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OPERATIONS WHICH DETECT $\mathscr{P}^1$ IN ODD PRIMARY CONNECTIVE K-THEORY

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Let G denote the Adams summand of connective unitary K-theory spectrum at the odd prime integer p. In this paper, we study maps ϕ: $G \to G$ which have two properties

(1) $\phi_* = 0$: $\pi_0(G) \to \pi_0(G)$,

(2) $\phi_*(v) = p\varepsilon v$ with the unit $\varepsilon \in Z_{(p)}^{\times}$, where $\pi_*(G) = Z_{(p)}[v]$ and $|v| = 2(p-1)$. An example of such operations is the Adams operation $\psi^{p+1} - 1$, and we will give an elementary proof of non-existence of elements of mod p Hopf invariant one.

0. Introduction. The purpose of this paper is to study a certain family of operations in the Adams summand of the connective unitary K-theory spectrum and to demonstrate their usefulness in analyzing the action of the Steenrod algebra on the mod p cohomology of certain spectra.

Although our technique follows closely the work given by M. Mahowald and R. J. Milgram [**13**], they treated only the mod 2 case and it seems useful to give the mod p version for an odd prime p.

This paper is organized as follows:

In §1, we consider the basic properties of the spectrum G.

In §2, we define the operations which detect $\mathscr{P}^1$ and give their basic properties.

In §3, we give the proof of Theorem 2.9, which is the key invariant property of operations which detect $\mathscr{P}^1$.

In §4, we given an elementary proof of the non-existence of non-zero mod p Hopf invariant and demonstrate their usefulness in the analysis of the action of mod p Steenrod algebra on the mod p cohomology of certain spectra with few cells.

In the final of this section, the author would like to take this opportunity to thank Professor M. Mahowald for his sincere valuable advice during his visiting Tokyo in 1985.

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1. Preliminaries. Throughout this paper, let p be a fixed odd prime and we work in the stable category of (p-local) CW complexes (spectra) with base points.

We will denote by $[X, Y]$ the abelian group of stable homotopy classes of maps from X to Y when X and Y are CW complexes (spectra).

We will identify $[X, Y]$ with $[\Sigma^n X, \Sigma^n Y]$ for any integer n and we will not distinguish between a map and its (stable) homotopy class. Let $Z_{(p)}$ be the ring of integers localized at p and $Z_{(p)}^{\times}$ the group of units in $Z_{(p)}$.

If E is a spectrum and we smash E with the Moore spectrum $M(Z_{(p)})$, we will denote the resulting spectrum by $E_{(p)}$.

Let S^0 and bu be the sphere spectrum and the connective unitary K-theory spectrum, respectively.

Let $\mathscr{A}$ be the mod p Steenrod algebra and $\mathscr{A}(\ )$ denote the left ideal in $\mathscr{A}$ by the set in parentheses.

Then the following is well-known.

THEOREM 1.1 ([3], [7], [9]). *There is a commutative ring spectrum G such that*

$$(1) \qquad bu_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2i} G,$$

$$(2) \qquad \pi_*(G) = Z_{(p)}[v] \quad \textit{with} \quad |v| = 2(p-1),$$

$$(3) \qquad H^*(G, Z/p) = \mathscr{A}/\mathscr{A}(Q_0, Q_1),$$

where $Q_0 = \beta$, *the* mod p *Bockstein, and* $Q_1 = \mathscr{P}^1\beta - \beta\mathscr{P}^1$.

COROLLARY 1.2. *If* $i > 0$, *then*

$$H^i(G, Z) = \begin{cases} \text{direct sum of } Z/p\text{'s} & \text{if } i \not\equiv 0 \bmod 2(p-1) \\ Z_{(p)} \oplus \text{direct sum of } Z/p\text{'s} & \text{if } i \equiv 0 \bmod 2(p-1). \end{cases}$$

Proof. Let $\mathscr{A}^*$ be the dual of the Steenrod algebra $\mathscr{A}$. Then it is the tensor algebra of an exterior algebra and a polynomial algebra:

$$\mathscr{A}^* = E[\tau_0, \tau_1, \tau_2, \dots] \otimes Z/p[\xi_1, \xi_2, \dots]$$

where $|\tau_n| = 2p^n - 1$ and $|\xi_n| = 2(p^n - 1)$. The dual of the quotient $\chi(\mathscr{A}/\mathscr{A}(Q_0, Q_1))^*$ is the subalgebra of $\mathscr{A}^*$:

$$\chi(\mathscr{A}/\mathscr{A}(Q_0, Q_1))^* = E[\tau_2, \tau_3, \dots] \otimes Z/p[\xi_1, \xi_2, \dots]$$

where χ denotes the canonical anti-automorphism.

