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SPACES OF SECTIONS OF EILENBERG-MAC LANE FIBRATIONS

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We show first that the space of sections of a fibration with an Eilenberg-Mac Lane space as fibre has the weak homotopy type of a product of Eilenberg-Mac Lane spaces. Secondly, mapping spaces with twisted Eilenberg-Mac Lane spaces as targets are shown to be generalized twisted Eilenberg-Mac Lane spaces.

1. Introduction. Let $p\colon Y \to B$ be a (Serre) fibration, $i\colon A \hookrightarrow X$ a cofibration and $u\colon X \to Y$ a (continuous) map. Using Switzer's notation from [**14**], let

$$F_u(X, A; Y, B)$$

be the space of all maps $f\colon X \to Y$ such that $f \circ i = u \circ i$ and $p \circ f = p \circ u$. In other words, $F_u(X, A; Y, B)$ is the solution space for the lifting extension problem

$$\begin{array}{ccc} A & \xrightarrow{ui} & Y \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow {\scriptstyle p} \\ X & \xrightarrow[pu]{} & B \end{array}$$

with data $u|A\colon A \to Y$ and $pu\colon X \to B$.

We shall be concerned with decompositions of $F_u(X, A; Y, B)$ when $p\colon Y \to B$ has an Eilenberg-Mac Lane space as fibre. Suppose for instance that $p\colon K(G, n) \to *$ is the trivial fibration mapping an Eilenberg-Mac Lane space onto a point. Then

$$(*) \qquad F_u(X, \varnothing; K(G, n), *) = \prod_{i=0}^{n} K\big(H^{n-i}(X; G), i\big)$$

by Haefliger's sharpened version [7] of a theorem of Thom [**15**] and independently Federer [**4**]. The main purpose of this paper is to establish a twisted version of $(*)$

2. Preliminaries. We shall work in the category of compactly generated spaces. For any two compactly generated spaces X and Y, we let $X \times Y$ and $F(X; Y)$ denote the compactly generated spaces associated to the Cartesian product of X and Y and the space of maps of X into Y with the compact-open topology, respectively. These constructions assure the continuity of the evaluation map $e\colon F(X;Y)\times X\to Y$ and the validity of the Exponential Law ([**16**], pp. 17–21) and thus eliminate the difficulties with the topology of function spaces as pointed out by Thom in the first paragraphs of [**15**].

Throughout this paper we let (X,A) denote an NDR-pair ([**16**], p. 22) with X 0-connected and $p\colon Y\to B$ a fibration with 0-connected base space B. Then $F_u(X,A;Y,B)$ is a closed subset of $F(X;Y)$ and thus compactly generated in the (usual) subspace topology.

Composition with maps from the right or from the left defines maps of function spaces. If for instance $A\subset X'\subset X$ is a nested sequence of NDR-pairs and $j\colon X'\to X$ the inclusion, then the induced map

$$\bar{j}\colon F_u(X,A;Y,B)\to F_{uj}(X',A;Y,B)$$

is a fibration with $F_u(X,X';Y,B)$ as fibre. Similarly, if $Y\to Y'\to B$ is a sequence of fibrations and $q\colon Y\to Y'$ the projection, then the induced map

$$\underline{q}\colon F_u(X,A;Y,B)\to F_{qu}(X,A;Y',B)$$

is a fibration with $F_u(X,A;Y,Y')$ as fibre ([**14**], Proposition, p. 528).

Let π be an abelian group. We shall be particularly interested in the $K(\pi,1)$-sectioned spaces [**10**] that arise in the following way. Suppose that G is a system of local coefficients in the Eilenberg-Mac Lane space $K(\pi,1)$ given by a homomorphism $\varphi\colon \pi_1(K(\pi,1))=\pi\to \mathrm{Aut}(G_0)$ of π into the automorphism group of a typical group G_0 of G. For any integer $n>0$, G may be realized, see ([**5**], Ch. III) or ([**10**], p. 7), as the system of local coefficients defined by the n-dimensional homotopy groups of the fibres of a sectioned fibration

