L_N) \cup U$ along the common boundary $S(L_N)$ with the natural Pin^c structure. If L_M is the orientation bundle of M, then L_M is the pull back of L_N over $D(L_N)$ and L_M is trivial over $M - N$. Let $f\colon N \to RP^{k-1}$ be the classifying map of L_N. Extend f as a fiber linear map $f\colon (D(L_N), N) \to (L_{RP^{k-1}}, RP^{k-1})$. Since $RP^k - RP^{k-1}$ is contractable, we can change f so $f^{-1}(RP^{k-1}) = N$ and $f(x) = \infty$ near the boundary $S(L)$. Extend f to M so $f(U) = \infty$. Since $f^{-1}(RP^{k-1}) = N$ with the given Spin^c structure, $\delta_2(M) = N$. This shows image$(\delta_2) = \tilde{\Omega}^{\mathrm{Spin}^c}_{m-2}(BZ_2) \oplus \mathrm{Tor}(\Omega^{\mathrm{Spin}^c}_{m-2})$.

If $N \in \nu(\Omega^{\mathrm{Spin}^c}_{m-1})$, then $L_N = 1$ so we may take $f(N) = \infty$ and $\delta_2(N) = 0$. Therefore $\nu(\Omega^{\mathrm{Spin}^c}_{m-1}) \subseteq \ker(\delta_2)$. Since $|\nu(\Omega^{\mathrm{Spin}^c}_{m-1})| = |RP^1 \times \Omega^{\mathrm{Spin}^c}_{m-1}| = |\Omega^{\mathrm{Spin}^c}_{m-1} \otimes Z_2|$ by Lemma 2.4, we may estimate

$$\left|\tilde{\Omega}^{\mathrm{Spin}^c}_m(BZ_2)\right| \geq \left|\Omega^{\mathrm{Pin}^c}_{m-1}\right| \geq \left|\tilde{\Omega}^{\mathrm{Spin}^c}_{m-2}(BZ_2)\right| \cdot \left|\mathrm{Tor}\left(\Omega^{\mathrm{Spin}^c}_{m-2}\right)\right| \cdot \left|\nu\left(\Omega^{\mathrm{Spin}^c}_{m-1}\right)\right|$$

$$\geq \Pi_{j\geq 0}\left|\mathrm{Tor}\left(\Omega^{\mathrm{Spin}^c}_{m-2j-2}\right)\right| \cdot \left|\Omega^{\mathrm{Spin}^c}_{m+1-2j} \otimes Z_2\right| = a(m,2)b(m)c(m).$$

By Lemma 1.4, $|\tilde{\Omega}^{\mathrm{Spin}^c}_m(BZ_2)| \leq a(m,2)b(m)c(m)$ so all the inequalities are equalities. This shows δ_1 is an isomorphism and $\ker(\delta_2) = \nu(\Omega^{\mathrm{Spin}^c}_{m-1})$. This proves (a,b); we combine (a) and (b) to prove (c).

Since $\ker(\delta_2) = \delta_1(RP^1 \times \Omega^{\mathrm{Spin}^c}_{m-1})$ consists of elements of order 2, $2\delta_2^{-1}$ is well defined. Let $M \in$ image(δ_2) and let L_M be the real line bundle over M corresponding to the Z_2 structure. Since $T(L_M \oplus 1) = T(M) \oplus L_M \oplus 1$, $D(L_M \oplus 1)$ has a natural Pin^c structure and $S(L_M \oplus 1)$ inherits a Pin^c structure that bounds. This is not the structure we wish to use. Define a complex line bundle H over $S(L_M \oplus 1)$ with transition functions $h_{ab} = e^{i(\theta_a - \theta_b)/2}$; $H^2 = 1$. Inequivalent Pin^c structures are parametrized by complex line bundles; we twist the bounding Pin^c structure on $S(L_M \oplus 1)$ by H to define a new Pin^c structure on $S(L_M \oplus 1)$; we denote this by $\alpha_2(M)$ and the map $M \to \alpha_2(M)$ extends as an $\Omega^{\mathrm{Spin}^c}_*$ module morphism in bordism.

A more geometric description of this structure can be given as follows. Let $S_\pm = \{(z,t) \in S(L_M \oplus 1)\colon\ t \geq 0 \text{ or } t \leq 0\}$ be the closed upper/lower hemispheres. Let $D_\pm$ be two copies of the unit disk bundle $D(L_M)$. Use the diffeomorphism $d_\pm \to (\pm d_\pm, \pm(1 - |d_\pm|^2)^{1/2}$ to identify $D_\pm$ with $S_\pm$ where the zero sections $M_\pm$ go to the north and south poles respectively. This decomposes $S(L_M \oplus 1) = D_+ \cup D_-$ where the glueing is the antipodal map -1 on the boundary $S(L_M)$. The antipodal map on $S(L_M \oplus 1)$ becomes the shift χ which interchanges D_+ and D_-. Give $D_\pm - M_\pm$ opposite orientations to define a Spin^c structure on $S(L_M \oplus 1) - M_+ - M_-$ and a Pin^c structure on $S(L_M \oplus 1)$. χ is a Pin^c involution reversing the orientation of $S(L_M) - M_+ - M_-$.

LEMMA 3.2. *If* $M \in \tilde{\Omega}^{\mathrm{Spin}^c}_{m-2}(BZ_2)$, *then* $2\delta_2^{-1}(M) = \alpha_2(M)$.