We calculate its homology for the boundary obtained by dualizing the right action of Q_0 in $\mathscr{A}/\mathscr{A}(Q_0, Q_1)$.

Since Q_0 is primitive, Q_0^* is a derivation and $Q_0^*\tau_n = \xi_n$ for $n \geq 2$, $Q_0^*\xi_1 = 0$. Thus $H(\chi(\mathscr{A}/\mathscr{A}(Q_0, Q_1))^*, Q_0) = Z/p[\xi_1]$ and the assertion follows. □

The above calculation also shows the following result:

COROLLARY 1.3. *An explicit description of the* mod p *restriction of the integral generator in* $H^{2(p-1)n}(G, Z)$ *is* $\chi(\mathscr{P}^n)\iota$.

REMARK 1.4. Since G is a commutative ring spectrum, there is a ring structure map μ: $G \wedge G \to G$ and a unit map ι_*: $S^0 \to G$. The Cartan formula induces the product in $\mathscr{A}^*$ and so induced homomorphism

$$\mu_*: H_*(G, Z/p) \otimes H_*(G, Z/p) \to H_*(G, Z/p)$$

coincides with the usual multiplication in $\chi(\mathscr{A}/\mathscr{A}(Q_0, Q_1))^*$. Consequently, if h_1 is the integral generator of $H_{2(p-1)}(G, Z)$, then

$$(1.5) \qquad h_n = (h_1)^n = \mu_*(h_1 \otimes h_1 \otimes \cdots \otimes h_1) \quad (n \text{ factors})$$

represents an integral generator in $H_{2(p-1)n}(G, Z)$.

COROLLARY 1.6. *In dimension* $2(p-1)n$, *the Hurewicz homomorphism*

$$h: \pi_{2(p-1)n}(G) \to H_{2(p-1)n}(G, Z)$$

is injective and $h(v^n) = p^n \varepsilon h_n$ *with some unit* $\varepsilon \in Z_{(p)}^{\times}$.

Proof. The spectrum G has the (stable p-local) cell structure

$$(1.7) \qquad G = S_{(p)}^0 \cup_{\alpha_1} e^{2(p-1)} \cup e^{2(p^2-1)} \cup \cdots$$

where α_1 generates $\pi_{2p-3}(S^0)_{(p)}$.

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\pi_{2(p-1)}(S^0)_{(p)} = 0 \to & \pi_{2(p-1)}(G) & \xrightarrow{j_*} & \pi_{2(p-1)}(G, S_{(p)}^0) & \xrightarrow{\partial} & \pi_{2p-3}(S^0)_{(p)} & \to 0 \\
& \downarrow h & & \cong \downarrow h & & & \\
& H_{2(p-1)}(G, Z) & \underset{\cong}{\xrightarrow{j_*}} & H_{2(p-1)}(G, S_{(p)}^0, Z) & & &
\end{array}$$

where h denotes the Hurewicz homomorphism.

Since the order of α_1 is p, $h(v) = p\varepsilon_1 h_1$ with some unit $\varepsilon_1 \in Z_{(p)}^\times$. Hence $h(v^n) = p^n \varepsilon h_n$ with $\varepsilon = (\varepsilon_1)^n \in Z_{(p)}^\times$. □

2. Operations which detect $\mathscr{P}^1$ on the spectrum G. For a spectrum E, we define the E-homology and E-cohomology by

(2.1) $$E_n(X) = \pi_n(E \wedge X) \quad \text{and} \quad E^n(X) = [X, \Sigma^n E].$$

In particular, the Steenrod algebra of E, $\mathscr{A}(E)^*$ is defined by

(2.2) $$\mathscr{A}(E)^* = E^*(E).$$

Note that it acts in E-homology by the following

(2.3) $$\phi(f) = (\phi \wedge 1) \circ f \quad \text{for } \phi \in \mathscr{A}(E)^r \text{ and } f \in E_i(X),$$

where

$$S^i \xrightarrow{f} E \wedge X \xrightarrow{\phi \wedge 1} \Sigma^r E \wedge X.$$