$$K(G_0,n)\to K(G_0,n;\varphi)\underset{\check{k}}{\overset{\hat{k}}{\rightleftarrows}} K(\pi,1)$$

over $K(\pi,1)$. This fibration, which we shall denote by $\kappa(G,n)$, classifies cohomology with local coefficients in the sense that by the Classification Theorem ([**16**], Theorem 6.13, p. 302), ([**13**], Theorem 3.6), ([**12**], Theorem II),

$$\pi_0\big(F_u(X,A;K(G_0,n,\varphi),\,K(\pi,1))\big)=H^n\big(X,A;u_1^*G\big)$$

for any map $u\colon X \to K(G_0, n; \varphi)$ with $u_1 = \hat{k}u$. Via pull-back of the path-space fibration in the category of $K(\pi, 1)$-sectioned spaces [**10**],

$$K(G_0, n-1) \to \overline{P}K(G_0, n; \varphi) \to K(G_0, n; \varphi),$$

this equality may be interpreted as a bijective correspondence between fibre homotopy equivalence classes of $K(G_0, n-1)$-fibrations over X with u_1^*G as associated system of local coefficients and the cohomology group $H^n(X; u_1^*G)$.

As a final subject of this mixed section we shall now discuss Künneth theorems for cohomology with local coefficients. First an algebraic lemma ([**1**], Theorem 2.8).

LEMMA 2.2. *Let $\underline{P}$ be a free positive and $\underline{N}$ a negative chain complex over $\mathbf{Z}$. Then there is an isomorphism*

$$\Phi_{\underline{N}}\colon H(\mathrm{Hom}(\underline{P}, \underline{N})) \to H(\mathrm{Hom}(\underline{P}, H(\underline{N})))$$

which is natural in the first variable.

Proof. Choose a free negative complex $\underline{N}'$ and chain maps

$$H(\underline{N}) \stackrel{\beta}{\leftarrow} \underline{N}' \stackrel{\alpha}{\rightarrow} \underline{N}$$

such that α is a quasi-isomorphism and $\beta_* = \alpha_*\colon H(\underline{N}') \to H(\underline{N})$; cf. ([**3**], p. 169). Since $\underline{P}$ is free (projective), the induced chain maps

$$\mathrm{Hom}(1, \alpha)\colon \mathrm{Hom}(\underline{P}, \underline{N}') \to \mathrm{Hom}(\underline{P}, \underline{N}),$$
$$\mathrm{Hom}(1, \beta)\colon \mathrm{Hom}(\underline{P}, \underline{N}') \to \mathrm{Hom}(\underline{P}, H(\underline{N}))$$

are again quasi-isomorphisms. Thus

$$\Phi_{\underline{N}} = \mathrm{Hom}(1, \beta)_* \circ \mathrm{Hom}(1, \alpha)_*^{-1}\colon H(\mathrm{Hom}(\underline{P}, \underline{N})) \to H(\mathrm{Hom}(\underline{P}, H(\underline{N})))$$

is an isomorphism. $\Phi_{\underline{N}}$ is easily seen to commute with $\mathrm{Hom}(\gamma, 1)_*$ for any chain map $\gamma\colon \underline{P} \to \underline{P}'$ between free positive chain complexes. □

Note that since the complex $H(\underline{N})$ has trivial differentiation,

$$H_n(\mathrm{Hom}(\underline{P}, H(\underline{N}))) = \coprod_{p+q=n} H_p\big(\mathrm{Hom}\big(\underline{P}, H_q(\underline{N})\big)\big)$$

where $H_q(\underline{N})$ is considered as a complex concentrated in degree 0.

As to cohomology of spaces, Lemma 2.2 has the following reformulation.

LEMMA 2.3. *Let (Z, C) and (X, A) be NDR-pairs, G a system of local coefficients in X, and* pr_2: $Z \times X \to X$ *the projection onto the second factor. Then there is an isomorphism*

$$\Phi_{(X,A)}: H^n\big((Z,C)\times(X,A); \mathrm{pr}_2^*G\big) \to \coprod_{p+q=n} H^p(Z,C; H^q(X,A;G))$$

which is natural in the first factor.