Proof. Let U be an $m - 1$ dimensional Spin^c manifold with boundary $\partial U = S(L_M)$. Let $U_\pm$ be two copies of U. Identify $\partial U_\pm \times (0,1]$ with $S_\pm(L_M) \times (0,1]$ in $D_\pm(L_M) - M_\pm$. Glue $\partial U_\pm \times [.2,.8]$ to $D_\pm(L_M) \times \{1\}$ in $S(L_M \oplus 1) \times \{1\}$ to construct the bordism

$$S(L_M \oplus 1) \times [0,1] \cup U_+ \times [.2,.8] \cup U_- \times [.2,.8].$$

This admits a Pin^c structure and is orientable off the core $M_\pm \times [0,1]$ using the natural orientation for $U_+ \times [.2,.8]$ and the reversed orientation for $U_- \times [.2,.8]$. The boundary of this bordism consists of four pieces. The first is $\alpha_2(M) = S(L_M \oplus 1) \times \{0\}$. The second and third pieces are $D_{\pm .2}(L_M) \cup U_\pm \times \{.2\}$ which yields $2\delta_2^{-1}(M)$ as was shown in the proof of Lemma 3.1. The final piece is an error term

$$E = \left(S(L_M \oplus 1) - \mathrm{int}(D_{+,.8}(L_M)) - \mathrm{int}(D_{-,.8}(L_M))\right)$$
$$\cup U_+ \times \{.8\} \cup U_- \times \{.8\}.$$

Since $E \cap M_\pm \times [0,1] = \varnothing$, E is orientable. Let the shift χ induce an action of Z_2 on E and let $\hat{E} = E/Z_2$ with the induced Pin^c structure. Let $\hat{L}$ be the orientation line bundle of $\hat{E}$. Since χ is orientation reversing, $S(\hat{L}) = E$. $T(\hat{L}) = \hat{L} \oplus T(\hat{E})$ so $\hat{T}(\hat{L})$ is orientable and has a Spin^c structure. Since E is the boundary of $D(\hat{L})$, it is zero in $\Omega_*^{\mathrm{Pin}^c}$. This completes the proof.

We now study the behavior of the eta invariant with respect to the δ_i. Let $M \in \Omega_m^{\mathrm{Spin}^c}(BZ_2)$ and let $N \in \Omega_{m-1}^{\mathrm{Pin}^c}$. Give N the Z_2 structure corresponding to the orientation class so $\rho_1(N) = L_N$. This extends η: $R(Z_2) \otimes R(\mathrm{Spin}^c) \otimes \Omega_*^{\mathrm{Pin}^c} \to Q/Z$. It is an easy exercise using Clifford algebras to show $\tau(\delta_1(M)) = \tau(M)|_{\delta_1(M)}$ and $\tau(\delta_2(N)) = \tau(N)|_{\delta_2(N)} \otimes L_N$; we omit details in the interests of brevity. Let $s_1(\Lambda^k) = 1 \otimes \Lambda^k + \rho_1 \otimes \Lambda^{k-1}$, $s_1(\tau) = \tau$, $s_2(\Lambda^k) = 1 \otimes \Lambda^k + \rho_1 \otimes \Lambda^{k-1}$, and $s_2(\tau) = \rho_1 \otimes \tau$ define $R(Z_2)$ module algebra isomorphisms s_i of $R(Z_2) \otimes R(\mathrm{Spin}^c)$. If $\theta \in R(Z_2) \otimes R(\mathrm{Spin}^c)$, then $\theta(M)|_{\delta_1(M)} = s_1(\theta)(\delta_1(M))$ and $\theta(N)|_{\delta_2(N)} = s_2(\theta)(\delta_2(N))$.

LEMMA 3.3. *Let* $\psi \in R(Z_2) \otimes R(\mathrm{Spin}^c)$, $M_1 \in \tilde{\Omega}^{\mathrm{Spin}^c}_{\mathrm{odd}}(BZ_2)$, *and* $M_2 \in \Omega^{\mathrm{Pin}^c}_{\mathrm{even}}$.

(a) $\eta((\rho_0 - \rho_1) \otimes \psi,\ M_1) = \eta(s_1(\psi), \delta_1(M_1))$. $M_1 \in \ker_*(\eta, 2)$ *iff* $\delta_1(M_1) \in \ker_*(\eta)$.

(b) $2\eta(\psi, M_2) = \eta((\rho_0 - \rho_1)\cdot s_2(\psi), \delta_2(M_2))$. *If* $2\cdot M_2 \in \ker_*(\eta)$, *then* $\delta_2(M_2) \in \ker_*(\eta, 2) \oplus \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})$.

(c) *If* $2M_1 \in \ker_*(\eta, 2)$, *then* $\Delta(M_1) \in \ker_*(\eta, 2) \oplus \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})$.

Proof. Since $\alpha_1 \cdot \delta_1 = \mathrm{id}$, we may assume $M_1 = RP(L_N \oplus 1)$. The parity involved plays an important role so we suppose first $s_1(\psi) \in R(\mathrm{Spin}^c)$. Let $X = \psi(M_1)$ so $X|_N = s_1(\psi)(N)$. We study the eta invariant on $RP(L_N \oplus 1)$ by working equivariantly over $S(L_N \oplus 1)$. Let $\bar{D}$: $C^\infty(V) \to C^\infty(V)$ be the tangential operator of the Pin^c complex over N with coefficients in X. Let β_a denote normalized Clifford multiplication by the local orientations ω_a; β_a defines an automorphism of V which anti-commutes with $\bar{D}$, see Gilkey **[11, 14]**. The tangential operator of the Spin^c complex with coefficients in X over $S(L_N \oplus 1)$ is $D = \bar{D} + \beta_a \cdot \partial/\partial\theta_a$: $C^\infty(V) \to C^\infty(V)$. Let $\{\lambda_k, f_k\}$ be a spectral resolution of $\bar{D}$. Since cos(sin) are even(odd) functions, $\cos(n\theta_a)\cdot f_k$ and $\sin(n\theta_a)\beta_a f_k$ are well defined sections to V over $S(L_N \oplus 1)$. If n is even(odd), these define sections to $V(V \otimes L_{M_1})$ over $RP(L_N \oplus 1)$. These functions form a complete orthogonal system for $L^2(V)$ over $S(L_N \oplus 1)$ if we omit the $\sin(n\theta)$ terms for $n = 0$. If $n > 0$, then