DEFINITION 2.4. An operation $\phi \in \mathscr{A}(G)^0$ is said to *detect* $\mathscr{P}^1$ if there exists a map τ: $G \to \Sigma^{2(p-1)} G$ such that,

(1) the diagram

$$\begin{tikzcd} G \arrow[r, "\tau"] \arrow[dr, "\phi"'] & \Sigma^{2(p-1)} G \arrow[d, "\pi"] \\ & G \end{tikzcd}$$

is homotopy commutative, where π: $\Sigma^{2(p-1)} G \to G$ is the Bott periodicity map,

(2) $\tau^*(\iota) = \mathscr{P}^1(\iota)$, where τ^* denotes the induced homomorphism

$$H^0(G, Z/p) = Z/p\{\iota\} \to H^{2(p-1)}(G, Z/p) = Z/p\{\mathscr{P}^1(\iota)\}.$$

REMARK 2.5. (1) By using the Bott periodicity, there is a fiber sequence:

$$\Sigma^{2(p-1)} G \xrightarrow{\pi} G \xrightarrow{\kappa} KZ_{(p)}$$

where $KZ_{(p)}$ denotes the Eilenberg-MacLane Spectrum for $Z_{(p)}$.

(2) From (1.6), $\pi_*(I_*) = p\varepsilon h_1$ with some unit $\varepsilon \in Z_{(p)}^\times$, where π_* denotes the induced homomorphism

$$H_{2(p-1)}(\Sigma^{2(p-1)} G, Z) = Z_{(p)}\{I_*\} \to H_{2(p-1)}(G, Z) = Z_{(p)}\{h_1\}.$$

LEMMA 2.6. *Let ϕ be the operation in $\mathscr{A}(G)^0$. Then ϕ detects $\mathscr{P}^1$ if and only if the following two conditions hold*:

(a) $\phi_* = 0\colon \pi_0(G) \to \pi_0(G)$.

(b) $\phi_*(v) = p\varepsilon v$ *with some unit* $\varepsilon \in Z_{(p)}^{\times}$.

Proof. First, suppose ϕ detects $\mathscr{P}^1$. Then the two conditions easily follow from $\pi_0(\Sigma^{2(p-1)}G) = 0$ and (2.5).

Conversely, we assume the operation ϕ satisfies two conditions (a) and (b). From the condition (a), $\phi^*(\kappa) = 0$ and the map $\kappa \circ \phi$ is null-homotopic. Hence there is a lifting $\tau\colon G \to \Sigma^{2(p-1)}G$ such that $\pi \circ \tau = \phi$. Similarly, using the diagram chasing, from (2.5) and (b) we can deduce the relation $\tau^*(\iota) = \mathscr{P}^1(\iota)$. Thus ϕ detects $\mathscr{P}^1$. □

EXAMPLE 2.7. Let $i'\colon G \to bu_{(p)}$ be the inclusion map. It is well-known that there is a map of ring spectra $\psi^n\colon G \to G$ which makes the diagram

(2.8)
$$\begin{array}{ccc} G & \xrightarrow{\psi^n} & G \\ i' \downarrow & & i' \downarrow \\ bu_{(p)} & \xrightarrow{\psi^n} & bu_{(p)} \end{array}$$

commute, where the lower map ψ^n is derived from the Adams operation in complex K-theory and $(n, p) = 1$. (See (0.2) in [22]). Furthermore, it is easy to see that $\psi^{p+1} - 1$ satisfies the conditions (a) and (b). Hence $\psi^{p+1} - 1$ detects $\mathscr{P}^1$.

The following is the key invariant property of operations which detect $\mathscr{P}^1$.

THEOREM 2.9 (*M. Mahowald and R. J. Milgram*, [13]). *Let ϕ be the operation in $\mathscr{A}(G)^0$ which detects $\mathscr{P}^1$. Then ϕ_* on $\pi_{2(p-1)n}(G)$ is multiplication by*

$$p^{f(n)}\varepsilon_n$$

where $f(n) = \nu_p(n) + 1$, $\varepsilon_n \in Z_{(p)}^{\times}$ and $\nu_p(n)$ is the power to which p is raised in the prime decomposition of n.