Proof. We may assume that Z and X are 0-connected spaces and that (Z,C) and (X,A) are CW-pairs. Let $(\tilde{Z},\tilde{C}) \to (Z,C)$ and $(\tilde{X},\tilde{A}) \to (X,A)$ be the universal covering spaces so that ([16], Theorem 4.9, p. 288)

$$\Gamma^*\big((Z,C)\times(X,A); \mathrm{pr}_2^*G\big) \cong \mathrm{Hom}_R(\Gamma(\tilde{Z},\tilde{C}) \otimes \Gamma(\tilde{X},\tilde{Z}), G_0)$$

where $R = \mathbf{Z}(\pi_1(Z)) \otimes \mathbf{Z}(\pi_1(X))$ acts on the typical group G_0 by $(\xi \otimes \eta)g = \eta g$ for $\xi \in \pi_1(Z)$, $\eta \in \pi_1(X)$ and $g \in G_0$. We use $(\Gamma^*)\Gamma$ to denote cellular (co-)chain complexes. Since

$$\begin{aligned}\mathrm{Hom}_R(\Gamma(\tilde{Z},\tilde{C}) &\otimes \Gamma(\tilde{X},\tilde{A}), G_0)\\ &= \mathrm{Hom}_{\pi_1(Z)}\big(\Gamma(\tilde{Z},\tilde{C}), \mathrm{Hom}_{\pi_1(X)}(\Gamma(\tilde{X},\tilde{A}),G_0)\big)\\ &= \mathrm{Hom}_{\pi_1(Z)}(\Gamma(\tilde{Z},\tilde{C}), \Gamma^*(X,A;G))\\ &= \mathrm{Hom}\big(\Gamma(Z,C),\ \Gamma^*(X,A;G)\big),\end{aligned}$$

Lemma 2.3 follows from Lemma 2.2. □

The isomorphisms of the last two lemmas are not uniquely defined.

3. Spaces of lifts in $K(G_0, n)$-fibrations. In this section we assume that p: $Y \to B$ is a fibration with an Eilenberg-Mac Lane space $K(G_0, n)$, where G_0 is an abelian group, as fibre. Let u: $X \to Y$ be any map and put $u_1 = pu$: $X \to B$.

First assume that p: $Y \to B$ is a principal $K(G_0,n)$-fibration. Then the pull-back $u_1^*(p)$ is a fibre homotopically trivial fibration ([15], II). Hence

$$F_u(X,A;Y,B) = F_{u'}(X,A;K(G_0,n),*)$$

for some map u': $X \to K(G_0,n)$, for $F_u(X,A;Y,B)$ may be interpreted as a space of sections of $u_1^*(p)$. The (relative version of the) theorem of Thom ([15], Théorème 3), ([7], Proposition, p. 609), ([8], Theorem 1) thus asserts that

$$F_u(X,A;Y,B) = \prod_{i=0}^{n} K\big(H^{n-i}(X,A;G_0),i\big)$$

up to weak homotopy type.

Now consider the general case of a not necessarily principal $K(G_0, n)$-fibration $p: Y \to B$. Let G denote the system of local coefficients in B defined by the n-dimensional homotopy groups of the fibres of p. Following the proof of Thom's theorem as it appears in [7], we consider the evaluation map

$$e: F_u(X, A; Y, B) \times X \to Y$$

given by $e(f, x) = f(x)$. Note that

$$e \in F_{u \circ \mathrm{pr}_2}\big((F_u(X, A; Y, B), u) \times (X, A); Y, B\big).$$

For $0 \leq i \leq n$, choose maps

$$e^i: \big(F_u(X, A, Y, B), u\big) \to \big(K\big(H^{n-i}(X, A; u_1^*G), i\big), *\big)$$

such that the array of homotopy classes $([e^0], [e^1], \ldots, [e^n])$ corresponds to the (vertical and relative) homotopy class $[e]$ of e under the composite bijection

$$\begin{aligned}\pi_0\big(F_{u \circ \mathrm{pr}_2}\big((F_u(X, A; Y, B), u) \times (X, A); Y, B\big)\big) \\ = H^n\big((F_u(X, A; Y, B), u) \times (X, A); \mathrm{pr}_2^* u_1^* G\big) \\ \xrightarrow{\Phi(X,A)} \coprod_{0 \leq i \leq n} H^i\big(F_u(X, A; Y, B), u; H^{n-i}(X, A; u_1^* G)\big).\end{aligned}$$

The main result of this section is the following generalization of Thom's theorem ([**15**], I) and the Classification Theorem ([**12**], Theorem II).