$$D(a\cos(n\theta)f_k + b\sin(n\theta)\beta f_k) = (\lambda_k a - nb)\cos(n\theta)f_k + (-na - \lambda_k b)\sin(n\theta)\beta f_k$$

so D is given by the matrix $\begin{pmatrix} \lambda_k & -n \\ -n & -\lambda_k \end{pmatrix}$ on this subspace. This matrix has two unequal non-zero eigenvalues $\pm(\lambda_k^2 + n^2)^{1/2}$ which cancel in pairs and contribute nothing to the eta invariant. If $n = 0$, then f_k is an eigenvector corresponding to the eigenvalue λ_k. Consequently $\eta(D) = \eta(\bar{D})$ and $\eta(D_{\rho_1}) = 0$. This shows $\eta(\bar{D}) = \eta(\rho_0 - \rho_1, D)$ and establishes the formula in this case. If $s_1(\psi) \in \rho_1 \cdot R(\mathrm{Spin}^c)$, then the parities involved are reversed owing to the twisting of the line bundle L_N. We use Gilkey **[11]** and the case previously considered to compute

$$\begin{aligned}\eta(s_1(\psi), N) &= -\eta(L_N \cdot s_1(\psi), N) = -\eta(s_1(\rho_1 \cdot \psi), N) \\ &= -\eta((\rho_0 - \rho_1)\cdot \rho_1 \cdot \psi, M) = \eta((\rho_0 - \rho_1)\cdot\psi, M)\end{aligned}$$

which establishes the formula in general. If $N \in \ker_*(\eta)$, then $\eta(\theta, N) = 0 \forall \theta \in R(Z_2) \otimes R(\mathrm{Spin}^c)$ since $\eta(\rho_1 \otimes \theta, N) = -\eta(\theta, N)$. Consequently $M \in \ker_*(\eta, 2)$ by the formula. Conversely, $1 \otimes R(\mathrm{Spin}^c) \subseteq \mathrm{image}(s_1)$ so $M \in \ker_*(\eta, 2)$ implies $N \in \ker_*(\eta)$. This proves (a).

We use Lemma 3.2 to prove (b). If $N = \delta_2(M_2)$, then $2M_2 = S(L_N \oplus 1)$. Let $X = \psi(S(L_N \oplus 1))$ so $X|_N = s_2(\psi)(N)$. Let D be the tangential operator of the Spin^c complex over N with coefficients in X and let D_1 be D with coefficients in L_N. The pull back of L_N is the orientation bundle of $S(L_N \oplus 1)$. Normalized Clifford multiplication by local sections ω_a of L_N define isomorphisms β_a: $1 \to L_N$ and β_a: $L_N \to 1$ which intertwine D and D_1. Let

$$\overline{D} = \begin{pmatrix} D \otimes 1 & 1 \otimes \beta\partial/\partial\theta \\ 1 \otimes \beta\partial/\partial\theta & -D_1 \otimes 1 \end{pmatrix}$$

be the tangential operator of the Pin^c complex over $S(L_N \oplus 1)$ with coefficients in $s_2(\psi)$. If $\{f_k, \lambda_k\}$ and $\{g_k, \mu_k\}$ is a spectral resolution of D and D_1, then the collection $\{(\cos(n\theta)f_k, 0),\ (0, \beta\sin(n\theta)f_k),\ (\beta\sin(n\theta)g_k, 0),\ (0, \cos(n\theta)g_k)\}$ is a complete orthogonal system for $L^2(V \oplus (V \otimes L_N))$ over $S(L_N \oplus 1)$ if we omit the terms in $\sin(n\theta)$ for $n = 0$. If $n \neq 0$,

$$\begin{aligned} &\overline{D}(a\cos(n\theta)f_k \oplus b\sin(n\theta)\beta f_k) \\ &\quad = (a\lambda_k - nb)\cos(n\theta)f_k \oplus (-na - \lambda_k b)\sin(n\theta)\beta f_k \\ &\overline{D}(a\sin(n\theta)\beta g_k \oplus b\cos(n\theta)g_k) \\ &\quad = (a\mu_k - nb)\sin(n\theta)\beta f_k \oplus (-nb - \mu_k b)\cos(n\theta)g_k \end{aligned}$$

so $\overline{D}$ is given by a matrix $\left(\begin{smallmatrix} \lambda_k & -n \\ -n & -\lambda_k \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \mu_k & -n \\ -n & -\mu_k \end{smallmatrix}\right)$. These matrices have two unequal non-zero eigenvalues $\pm(\lambda_k^2 + n^2)^{1/2}$ or $\pm(\mu_k^2 + n^2)^{1/2}$ which cancel in pairs and make no contribution to the η invariant. If $n = 0$, only the eigenvalues of D and $-D_1$ contribute. Since $2M_2 = S(L_N \oplus 1)$,

$$\begin{aligned} 2\eta(\psi, M_2) &= \eta(\overline{D})(S(L_N \oplus 1)) = \eta(D) - \eta(D_1) \\ &= \eta((\rho_0 - \rho_1) \cdot s_2(\psi), N). \end{aligned}$$

If $2 \cdot M_2 \in \ker_*(\eta, 2)$, then $\eta(\theta, N) = 0 \forall\theta$ since s_2 is an isomorphism. Let $N = N_1 + N_2$ for $N_1 \in \tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_2)$ and $N_2 \in \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})$. Since $\eta(\theta, N_2) = 0$, $N_1 \in \ker_*(\eta, 2)$ which completes the proof of (b). We combine (a, b) to prove (c).