REMARK 2.10. The above result was stated in [13] without proof. Although the idea of its proof is essentially derived from [13], for the sake of completeness we will show it in the next section.

3. Proof of Theorem 2.9. Throughout this section we assume that $\phi \in \mathscr{A}(G)^0$ detects $\mathscr{P}^1$.

First, consider the (stable) 2-cell complex

$$M = S^0 \cup_{\alpha_1} e^{2(p-1)} \tag{3.1}$$

where α_1 generates $\pi_{2p-3}(S^0)_{(p)} \cong Z/p$.

The complex M has the property that $\mathscr{P}^1$ is non-trivial in mod p cohomology. There is a cofiber sequence

$$S^{2p-3} \xrightarrow{\alpha_1} S^0 \xrightarrow{i} M \xrightarrow{q} S^{2(p-1)}. \tag{3.2}$$

By using (1.7) there is an inclusion map

$$\alpha\colon M \to G.$$

We denote by β the composite of maps

$$\beta\colon M \xrightarrow{q} S^{2(p-1)} \xrightarrow{v} G.$$

We put $\wedge^n M = M \wedge M \wedge \cdots \wedge M$ (n times).

Then, using the Atiyah-Hirzebruch spectral sequence, we have

LEMMA 3.3. *As an abelian group $G^0(\wedge^n M)$ is free over $Z_{(p)}$ and has the basis*

$$\mu(\delta_1 \wedge \delta_2 \wedge \cdots \wedge \delta_n),$$

where δ_i is either α or β.

Similarly, using (1.6), (2.5) and (3.2), we obtain

LEMMA 3.4. *There is a unit $\varepsilon_0 \in Z_{(p)}^{\times}$ such that,*

$$\phi(\alpha) = \varepsilon_0 \beta \quad \text{in } G^0(M).$$

LEMMA 3.5. *Let $n \geq 2$. Then there are elements $\varepsilon_i \in Z_{(p)}$ $(2 \leq i \leq n)$ such that,*

$$\begin{aligned}\phi\Big(\mu\big(\underbrace{\alpha \wedge \alpha \wedge \cdots \wedge \alpha}_{n}\big)\Big) &= \varepsilon_0 S\Big(\mu\big(\beta \wedge \underbrace{\alpha \wedge \cdots \wedge \alpha}_{n-1}\big)\Big) \\ &\quad + \sum_{k=2}^{n} \varepsilon_k S\Big(\mu\big(\underbrace{\beta \wedge \cdots \wedge \beta}_{k} \wedge \underbrace{\alpha \wedge \cdots \wedge \alpha}_{n-k}\big)\Big)\end{aligned}$$

where ε_0 is given in (3.4) *and $S(\ )$ denotes the symmetric sum.*

Proof. The proof is similar to (2.7) in **[13]** and left to the reader. □

Now consider the induced homomorphism

$$\mu(\alpha \wedge \alpha \wedge \cdots \wedge \alpha)_*: H_*\left(\bigwedge^n M, Z\right) \to H_*(G, Z).$$

From (1.5), we have

$$\mu(\alpha \wedge \cdots \wedge \alpha)_*(e_{2(p-1)} \otimes e_{2(p-1)} \otimes \cdots \otimes e_{2(p-1)}) = (h_1)^n = h_n \quad \text{in } H_{2(p-1)n}(G, Z).$$

Thus

$$\phi_* \mu(\alpha \wedge \cdots \wedge \alpha)_*(e_{2(p-1)} \otimes \cdots \otimes e_{2(p-1)}) = \phi_*(h_n). \tag{3.6}$$

On the other hand, if we put

$$N = pn\varepsilon_0 + \sum_{k=2}^{n} \varepsilon_k p^k \binom{n}{k}, \tag{3.7}$$

then it follows from (3.5) that we have

$$\phi_* \mu(\alpha \wedge \cdots \wedge \alpha)_*(e_{2(p-1)} \otimes \cdots \otimes e_{2(p-1)}) = Nh_n. \tag{3.8}$$

Thus we have the following result.

PROPOSITION 3.9. *Let ϕ be the operation in $\mathscr{A}(G)^0$ which detects $\mathscr{P}^1$. Then ϕ_* on $H_{2(p-1)n}(G, Z)$ is multiplication by N where N is given in* (3.7).