THEOREM 3.1. *The map*

$$(e^0, e^1, \ldots, e^n): F_u(X, A; Y, B) \to \prod_{i=0}^{n} K\big(H^{n-i}(X, A; u_1^* G), i\big)$$

is a weak homotopy equivalence.

Proof. For $i \geq 0$, the Exponential Law

$$F_u\big(S^i, *; F_u(X, A; Y, B), u\big) = F_{u \circ \mathrm{pr}_2}\big((S^i, *) \times (X, A); Y, B\big)$$

$$\alpha \to e \circ (\alpha \times 1)$$

induces a bijection

$$\psi^i: \pi_i\big(F_u(X, A; Y, B), u\big) \to H^n\big((S^i, *) \times (X, A); \mathrm{pr}_2^* u_1^* G\big)$$

$$[\alpha] \to (\alpha \times 1)^*[e]$$

between path-components. According to Lemma 2.3 there is a commutative diagram (with $F_u = F_u(X, A; Y, B)$)

$$\begin{array}{ccc} H^n((F_u, u) \times (X, A); \mathrm{pr}_2^* u_1^* G) & \xrightarrow{\Phi_{(X,A)}} & \coprod_{0 \le j \le n} H^j(F_u, u; H^{n-j}(X, A; u_1^* G)) \\ (\alpha \times 1)^* \downarrow & & \downarrow \alpha^* \circ \mathrm{pr}_i \\ H^n((S^i, *) \times (X, A); \mathrm{pr}_2^* u_1^* G) & \xrightarrow{\Phi_{(X,A)}} & H^i(S^i, *; H^{n-i}(X, A; u_1^* G)) \end{array}$$

showing that

$$\begin{aligned} \Phi_{(X,A)} \psi^i([\alpha]) &= \Phi_{(X,A)}(\alpha \times 1)^*[e] \\ &= \alpha^* \circ \mathrm{pr}_i \circ \Phi_{(X,A)}([e]) = \alpha^*([e^i]). \end{aligned}$$

In other words, the bijection

$$\Phi_{(X,A)} \psi^i \colon \pi_i(F_u, u) \to H^i(S^i, *; H^{n-i}(X, A; u_1^* G)) = H^{n-i}(X, A; u_1^* G)$$

equals the homomorphism

$$(e_i)_* \colon \pi_i(F_u, u) \to \pi_i(K(H^{n-i}(X, A; u_1^* G), i), *) = H^{n-i}(X, A; u_1^* G)$$

induced by e_i. Hence $(e_i)_*$ is an isomorphism (for $i \ge 1$) of homotopy groups. □

REMARK 3.2. Let (Z, C) be an NDR-pair and $\alpha \colon (Z, C) \to (F_u(X, A; Y, B), u)$ a map. Then

$$[e \circ (\alpha \times 1)] \in H^n((Z, C) \times (X, A); \mathrm{pr}_2^* u_1^* G)$$

and $e^i \circ \alpha \colon (Z, C) \to (K(H^{n-i}(X, A; u_1^* G), i), *)$ represents

$$\mathrm{pr}_i(\Phi_{(X,A)}([e \circ (\alpha \times 1)])) \in H^i(Z, C; H^{n-i}(X, A; u_1^* G)).$$

An application of Theorem 3.1 to the classifying fibration $\kappa(G, n)$ over $K(\pi, 1)$ yields

COROLLARY 3.3. *The space $\Gamma(\kappa(G, n))$ of sections of $\kappa(G, n)$ has the weak homotopy type of the product*

$$\prod_{i=0}^{n} K(\mathrm{Ext}_\pi^{n-i}(\mathbf{Z}, G_0), i)$$

where $\mathbf{Z}$ is considered as a trivial π-module.