It is easier to study the behavior of the Stiefel-Whitney numbers with respect to the δ_i. Let $M \in \Omega_m^0(BZ_2)$ and let f: $M \to RP^k$ be the classifying map. Make f transverse to RP^{k-1} and let $\delta(M) = f^{-1}(RP^{k-1})$. Let $s(w_k) = w_k \otimes 1 + w_{k-1} \otimes x_1$ define an $H^*(BZ_2;\ Z_2)$ module algebra isomorphism of $W^*(BZ_2)$. Since $T(M)|_{\delta(M)} = T(\delta(M)) \oplus L_M$, the cohomology classes $x(M)|_{\delta(M)}$ and $s(x)(\delta(M))$ agree.

LEMMA 3.4. (a) *If* $M \in \Omega_m^0(BZ_2)$ *and if* $x \in W^{m-1}(BZ_2)$, *then* $(x_1(L_M) \cdot x)(M) = s(x)(\delta(M))$.

(b) $M \in \ker_m(SW, 2)$ *iff* $\delta_1(M) \in \ker_{m-1}(SW)$.

(c) *If* $N \in \Omega_{m-1}^{\mathrm{Pin}^c} \cap \ker_{m-1}(SW)$, *then* $\delta_2(N) \in \ker_{m-2}(SW, 2) \subseteq \tilde{\Omega}_{m-2}^{\mathrm{Spin}^c}(BZ_2)$.

(d) *If* $M \in \ker_m(SW, n)$, *then* $\Delta(M) \in \ker_{m-2}(SW, n) \subseteq \tilde{\Omega}_{m-2}^{\mathrm{Spin}^c}(BZ_n)$.

Proof. Let i: $\delta_1(M) \to M$ be the inclusion and let $\theta = x(L_M) \in H^{m-1}(M;\ Z_2)$. Since $[M]$ is the Poincaré dual of $w_1(L_M)$, $i_*[\delta_1(M)] = w_1(L_M) \cap [M]$ and hence

$$\theta w_1(L_M)(M) = \langle \theta \cup w_1(L_m), [M] \rangle = \langle \theta, w_1(L_M) \cap [M] \rangle$$
$$= \langle \theta, i_*[\Delta(M)] \rangle = \langle i^*(\theta), [\Delta(M)] \rangle.$$

This proves (a) since $i^*(\theta) = s(x)(\delta(M))$. To prove (b, c), let $M \in \tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_2)$ and let $N = \delta_1(M)$. Suppose $M \in \ker_m(SW, n)$. If $x \in W^{m-1}$, choose $y \in W^{m-1}(BZ_2)$ so $s(y) = x$. Then $0 = (x_1 \cdot y)(M) = x(N)$ so $N \in \ker_{m-1}(SW)$. Suppose $N \in \ker_{m-1}(SW)$. Let the orientation line bundle give a Z_2 structure to N. The equivariant Stiefel-Whitney numbers of N can be computed in terms of the ordinary Stiefel-Whitney numbers of N so all the equivariant Stiefel-Whitney numbers of N vanish. If $x \in W^m$, then $x(M) = 0$ since $M = 0$ in $\Omega_*^{\mathrm{Spin}^c}$. We therefore suppose $x = x_1 \cdot y$ for $y \in W^{m-1}(BZ_2)$. Then $0 = y(N) = (x_1 \cdot y)(M)$ so $M \in \ker_m(SW, n)$ which proves (b). If $y \in W^{m-2}(BZ_2)$, choose x so $s(x) = y$. Then $y(\delta_2(N)) = (x_1 \cdot x)(N) = 0$. Decompose $\delta_2(N) = X_1 + X_2$ for $X_1 \in \tilde{\Omega}_{m-2}^{\mathrm{Spin}^c}(BZ_2)$ and $X_2 \in \mathrm{Tor}(\Omega_{m-2}^{\mathrm{Spin}^c})$. Since X_1 bounds, $X_2 \in \ker_{m-2}(SW)$ so $X_2 = 0$. This shows $\delta_2(N) \in \ker_{m-2}(SW, 2)$ and proves (c). The proof of (d) is the same as that of (c) and is omitted in the interests of brevity. This completes the proof of the Lemma.

4. $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$ and $\Omega_*^{\mathrm{Pin}^c}$. Let $\pi_2(m)$: $\Omega_m^{\mathrm{Spin}^c}(BZ_2) \to \mathrm{Tor}(\Omega_m^{\mathrm{Spin}^c})$ be the natural projection and let $\pi = \oplus_{k>0}\pi_2(m-2k)\Delta^k$. Let $R_*(n)$ be the $\Omega_*^{\mathrm{Spin}^c}$ submodule of $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$ generated by the $L^k(n)$ and let $E(m) = \oplus_{k \geq 0}\mathrm{Tor}(\Omega_{m-2k-2}^{\mathrm{Spin}^c})$.

LEMMA 4.1. (a) $0 \to R_m(n) \to \tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n) \xrightarrow{\pi} E(m) \to 0$ *is exact.*

(b) $0 \to L^1(n) \times \Omega_{m-1}^{\mathrm{Spin}^c} \to R_m(n) \xrightarrow{\Delta} R_{m-2}(n) \to 0$ *is exact.*

(c) $|R_m(n)| = a(m,n)b(m)$ *and* $|\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)| = a(m,n)b(m)c(m)$.

(d) $\ker_m(SW, n) \cap \ker_m(\eta, n) = 0$.

REMARK. (d) completes the proof of Theorem 0.1(a); we apply δ_1 to (d) and use the results of the third section to complete the proof of Theorem 0.2(a).