Similarly, using (1.6), we also obtain

COROLLARY 3.10. *Under the same assumptions as* (3.9), *ϕ_* on $\pi_{2(p-1)n}(G)$ is multiplication by N where N is given in* (3.7).

To prove Theorem 2.9, without loss of generalities we may assume ε_k is an integer for $2 \le k \le n$, and so it suffices only to show the following

LEMMA 3.11. *For an integer k with $2 \le k \le n$,*

$$\nu_p(n) + 1 < k + \nu_p\left(\binom{n}{k}\right).$$

DEFINITION 3.12. For a positive integer m, it is possible to write $m = \sum a_k p^k$ $(0 \le a_k \le p - 1)$ for unique integer a_k, almost all of which vanish.

Then we define

$$\alpha(m) = \sum a_k.$$

Then the following is well-known:

LEMMA 3.13. (a) $\nu_p(m!) = (m - \alpha(m))/(p-1)$.

(b) $\nu_p(\binom{n}{k}) = (\alpha(k) + \alpha(n-k) - \alpha(n))/(p-1)$.

(c) *For integers a, j and θ with* $1 \le a \le p-1$, $j \ge 0$, $\theta \ge 1$, $(\theta, p) = 1$, *we have the relation*

$$\alpha(ap^j - \theta) = a - \alpha(\theta) + (p-1)j.$$

Proof of Lemma 3.11. If $k > \nu_p(n) + 1$, then $\nu_p(n) + 1 < k \le k + \nu_p(\binom{n}{k})$ and the assertion holds. Thus we may suppose $k \le \nu_p(n) + 1$.

We put $\nu = \nu_p(n)$, $n = ap^\nu + p^{\nu+1}\lambda$, $k = p^s\theta \ge 2$, where θ and a are positive integers with $(\theta, p) = (a, p) = 1$. Then

$$\alpha(\lambda) = \alpha(n) - a, \qquad \alpha(k) = \alpha(\theta). \tag{3.14}$$

Since $2 \le k \le \nu + 1$, $p^s\theta = k \le \nu + 1 < p^\nu \le ap^\nu$. Hence,

$$n - k = (ap^\nu - p^s\theta) + p^{\nu+1}\lambda \quad \text{with } 0 < ap^\nu - p^s\theta < p^{\nu+1}\lambda.$$

Therefore,

$$\begin{aligned} \alpha(n-k) &= \alpha(ap^\nu - p^s\theta) + \alpha(p^{\nu+1}\lambda) = \alpha(ap^{\nu-s} - \theta) + \alpha(\lambda) \\ &= (a - \alpha(\theta) + (p-1)(\nu - s)) + \alpha(\lambda) \quad \text{(by (3.3), (c))} \\ &= \alpha(n) - \alpha(k) + (p-1)(\nu - s) \qquad \text{(by (3.14))}. \end{aligned}$$

Hence

$$\alpha(k) + \alpha(n-k) - \alpha(n) = (p-1)(\nu - s). \tag{3.15}$$

Thus,

$$\begin{aligned} D &= \left(k + \nu_p\left(\binom{n}{k}\right)\right) - (\nu_p(n) + 1) \\ &= k + (\alpha(k) + \alpha(n-k) - \alpha(n))/(p-1) - (\nu + 1) \quad \text{(by (3.13))} \\ &= k + (\nu - s) - (\nu + 1) \quad \text{(by (3.15))} \\ &= k - s - 1. \end{aligned}$$

That is,

$$D = k + \nu_p\left(\binom{n}{k}\right) - (\nu_p(n) + 1) = k - s - 1. \tag{3.16}$$

If $s = 0$, then $k = \theta \geq 2$ and so $D = k - s - 1 = \theta - 1 > 0$. If $s > 0$, using $p \geq 3$ we have $s + 1 < p^s$. Hence, $D = k - s - 1 = p^s\theta - s - 1 \geq p^s - s - 1 > 0$. Thus, from (3.16), we have the desired result. □

COROLLARY 3.17. *Let ϕ be the operation in $\mathscr{A}(G)^0$ which detects $\mathscr{P}^1$. Then ϕ_* on $H_{2(p-1)n}(G, Z)$ is multiplicaton by*

$$p^{f(n)}\varepsilon_n$$

where $f(n) = \nu_p(n) + 1$ and $\varepsilon_n \in Z_{(p)}^{\times}$.