Proof. $H^{n-i}(K(\pi, 1); G) = \mathrm{Ext}_\pi^{n-i}(\mathbf{Z}, G_0)$ by a theorem of Eilenberg ([**16**], Theorem 3.5*, p. 281). □

Note that the additive structure of $H^*(X, A; G)$ suffices to determine the weak homotopy type of $F_u(X, A; Y, B)$ when $p\colon Y \to B$ is a $K(G_0, n)$-fibration; cf. ([**15**], I). This is not true in general.

4. Change of base point. Let $p\colon Y \to B$ be the $K(G_0, n)$-fibration of the previous section and let $u, v\colon X \to Y$ be two maps such that $u \mid A = v \mid A$ and $pu = pv$. Then $F_u(X, A; Y, B) = F_v(X, A; Y, B)$ as free spaces. The purpose of this section is to discuss the relation between the pointed spaces $(F_u(X, A; Y, B), u)$ and $(F_v(X, A; Y, B), v)$.

To clarify the role of the chosen base point, we now write ψ^i_u for the homomorphism ψ^i introduced in the proof of Theorem 3.1. Explicitly,

$$\psi^i_u\colon \pi_i(F_u(X, A; Y, B), u) \to H^n((S^i, *) \times (X, A); \mathrm{pr}_2^* u_1^* G)$$

takes $[\alpha] \in \pi_i(F_u(X, A; Y, B), u)$ to the primary difference

$$\psi^i_u([\alpha]) = \delta^n(u \circ \mathrm{pr}_2, e \circ (\alpha \times 1))$$

of $u \circ \mathrm{pr}_2$ and the adjoint $e \circ (\alpha \times 1)$ of α.

In order to compare ψ^i_u and ψ^i_v, we introduce the set $[S^i, F_u(X, A; Y, B)]$ of free homotopy classes of free maps of S^i into $F_u(X, A; Y, B)$. (Note in this connection that $F_u(X, A; Y, B)$ is a simple space by Theorem 3.1.) Also in this case we get a bijection

$$\psi^i_u\colon [S^i, F_u(X, A; Y, B)] \to H^n(S^i \times (X, A); \mathrm{pr}_2^* u_1^* G)$$

by forming primary differences as above.

Let $\bar{j}\colon \pi_i(F_u(X, A; Y, B), u) \to [S^i, F_u(X, A; Y, B)]$ be the inclusion induced by the inclusion $j\colon S^i \to (S^i, *)$. Then one easily proves:

LEMMA 4.1. *The deviation from commutativity of the diagram*

$$\begin{array}{ccc} \pi_i(F_u(X, A; Y, B), u) & \overset{\psi^i_u}{\to} & H^n((S^i, *) \times (X, A); \mathrm{pr}_2^* u_1^* G) \\ \bar{j} \downarrow & & \downarrow (j \times 1)^* \\ [S^i, F_u(X, A; Y, B)] & \overset{\psi^i_v}{\to} & H^n(S^i \times (X, A); \mathrm{pr}_2^* u_1^* G) \end{array}$$

is given by

$$(j \times 1)^* \circ \psi^i_u - \psi^i_v \circ \bar{j} = \mathrm{pr}_2^* \delta^n(u, v)$$

where $\delta^n(u, v) \in H^n(X, A; u_1^* G)$ *is the primary difference of* u *and* v.

Now assume that $p_1\colon Y_1 \to B_1$ is another fibration with an Eilenberg-Mac Lane space $K(G_0', q)$, G_0' abelian, $q \geq 1$, as fibre and that

$$\begin{array}{ccc} Y & \overset{k}{\to} & Y_1 \\ p\downarrow & & \downarrow p_1 \\ B & \overset{k_1}{\to} & B_1 \end{array}$$

is a fibre map of p into p_1. Let G_1 denote the local coefficient system in B_1 determined by p_1.