Proof. If $n = 2$, image $(\Delta) = \tilde{\Omega}_{m-2}^{\mathrm{Spin}^c}(BZ_2) \oplus \mathrm{Tor}(\Omega_{m-2}^{\mathrm{Spin}^c})$ by Lemma 3.1 so π is surjective. The inclusion $Z_2 \to Z_n$ induces an $\Omega_*^{\mathrm{Spin}^c}$ module morphism called induction i: $\Omega_*^{\mathrm{Spin}^c}(BZ_2) \to \Omega_*^{\mathrm{Spin}^c}(BZ_n)$. Let p: $S^{2k-1}/Z_2 \to S^{2k-1}/Z_n$ be the natural covering projection; p is compatible with the inclusions defining BZ_2 and BZ_n. If $M \in \Omega_*^{\mathrm{Spin}^c}(BZ_2)$ and if f: $M \to RP^{2k-1}$ is the classifying map, then pf is the classifying map for $i(M)$ so

$i \cdot \Delta = \Delta \cdot i$. Since $i \cdot \pi_2(m) = \pi_2(m) \cdot i$, $i \cdot \pi = \pi \cdot i$ and π is surjective for all n. Since $\Delta(L^k(n)) = L^{k-1}(n)$ and since Δ is an $\Omega_*^{\mathrm{Spin}^c}$ module morphism, $\Delta\colon R_m(n) \to R_{m-2}(n) \to 0$. Since $R_{m-2}(n) \subseteq \tilde{\Omega}_{m-2}^{\mathrm{Spin}^c}(BZ_n)$, $R_m(n) \subseteq \ker(\pi)$. Since $\Delta(L^1(n)) = 0$, $L^1(n) \times \Omega_{m-1}^{\mathrm{Spin}^c} \subseteq \ker(\Delta)$. We use Lemma 2.4 to estimate:

$$\begin{aligned} |R_m(n)| &= \prod_{j \geq 0} |\ker(\Delta) \cap R_{m-2j}(n)| \\ &\geq \prod_{j \geq 0} \left| L^1(n) \times \Omega_{m-2j-1}^{\mathrm{Spin}^c} \right| = a(m,n)b(m), \end{aligned}$$

$$\begin{aligned} |\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)| &= |\ker(\pi)| \cdot |E(m)| \\ &\geq |R_m(n)| \cdot |E(m)| \geq a(m,n)b(m)c(m). \end{aligned}$$

Since $|\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n)| \leq a(m,n)b(m)c(m)$ by Lemma 1.4, all the inequalities must have been equalities. Thus $\ker(\Delta) \cap R_m(n) = L^1(n) \times \Omega_{m-1}^{\mathrm{Spin}^c}$ and $\ker(\pi) = R_*(n)$. This completes the proof of (a, b, c).

We prove (d) by induction on m; it is immediate if $m < 0$. Let $M \in \ker_m(SW, n) \cap \ker_m(\eta, n)$. By Lemma 3.4, $\Delta^k(M) \in \ker_{m-2k}(SW, n) \subseteq \tilde{\Omega}_{m-2k}^{\mathrm{Spin}^c}(BZ_n)$. Thus $\pi(M) = 0$ and $M \in R_m(n)$. We therefore apply Lemma 2.6 to see $\Delta(M) \in \ker(\eta, n)$ so $\Delta(M) = 0$ by induction. Since $M \in \ker(\Delta) \cap R_m(n)$, $M \in L^1(n) \times \Omega_{m-1}^{\mathrm{Spin}^c}$. We use Lemma 2.4 to conclude $M = 0$ which completes the proof.

Let $\vec{q} = (q_1, \ldots, q_k)$ be a collection of odd integers. Let $\rho(\vec{q}) = \mathrm{diag}(\lambda^{q_1}, \ldots, \lambda^{q_k})$ be a fixed point free representation of Z_n in $\mathrm{U}(k)$. Let $L(n;\ \vec{q}) = S^{2k-1}/\rho(\vec{q})(Z_n)$ be a generalized lens space. $L(n;\ \vec{q}) \in \tilde{\Omega}_{2k-1}^{\mathrm{Spin}^c}(BZ_n)$ and $L^k(n) = L(n; 1, \ldots, 1)$. Let

$$M_p(k+1) = L(n; -1, \ldots, -1, 1, \ldots, 1)$$

for $q_1 = \cdots = q_p = -1$ and $q_{p+1} = \cdots = q_{k+1} = 1$.

LEMMA 4.2. *$\sum_{p=0}^{k} \binom{p}{k} M_p(k+1) = L^1(n) \times (CP^1)^k$ in $\tilde{\Omega}_{2k+1}^{\mathrm{Spin}^c}(BZ_n)$.*

Proof We use the circle trick of Conner-Floyd [7]. If $\lambda \in Z_n$, let $T_1(\lambda)z = \lambda z$ and let $T_2(\lambda)z = z$ for $z \in S^1$, $z \in D^2$, or $z \in C \cup \infty = CP^1$. Let

$$N_1 = S^1 \times (CP^1)^k / (T_1 \times T_2^k)(Z_n) \quad \text{and}$$

$$N_2 = S^1 \times (CP^1)^k / (T_1 \times T_1^k)(Z_n).$$

Since T_2 is trivial, $N_1 = L^1(n) \times (CP^1)^k$. Let $f(z,w) = (z, zw)$ intertwine these two actions so $N_1 = N_2$ in $\tilde{\Omega}_{2k+1}^{\mathrm{Spin}^c}(BZ_2)$. The action $T_1 \times T_1^k$ on $D_2 \times (CP^1)^k$ has 2^k isolated fixed points at $0 \times \{0, \infty\} \times \cdots \times \{0, \infty\}$.

Cut out small Z_n equivariant spheres about each fixed point to construct a manifold on which Z_n acts freely. The quotient is an equivariant bordism between N_2 and the sum of 2^k lens spaces. At $0 \in D_2$ or $0 \in CP^1$, the action is ρ_1. At $\infty \in CP^1$, the action is ρ_{-1} owing to the change of coordinates $z = 1/w$. There are $\binom{k}{p}$ fixed points corresponding to $p \cdot \infty$ and $(k+1-p) \cdot 0$ so we get $\binom{k}{p}$ copies of $M_p(k+1)$ which completes the proof.