4. Applications. In this section we show how any operation which detects $\mathscr{P}^1$ gives an elementary proof of the non-existence of mod p Hopf invariant and demonstrate its usefulness in the analysis of the action of mod p Steenrod algebra on the mod p cohomology of certain spectra with few cells.

Let KZ be the Eilenberg-MacLane spectrum for Z. We note that $H_*(G, Z) = G_*(KZ) = \pi_*(G \wedge KZ)$.

LEMMA 4.1. *The* mod p *restriction of h_n is dual to $1 \otimes \mathscr{P}^n(\iota)$ in $H^{2(p-1)n}(G \wedge KZ, Z/p)$.*

Proof. Since $\chi(\mathscr{P}^n) \otimes 1 + 1 \otimes \mathscr{P}^n$ is decomposable over $\mathscr{A} \otimes \mathscr{A}$, the assertion easily follows from (1.2). □

Note that KZ has the (stable) cell-structure

$$KZ = S^0 \cup_\eta e^2 \cup \cdots . \tag{4.2}$$

Thus there is a cofiber sequence

$$S^0 \xrightarrow{\omega} KZ \xrightarrow{\rho} K = KZ/S^0. \tag{4.3}$$

Then we have the exact sequence

$$\to \pi_j(G) \xrightarrow{h} H_j(G, Z) = G_j(KZ) \xrightarrow{\rho_*} G_j(K) \xrightarrow{\partial} \pi_{j-1}(G) \to \tag{4.4}$$

LEMMA 4.5. *If $j > 0$, then*

$$G_j(K) = \begin{cases} Z/p^n\{\rho_*(h_n)\} \oplus \text{direct sum of } Z/p\text{'s} & \text{if } j = 2(p-1)n \\ \text{direct sum of } Z/p\text{'s} & \text{otherwise.} \end{cases}$$

Proof. The assertion follows from (1.2), (1.6) and (4.4) □

LEMMA 4.6. *Let ϕ be the operation in $\mathscr{A}(G)^0$ which detects $\mathscr{P}^1$. Then $\phi(\rho_*(h_n)) = p^{f(n)}a_n\rho_*(h_n)$ in $G_{2(p-1)n}(K)$, where $f(n) = \nu_p(n) + 1$ and a_n is a positive integer with $(a_n, p) = 1$.*

Proof. Since $\phi(\rho_*(h_n)) = \rho_*(\phi(h_n))$, the assertion follows from (3.17). □

REMARK 4.7. In general, $p^m > m + 1$ for $m \geq 1$.
Thus, if n is a positive integer with $\nu_p(n) \geq 1$, then $p^n > p^{f(n)}$ and so

$$\phi(\rho_*(h_n)) = p^{f(n)}a_n\rho_*(h_n) \neq 0.$$

THEOREM 4.8. (*The non-existence of the* mod p *Hopf invariant*; **[5]**, **[12]**, **[17]**). *Let p be an odd prime. Then for $i \geq 1$, there does not exist a (stable) two cell complex $X = S^0 \cup e^m$ with $\mathscr{P}^n$ non-trivial, where $n = p^i$ and $m = 2(p-1)n$.*

Proof. Suppose X exists, then there exists a map $\lambda\colon X \to KZ$ with $\lambda^*(\iota) = e^0$ and $\lambda_*(e_m)$ dual to $\mathscr{P}^n$, where we put

$$H_*(X, R) = R\{e_0, e_m\} \quad \text{and} \quad H^*(X, R) = R\{e^0, e^m\}$$

for $R = Z$ or Z/p.

It is easy to see that there is a map $\tau\colon S^m \to K$ such that, the diagram

$$(4.9)\qquad \begin{array}{ccccc} S^0 & \xrightarrow{j} & X & \rightarrow & S^m \\ \| & & \downarrow\lambda & & \downarrow\tau \\ S^0 & \xrightarrow{\omega} & KZ & \xrightarrow{\rho} & K \end{array}$$

is homotopy commutative.

Thus, applying the functor $\pi_*(G \wedge -)$ we have the commutative diagram

$$(4.10)\qquad \begin{array}{ccccc} \pi_m(G) & \xrightarrow{j_*} & G_m(X) & \rightarrow & G_m(S^m) \\ \| & & \downarrow\lambda_* & & \downarrow\tau_* \\ \pi_m(G) & \xrightarrow{h} & H_m(G, Z) & \xrightarrow{\rho_*} & G_m(K) \end{array}$$

Let ϕ be an operation in $\mathscr{A}(G)^0$ which detects $\mathscr{P}^1$.