For any pair $(Z, C; f)$ over Y and any integer $i \geq 0$, let $\sigma^i[k]_f$ denote the primary twisted cohomology operation that makes the diagram

$$\begin{array}{ccc} \pi_i\big(F_f(Z,C;Y,B),f\big) & \overset{k_*}{\to} & \pi_i\big(F_{kf}(Z,C;Y_1,B_1),kf\big) \\ \Phi_{(Z,C)}\psi^i_f \downarrow \cong & & \cong \downarrow \Phi_{(Z,C)}\psi^i_{kf} \\ H^{n-i}\big(Z,C;f_1^*G\big) & \overset{\sigma^i[k]_f}{\to} & H^{q-i}\big(Z,C;f_1^*k_1^*G_1\big) \end{array}$$

commute. The operation $[k]_f := \sigma^0[k]_f$ is given by $[k]_f\delta^n(f,g) = \delta^q(kf,kg)$ for any $g \in F_f(Z,C;Y,B)$.

In particular, $u\colon X \to Y$ determines operations

$$\sigma^i[k]_u\colon H^{n-i}\big(X,A;u_1^*G\big) \to H^{q-i}\big(X,A;u_1^*k_1^*G_1\big), \qquad i \geq 0,$$

and the maps $u \circ \mathrm{pr}_2\colon X \times S^i \rightarrowtail Y$, $i \geq 0$, determine operations $[k]_{u\circ \mathrm{pr}_2}$ such that

$$\begin{array}{ccc} \big[S^i, F_u(X,A;Y,B)\big] & \overset{k_*}{\to} & \big[S^i, F_{ku}(X,A;Y,B)\big] \\ \psi^i_u \downarrow & & \downarrow \psi^i_{ku} \\ H^n\big(S^i \times (X,A);\mathrm{pr}_2^*u_1^*G\big) & \overset{[k]_{u\circ\mathrm{pr}_2}}{\to} & H^q\big(S^i \times (X,A);\mathrm{pr}_2^*u_1^*k_1^*G_1\big) \end{array}$$

commutes. If $s^i \times -$ denotes the homomorphism that renders

$$\begin{array}{ccc} H^n\big((S^i,*) \times (X,A);\mathrm{pr}_2^*u_1^*G\big) & \overset{(j\times 1)^*}{\to} & H^n\big(S^i \times (X,A);\mathrm{pr}_2^*u_1^*G\big) \\ \Phi_{(X,A)} \downarrow & \nearrow{\scriptstyle s^i \times -} & \\ H^{n-i}\big(X,A;u_1^*G\big) & & \end{array}$$

commutative, then the equation

$$[k]_{u\circ\mathrm{pr}_2}\big(s^i \times \chi\big) = s^i \times \sigma^i[k]_u(\chi), \qquad \chi \in H^{n-i}\big(X,A;u_1^*G\big)$$

shows the relation between $[k]_u$ and $[k]_{u\circ\mathrm{pr}_2}$.

The object of the next theorem is to compare the operations $\sigma^i[k]_u$ and $\sigma^i[k]_v$ induced by two different maps u and v.

THEOREM 4.2. *For any* $\chi \in H^{n-i}(X, A; u_1^*G)$, $i > 0$, *the equality*

$$[k]_{u\circ \mathrm{pr}_2}\big(s^i \times \chi + \mathrm{pr}_2^*\delta^n(u, v)\big) = s^i \times \sigma^i[k]_v(\chi) + \mathrm{pr}_2^*([k]_u\delta^n(u, v))$$

holds in

$$H^q\big(S^i \times (X, A); \mathrm{pr}_2^* u_1^* k_1^* G_1\big) \cong H^{q-i}\big(X, A; u_1^* k_1^* G_1\big) \oplus H^q\big(X, A; u_1^* k_1^* G_1\big).$$