Let $M \in \tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$. Inequivalent Spin^c structures on M are parametrized by complex line bundles over M. Let $c(s) \cdot M$ be M with the same Z_n structure and with the Spin^c structure twisted by the complex line bundle which corresponds to the representation ρ_s. Since $M = 0$ in $\Omega_*^{\mathrm{Spin}^c}$, the Stiefel-Whitney numbers and rational Chern/Pontrjagin numbers of M vanish. Since $M = c(s) \cdot M$ in Ω_*^0, the Stiefel-Whitney numbers of $c(s) \cdot M$ vanish. Since the line bundle defined by ρ_s is flat, the Chern/Pontrjagin numbers are unchanged so $c(s) \cdot M = 0$ in $\Omega_*^{\mathrm{Spin}^c}$ by Theorem 1.2. The map $M \to c(s) \cdot M$ defines an $\Omega_*^{\mathrm{Spin}^c}$ module isomorphism of $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$. We extend $c(s)$ as the identity on $\Omega_*^{\mathrm{Spin}^c}$. Identify the group algebra $Z[Z_n]$ with $R(Z_n)$. Since $c(s)c(t)$ corresponds to twisting by $\rho_s\rho_t = \rho_{s+t}$, $c(s)c(t) = c(s+t)$. We define a representation of $R(Z_n)$ on $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$ by $c(\Sigma_s n_s \rho_s) = \Sigma_s n_s c(s)$. Dually, let $c(s) \cdot \Lambda^k = \Lambda^k$ and $c(s) \cdot \tau = \rho_{2s} \cdot \tau$ define a representation of $R(Z_n)$ on $R_0(Z_n) \otimes R(\mathrm{Spin}^c)$.

LEMMA 4.3. *Let* $M \in \tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$ *and let* $\rho \in R_0(Z_n)$. (a) $\Delta(c(\rho) \cdot M) = c(\rho) \cdot \Delta(M)$.

(b) *If* $\theta \in R_0(Z_n) \otimes R(\mathrm{Spin}^c)$, *then*

$$\eta(\theta, c(s) \cdot M) = \eta(\rho_s \cdot (c(s) \cdot \theta), M).$$

(c) $\rho \in R_0(Z_n)^k$ *iff* $\eta(\rho \cdot \bar{\rho}, L^k(n)) = 0\ \forall \bar{\rho} \in R_0(Z_n)$.

(d) $c(\rho) \cdot L^k(n) = 0$ *iff* $\rho \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$.

REMARK. It is in (d) that the analysis is used at last to embed the K-theory groups into the bordism groups.

Proof. The classifying map is not changed if we change the Spin^c structure so $\Delta(c(s) \cdot M) = \Delta(M)$ in $\Omega_*^{\mathrm{SO}}(BZ_n)$. If we twist the Spin^c structure on M by a line bundle L, then the Spin^c structure on $\Delta(M)$ is twisted by $L|_{\Delta(M)}$. Consequently $c(s) \cdot \Delta(M) = \Delta(c(s) \cdot M)$ which proves (a). Let $N = c(s) \cdot M$. Since only the Spin^c structure is changed, $\rho(N) = \rho(M)$ and $\Lambda^k(N) = \Lambda^k(M)$. Since τ involves a square, $\tau(N) = \rho_{2s}(M) \otimes \tau(M)$. Therefore $c(s) \cdot \theta(M) = \theta(N)$. Let $D(N)$ and $D(M)$ be the tangential operators of the Spin^c complex over N and M. Since the Spin^c

structure on N is twisted by ρ_s, $D(N)_\theta = D(M)_{\rho_s \cdot c(\theta)}$ which completes the proof of (b). We refer to Gilkey [12] for the proof of (c).

We prove (d) as follows. Let $\rho \in R(Z_n)$, let $\beta = (\rho_1 - \rho_0)/\rho_1 \in R_0(Z_n)$, and let $\bar{\rho} \in R_0(Z_n)$. Since $\beta \cdot \alpha^{k+1} = \alpha^k$ for $k > 0$, we use (a) and Lemma 2.3 to compute

$$0 = \eta(\bar{\rho}, c(\rho) \cdot L^k(n)) = \eta(\bar{\rho} \cdot \rho, L^k(n)) = \eta(\bar{\rho} \cdot (\rho \cdot \beta), L^{k+1}(n)).$$

This implies $\rho \cdot \beta \in R_0(Z_n)^{k+1}$ by (c). Since $\beta R(Z_n) = R_0(Z_n)$, $\rho \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$. Conversely, let $M = c(\rho) \cdot L^k(n)$ and assume $\rho \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$. We prove $M = 0$ by showing $M \in \ker_{2k-1}(\eta, n) \cap \ker_{2k-1}(SW, n)$. Let $\delta = \Sigma_s \rho_s$ be the regular representation. Since $\beta \cdot \rho \in R_0(Z_n)^{k+1} = \beta \cdot R_0(Z_n)^k$, choose $\sigma \in R_0(Z_n)^k$ so $\beta \cdot \rho = \beta \cdot \sigma$. Since the only zero divisors of β are multiples of δ and since $\beta \cdot (\rho - \sigma) = 0$, $\rho = \sigma + a \cdot \delta$. If $w \in W^{2k-1}(n)$, then $w(M) = \dim(\rho) \cdot w(L^k(n))$. Since $\dim(\rho) = a \cdot \dim(\delta) = a \cdot n$ is even, $M \in \ker_{2k-1}(SW, n)$. Let $\Psi = R_0(Z_n) \otimes Z[\Lambda^k]$ so $R_0(Z_n) \otimes R(\mathrm{Spin}^c) = \Psi[\tau]$. Suppose $\theta = \psi \cdot \tau^w$ for $\psi \in \Psi$. Choose a representation $\phi \in R_0(Z_n)$ so $\phi(L^k(n))$ and $\theta(L^k(n))$ define the same locally flat bundles. Let $m(2w+1)$ be the algebra morphism of $R(Z_n)$ defined by $m(2w+1)(\rho_s) = \rho_{(2w+1)s}$. Since $2w+1$ is odd, it is coprime to n and $m(2w+1)$ defines a ring isomorphism of $R(Z_n)$ preserving $R_0(Z_n)^j$ $\forall j$. Consequently $m(2w+1)(\rho) \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$ so $\beta \cdot m(2w+1)(\rho) \in R_0(Z_n)^{k+1}$. Let $\rho = \Sigma_s n_s \rho_s$. We use (b) and Lemma 2.3 to compute:

$$\begin{aligned}\eta(\psi \cdot \tau^w, c(\rho)M) &= \sum_s n_s \eta\big(\psi \cdot \rho_{(2w+1)s}, L^k(n)\big)\\ &= \sum_s \eta\big(\psi \cdot n_s \rho_{(2w+1)s}, L^k(n)\big) - \eta\big(\psi \cdot m(2w+1)(\rho), L^k(n)\big)\\ &= \eta\big(\phi \cdot m(2w+1)(\rho) \cdot \beta, L^{k+1}(n)\big)\end{aligned}$$

which vanishes by (c). This shows $M \in \ker_{2k-1}(\eta, n)$ and completes the proof.

Let $A_{2k}(n) = 0$ and let $A_{2k-1}(n)$ be the subgroup of $\tilde{\Omega}_{2k-1}(BZ_n)$ generated by all possible Spin^c structures on $L^k(n)$.

LEMMA 4.4. (a) $A_{2k-1}(n) = R_0(Z_n)/R_0(Z_n)^{k+1} = \tilde{K}(S^{2k+1}/Z_n)$ *is a finite group of order* n^k.

(b) $A_{2k-1}(2) = Z_{2^k}$ *is generated by* RP^{2k-1}.

(c) $L^1(n) \times (CP^1)^k \in A_{2k+1}(n)$.

Proof. Since any complex line bundles over $L^k(n)$ correspond to one of the ρ_s, $A_{2k-1}(n) = c(R(Z_n)) \cdot L^k(n)$. If $\rho \in R(Z_n)$, let $f(\rho) = \rho \cdot \beta \in R_0(Z_n)/R_0(Z_n)^{k+1}$ so $f(\rho) = 0$ iff $\rho \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$. By

Lemma 4.3, the map $c(\rho)(M) \to f(\rho)$ is well defined and provides an isomorphism between $A_{2k-1}(n)$ and $R_0(Z_n)/R_0(Z_n)^{k+1}$. This group has order n^k and is isomorphic to $\tilde{K}(S^{2k+1}/Z_n)$ (see Atiyah [2]). Since the two Spin^c structures on RP^{2k-1} are just $\pm RP^{2k-1}$, $A_{2k-1}(2)$ is a group of order 2^k generated by RP^{2k-1} which proves (b). If $M_p(k+1) = L(n;\ -1, \ldots, -1, 1, \ldots, 1)$, then $M_p(k+1) = (-1)^p L^{k+1}(n)$ in $\Omega_{2k+1}^{\mathrm{SO}}(BZ_n)$ since we have reversed the orientation of p complex coordinates. Consequently $M_p(k+1)$ and $\pm L^{k+1}(n)$ only differ by the choice of Spin^c structure so $M_p(k+1) \in A_{2k+1}(n)$. By Lemma 4.2, $L^1(n) \times (CP^1)^k$ is a linear combination of the $M_p(k+1)$ which completes the proof of the Lemma.

Remark. Using the description of the Spin^c structure associated to the unitary structure given in Hitchin [15] we see $M_p(k) = (-1)^p c(p) L^k(n)$ so the $\{M_p(k)\}$ generate A_{2k-1} for $0 \le p < n$; $\Sigma_p n_p M_p(k) = 0$ iff $\Sigma_p (-1)^p n_p(\rho_p - \rho_{p+1}) \in R_0(Z_n)^{k+1}$.

Let $S_*(n)$ be the Q_* submodule of $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$ generated by the $A_*(n)$ and let $T_*(n)$ be the $\mathrm{Tor}(\Omega_*^{\mathrm{Spin}^c})$ submodule of $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n)$ generated by the $L^k(n)$.

Lemma 4.5. (a) $R_m(n) = S_m(n) \oplus T_m(n)$, $S_*(n) = A_*(n) \otimes Q_*$, $T_*(n) = \oplus_{j \ge 0} \mathrm{Tor}(\Omega_{m-2j-1}^{\mathrm{Spin}^c})$.

(b) *If* $M \in R_m(2) \cap \ker_m(SW, 2)$, *then* $M \in 2 \cdot R_m(2)$.

(c) π: $\tilde{\Omega}_m^{\mathrm{Spin}^c}(BZ_n) \to E(m) \to 0$ *splits. The splitting* $\bar{E}(m) = \pi^{-1}E(m)$ *can be chosen so* $\bar{E}(m) \subseteq \ker_m(\eta, 2)$ *and so* Δ: $\bar{E}(m) \to \bar{E}(m-2) \oplus \mathrm{Tor}(\Omega_{m-2}^{\mathrm{Spin}^c})$.

(d) $S_m(n) \cap \ker_m(\eta, n) = 0$.