Then it is easy to see that $\tau_*(\iota_* \otimes e_m) = a\rho_*(h_n)$ for some unit $a \in (Z/p^n)^\times$. Hence $\tau_*(\phi(\iota_* \otimes e_m)) = aa_n p^{i+1}\rho_*(h_n) \neq 0$.

On the other hand, since ϕ detects $\mathscr{P}^1$, ϕ factors through $\Sigma^{2(p-1)}G$ and $\phi(\iota_* \otimes e_m) = 0$.

This is a contradiction and completes the proof. □

THEOREM 4.11. *Let p be an odd prime.*

(1) *If $r \not\equiv 0 \pmod{2(p-1)}$, then for $i \geq 1$, there does not exist a (stable) three cell complex $X = S^0 \cup e^{m-r} \cup e^m$ with $\mathscr{P}^n$ non-trivial, where $n = p^i$, $m = 2(p-1)n = 2(p-1)p^i$, $0 < r < n$ and $H^*(X, \mathbf{Z}/p)$ is torsion-free.*

(2) *If $r = 2(p-1)k$, then for $p = 3$ and $i \geq 2$, or $p \geq 5$ and $i \geq 1$, there does not exist a (stable) three cell complex $X = S^0 \cup e^{m-r} \cup e^m$ with $\mathscr{P}^n$ non-trivial, where $n = p^i$, $m = 2(p-1)n = 2(p-1)p^i$ and $0 < r < n$.*

Proof. The proof of the statement (1) is similar to (4.8) and we show (2). Suppose X exists, then using (4.1) there exists a map $\lambda\colon X \to KZ$ with $\lambda^*(\iota) = e^0$ and $\lambda_*(e_m)$ dual to $\mathscr{P}^n$, where we put $L = X/S^0 = S^{m-r} \cup e^m$,

$$H_*(X, R) = R\{e_0, e_{m-r}, e_m\} \quad \text{and} \quad H^*(X, R) = R\{e^0, e^{m-r}, e^m\}$$
$$\text{for } R = Z \text{ or } Z/p.$$

It is easy to see that there is a map $\tau\colon L \to K$ such that, the diagram

$$(4.12)\qquad \begin{array}{ccccc} S^0 & \xrightarrow{j} & X & \xrightarrow{\pi'} & L = X/S^0 \\ \| & & \downarrow \lambda & & \downarrow \tau \\ S^0 & \xrightarrow{\omega} & KZ & \xrightarrow{\rho} & K = KZ/S^0 \end{array}$$

is homotopy commutative.

Thus, applying the functor $\pi_*(G \wedge -)$ we have the commutative diagram

$$(4.13)\qquad \begin{array}{ccccccc} \pi_m(G) & \xrightarrow{j_*} & G_m(X) & \xrightarrow{\pi'_*} & G_m(L) & \to & 0 \\ \| & & \downarrow \lambda_* & & \downarrow \tau_* & & \\ \pi_m(G) & \xrightarrow{h} & H_m(G, Z) & \xrightarrow{\rho_*} & G_m(K) & \to & 0 \end{array}$$

where the horizontal sequences are exact.

Let ϕ be an operation in $\mathscr{A}(G)^0$ which detects $\mathscr{P}^1$.

Then, for some unit $a \in Z_{(p)}^{\times}$, $\tau_*(\iota_* \otimes e_m) = a\rho_*(h_n)$.

Hence, from (4.6) we have

$$\begin{aligned} &\tau_*\big(\phi(\iota_* \otimes e_m)\big) = aa_n p^{i+1}\rho_*(h_n) \neq 0, \\ &\tau_*\big(\phi^2(\iota_* \otimes e_m)\big) = aa_n^2 p^{2i+2}\rho_*(h_n) \neq 0, \end{aligned} \tag{4.14}$$

where we put $\phi^2 = \phi \circ \phi$.