Proof. Some of the introduced maps are related by the following commutative diagram

$$\begin{array}{ccccccc}
\pi_i(F_u, v) & & & \xrightarrow{\underline{k}_*} & & & \pi_i(F_{ku}, kv) \\
 & \searrow^{\bar{j}} & & & & \swarrow^{\bar{j}} & \\
\Big\downarrow \psi_u^i & & [S^i, F_u] & \xrightarrow{\underline{k}_*} & [S^i, F_{ku}] & & \Big\downarrow \psi_{ku}^i \\
 & & \psi_u^i \downarrow & & \downarrow \psi_{ku}^i & & \\
 & & H^n(S^i \times (X, A)) & \xrightarrow{[k]_{u\circ \mathrm{pr}_2}} & H^q(S^i \times (X, A)) & & \\
 & \nearrow_{(j\times 1)^*} & & & & \nwarrow_{(j\times 1)^*} & \\
H^n((S^i, *) \times (X, A)) & & & \longrightarrow & & & H^q((S^i, *) \times (X, A))
\end{array}$$

in which some self explanatory abbreviations occur. In particular

$$(1) \qquad [k]_{u\circ \mathrm{pr}_2}\big(\psi_u^i \bar{j}[\alpha]\big) = (j \times 1)^* \psi_{ku}^i \underline{k}_*[\alpha]$$

for any homotopy class $[\alpha] \in \pi_i(F_u(X, A; Y, B), v)$. If $\psi_v^i \bar{j}[\alpha] = \chi$, then by Lemma 4.1,

$$\psi_u^i \bar{j}[\alpha] = (j \times 1)^* \psi_v^i[\alpha] + \mathrm{pr}_2^* \delta^n(u, v) = s^i \times \chi + \mathrm{pr}_2^* \delta^n(u, v),$$

so the left hand side of (1) becomes

$$[k]_{u\circ \mathrm{pr}_2}\big(\psi_u^i \bar{j}[\alpha]\big) = [k]_{u\circ \mathrm{pr}_2}\big(s^i \times \chi + \mathrm{pr}_2^* \delta^n(u, v)\big).$$

The right hand side of (1) can be rewritten, using Lemma 4.1 for the first equality, as follows

$$\begin{aligned}
(j \times 1)^* \psi_{ku}^i \underline{k}_*[\alpha] &= \psi_{kv}^i \bar{j} \underline{k}_*[\alpha] + \mathrm{pr}_2^* \delta^q(ku, kv) \\
&= [k]_{v\circ \mathrm{pr}_2}\big((j \times 1)^* \psi_v^i[\alpha]\big) + \mathrm{pr}_2^*[k]_u \delta^n(u, v) \\
&= [k]_{v\circ \mathrm{pr}_2}\big(s^i \times \chi\big) + \mathrm{pr}_2^*[k]_u \delta^n(u, v) \\
&= s^i \times \sigma^i[k]_v(\chi) + \mathrm{pr}_2^*[k]_u \delta^n(u, v). \qquad \square
\end{aligned}$$

Consequently, $[k]_u = [k]_v$ if $[k]_{u \circ \mathrm{pr}_2}$ happens to be an additive operation. On the other hand, examples do occur, see e.g. [**11**], where $[k]_u \neq [k]_v$.

5. Spaces of maps into twisted Eilenberg-Mac Lane spaces. Suppose that both π and G_0 are abelian groups, $\varphi\colon \pi \to \mathrm{Aut}(G_0)$ an action of π on G_0, and

$$K(G_0, n) \to K(G_0, n; \varphi) \overset{\hat{k}}{\rightleftarrows} K(\pi, 1)$$

the associated classifying fibration $\kappa(G, n)$. The purpose of this section is to describe mapping spaces with the total space $K(G_0, n; \varphi)$ as target.

The classifying fibration $\kappa(G, n)$ can be constructed more explicitly as follows. The Eilenberg-Mac Lane space $K(G_0, n)$ can be made into a left π-space in such a way that each $\xi \in \pi$ acts as a base-point preserving homeomorphism with the induced map

$$\xi_*\colon \pi_n(K(G_0, n), *) \to \pi_n(K(G_0, n), *)$$

equal to $\xi\colon G_0 \to G_0$ under some fixed isomorphism $\pi_n(K(G_0, n), *) \cong G_0$. The fibre bundle

$$K(G_0, n) \to E\pi \times_\pi K(G_0, n) \overset{\hat{k}}{\to} B\pi$$

associated to the universal principal π-bundle $\omega\colon E\pi \to B\pi$ is then a $\kappa(G, n)$.