Remark. The structure of A_* is given in Lemma 4.3. By Lemma 2.5, $T_m(n) \subseteq \ker_m(\eta, n)$. By construction $\bar{E}(m) \subseteq \ker_m(\eta, n)$. Thus by (d), $\tilde{\Omega}_*^{\mathrm{Spin}^c}(BZ_n) = A_*(n) \otimes Q_* \oplus \ker_*(\eta, n)$ and $\ker_m(\eta, n) = T_m(n) \oplus \bar{E}(m) = \oplus_{j>0} \mathrm{Tor}(\Omega_{m-j}^{\mathrm{Spin}^c})$. This completes the proof of Theorem 0.1(b). If $B_{m-1} = \delta_1 A_m(2)$, then $B_{2k} = Z_{2^{k+1}}$ is generated by RP^{2k} by Lemma 4.3. We apply δ_1 to Theorem 0.1(b) and use the results of the third section to prove Theorem 0.2(b).

Proof. Δ: $T_m(n) \to T_{m-2}(n) \to 0$. Since Δ and c commute, Δ: $S_m(n) \to S_{m-2}(n) \to 0$. Therefore $S_m + T_m(n) \subseteq \ker(\pi) = R_m(n)$ by Lemma 4.1. By Lemma 4.2 $L^1(n) \times (CP^1)^k \subseteq S_*(n)$. Since $P_* = Q_*[CP^1]$, $L^1(n) \times \Omega_*^{\mathrm{Spin}^c} \subseteq S_*(n) + T_*(n)$ by Lemma 2.4. Consequently $0 \to L^1(n) \times \Omega_{m-1}^{\mathrm{Spin}^c} \to S_m(n) + T_m(n) \to S_{m-2}(n) + T_{m-2}(n) \to 0$ so that $|S_m(n) + T_m(n)| = a(m, n)b(m)$ and $S_m(n) + T_m(n) = R_m(n)$. Since $|A_{2k}(n)| = 1$

and $|A_{2k-1}(n)| = n^k$, $|\{A_*(n) \otimes Q_*\}_m| = a(m,n)$ so $|S_m(n)| \le a(m,n)$. The map $(N_k) \to \Sigma_k L^k(n) \times N_k$ defines a surjection $\oplus_{j\ge 0}\mathrm{Tor}(\Omega^{\mathrm{Spin}^c}_{m-2j-1}) \to T_m(n)$ so $|T_m(n)| \le b(m)$ and $|S_m(n) + T_m(n)| \le a(m,n)b(m)$. Since the inequality is an equality, there are no additional relations and this completes the proof of (a).

Let $M \in R_*(2) \cap \ker_*(SW,2)$. We use (a) to decompose $M = \Sigma_k RP^{2k+1} \times N(k)$ for $N(k) \in Q_{m-2k-1} + \mathrm{Tor}(\Omega_{m-2k-1})$. If $N(k) \in \ker_{m-2k-1}(SW)\forall k$, then $N(k) \in 2Q_{m-2k-1}$ and $M \in 2R_m(n)$ by Lemma 1.3 which will prove (b). If this is false, choose j maximal so $N(j) \notin \ker(SW)$. Let $x \in W^{m-2j-1}$ so $x(N(j)) \ne 0$ and let $y = x_1^{2j+1} \cdot x$. If $k < j$, then $y(RP^{2k+1} \times N(k)) = 0$ since the x_1^{2j+1} term vanishes. If $k > j$, then $y(RP^{2k+1} \times N(k)) = 0$ since $N(k) = 0$ in Ω^0_*. Consequently $y(M) = y(RP^{2j+1} \times N(j)) = x(N(j)) \ne 0$. This contradiction proves (b).

Let i be induction and let r: $R_0(Z_n) \to R_0(Z_2)$ be restriction. If $M \in \Omega^{\mathrm{Spin}^c}_*(BZ_2)$ and if $\psi \in R_0(Z_n) \otimes R(\mathrm{Spin}^c)$, then the bundles $\psi(i(M))$ and $r(\psi)(M)$ agree so $\eta(\psi, i(M)) = \eta(r(\psi), M)$ and i: $\ker_*(\eta, 2) \to \ker_*(\eta, n)$. Consequently it suffices to prove (c) in the special case $n = 2$. Let $X \in E(m)$ and let $0 \oplus X \in E(m+2)$. Choose $Y \in \tilde\Omega^{\mathrm{Spin}^c}_{m+2}(BZ_2)$ so $\pi(Y) = 0 \oplus X$. Since $2X = 0$, $\pi(2Y) = 0$ so $2Y \in R_{m+2}(n)$. Since the Stiefel-Whitney numbers of $2Y$ vanish, $\exists\, Z \in R_{m+2}(2)$ so $2Y = 2Z$. Let $M = \Delta(Y - Z)$ so $2M = 0$ and $\pi(M) = X$. Since $\delta_1(2(Y-Z)) = 0$, $M \in \ker_m(\eta, 2)$ by Lemma 3.3 and the map $X \to M$ provides a suitable splitting. If we fix m large, we can choose $\bar{E}(m-2k) = (1-\pi_2)\Delta^k\bar{E}(m)$ for $k = 0, 1, \ldots$ Since $\tilde\Omega^{\mathrm{Spin}^c}_m(BZ_n)$ is finite, we use the pigeon hole principal to choose $\bar{E}(m)$ consistently for all m with this property. This completes the proof of (c).

Let $M \in S_m(n) \cap \ker_m(\eta, n)$; we show $M = 0$ by induction; this is trivial for $m < 0$. We apply Δ and Lemma 2.6 to see $\Delta(M) \in S_{m-2}(n) \cap \ker_{m-2}(\eta, n)$ so $\Delta(M) = 0$. Consequently $M = L^1(n) \times (N_1 + N_2)$ for $N_1 \in P_{m-1}$ and $N_2 \in \mathrm{Tor}(\Omega^{\mathrm{Spin}^c}_{m-1})$. We apply Lemma 2.4 to see $N_1 \in n \cdot P_{m-1}$ so $M = L^1(n) \times N_2$. Consequently $M \in S_m(n) \cap T_m(n) = 0$. This completes the proof of all the assertions in the paper.

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