On the other hand, $L = S^{m-r} \cup e^m = \Sigma^{m-r}(S^0 \cup e^r)$ and there is a cofiber sequence

$$S^0 \xrightarrow{i'} \Sigma^{r-m}L \xrightarrow{q} S^r. \tag{4.15}$$

Since $r = 2(p-1)k$, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & G_r(S^0) & \overset{i'_*}{\to} & G_r(\Sigma^{r-m}L) & \overset{q_*}{\to} & G_r(S^r) & \to & 0 \\ & & \downarrow \phi'_* & & \downarrow \phi & & \downarrow \phi''_* & & \\ 0 & \to & G_r(S^0) & \overset{i'_*}{\to} & G_r(\Sigma^{r-m}L) & \overset{q_*}{\to} & G_r(S^r) & \to & 0 \\ & & \downarrow \phi'_* & & \downarrow \phi & & \downarrow \phi''_* & & \\ 0 & \to & G_r(S^0) & \overset{i'_*}{\to} & G_r(\Sigma^{r-m}L) & \overset{q_*}{\to} & G_r(S^r) & \to & 0 \end{array} \tag{4.16}$$

where ϕ'_* and ϕ''_* are induced from ϕ and three horizontal sequences are exact. Let σ^{m-r}: $G_r(\Sigma^{r-m}L) \overset{\cong}{\to} G_m(L)$ be the iterated suspension isomorphism, and we put $\sigma^{m-r}(\iota_* \otimes e'_r) = \iota_* \otimes e_m$ for $\iota_* \otimes e'_r \in G_r(\Sigma^{r-m}L)$.

Since ϕ detects $\mathscr{P}^1$, it factors through $\Sigma^{2(p-1)}G$ and the induced homomorphism ϕ''_* is trivial.

Hence there is a unique element $b \in Z_{(p)}$ such that, $\phi(\iota_* \otimes e'_r) = bi'_*(v^k)$, since $G_r(S^0) = \pi_r(G) = Z_{(p)}\{v^k\}$.

Thus, using (2.9) we have

$$\phi^2\big(\iota_* \otimes e'_r\big) = b\varepsilon_k p^{f(k)} i'_*(v^k) \quad \text{with } f(k) = \nu_p(k) + 1.$$

Since $\sigma^{m-r}(i'_*(v^k)) = v^k \otimes e_{m-r}$, we have

$$\begin{aligned} &\phi(\iota_* \otimes e_m) = b\big(v^k \otimes e_{m-r}\big), \\ &\phi^2(\iota_* \otimes e_m) = b\varepsilon_k p^{f(k)}\big(v^k \otimes e_{m-r}\big), \end{aligned} \tag{4.17}$$

where $f(k) = \nu_p(k) + 1$.

Now we put $\lambda_*(v^k \otimes e_{m-r}) \equiv c \cdot h_n$ (mod direct sum of Z/p's) for some $c \in Z_{(p)}$. Then, from (4.13) and (4.17), we have

$$\begin{aligned} &\tau_*(\phi(\iota_* \otimes e_m)) = bc\rho_*(h_n) && \text{(mod direct sum of } Z/p\text{'s)}, \\ &\tau_*\big(\phi^2(\iota_* \otimes e_m)\big) = bc\varepsilon_k p^{f(k)}\rho_*(h_n) && \text{(mod direct sum of } Z/p\text{'s)}, \end{aligned} \tag{4.18}$$

where $f(k) = \nu_p(k) + 1$.

Since the order of $\rho_*(h_n)$ is p^n, using (4.14) and (4.18), we have

$$\begin{aligned} i+1 &\equiv \nu_p(bc) && \pmod{p^n}, \\ 2i+2 &\equiv \nu_p(bc)+\nu_p(k)+1 && \pmod{p^n}. \end{aligned}$$

Hence $\nu_p(k) \equiv i \pmod{p^n}$.

Since $0 < r = 2(p-1)k < 2(p-1)n = m$, $0 \le \nu_p(k) < i$. Thus, there is a positive integer d such that,

$$i - \nu_p(k) = dp^n.$$

Therefore, $p^n > n = p^i > i \ge i - \nu_p(k) = dp^n \ge p^n$. Thus $p^n > p^n$ and this is a contradiction. □

REMARK 4.19. When $p = 3$ and $i = 1$, we check that a (stable) three cell complex $X = S^0 \cup_{\alpha_1} e^4 \cup e^{12}$ with $\mathscr{P}^3$ non-trivial on the mod 3 cohomology is indeed possible, where $\alpha_1 = \alpha_1(3)$ generates $\pi_3(S^0)_{(3)} \cong Z/3$.

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