Let $u\colon X \to K(G_0, n; \varphi) = E\pi \times_\pi K(G_0, n)$ be any map into the total space of $\kappa(G, n)$. Put $u_1 = \hat{k}u$. Consider the fibration of function spaces

$$F_u(X; K(G_0, n; \varphi), B\pi) \to F_u(X; K(G_0, n; \varphi), *) \overset{\hat{k}}{\to} F_{u_1}(X; B\pi, *)$$

induced by the projection $\hat{k}$. The base space $F_u(X; B\pi, *) = H^1(X; \pi) \times K(\pi, 1)$ is disconnected (in general), so we let $F^0_{u_1}(X; B\pi, *) = K(\pi, 1)$ denote the path-component of $F_u(X; B\pi, *)$ containing u_1 and concentrate our attention on the pre-image $F_u(X; K(G_0, n; \varphi), *) \mid u_1 = \underline{\hat{k}}^{-1}(F^0_{u_1}(X; B\pi, *))$. By restriction of $\underline{\hat{k}}$ we then get the fibration

$$\prod_{i=0}^{n} K\big(H^{n-i}(X; u_1^*G), i\big) \to F_u(X; K(G_0, n; \varphi), *) \mid u_1 \to K(\pi, 1)$$

where Theorem 3.1 has been used to identify the fibre.

Since π is abelian, $\xi\colon G_0 \to G_0$, $\xi \in \pi$, is an operator automorphism, i.e. an automorphism of the local coefficient system G in $K(\pi, 1)$, and hence ξ induces a coefficient group automorphism ξ_* of $H^{n-i}(X; u_1^*G)$, $0 \leq i \leq n$.

After these preliminaries we can now state

THEOREM 5.1. *There is a weak (fibre) homotopy equivalence*

$$F_u(X; K(G_0, n; \varphi), *) \mid u_1 \to E\pi \times_\pi \left(\prod_{i=0}^{n} K\left(H^{n-i}(X; u_1^*G), i\right)\right),$$

where π acts on $H^{n-i}(X; u_1^*G)$, $0 \leq i \leq n$, through coefficient group automorphisms.

Proof. The cohomology operation ξ_* can be realized geometrically as in §4. For the based automorphism ξ of $K(G_0, n)$ is a π-map, and hence it extends to a homeomorphism ξ: $K \to K$ over and under $B\pi$. (Here, and in the following, $K = K(G_0, n; \varphi) = E\pi \times_\pi K(G_0, n)$.) As is easily seen, the i-fold suspension $\sigma^i[\xi]_u$ of the corresponding cohomology operation $[\xi]_u$ is the coefficient group automorphism ξ_*: $H^{n-i}(X; u_1^*G) \to H^{n-i}(X; u_1^*G)$, $0 \leq i \leq n$.

Since π is abelian, there exist H-space structures $\bar{\mu}$: $E\pi \times E\pi \to E\pi$, μ: $B\pi \times B\pi \to B\pi$ with strict units $e_0 \in E\pi$, $b_0 = \omega(e_0) \in B\pi$ such that $\mu \circ (\omega \times \omega) = \omega \circ \bar{\mu}$. The unique path lifting property implies that $\bar{\mu}(e_1\xi, e_2) = \bar{\mu}(e_1, e_2)\xi = \bar{\mu}(e_1, e_2\xi)$ for all $e_1, e_2 \in E\pi$, $\xi \in \pi$.

The space $F^1(X; K, B\pi)$ of lifts of u_1 is a left π-space under composition with the fibre maps ξ: $K \to K$, $\xi \in \pi$. Let

$$\bar{\psi}\colon E\pi \times_\pi F_u(X; K, B\pi) \to F_u(X; K, *)$$

be the map given by

$$\bar{\psi}((e, v)\pi)(x) = \left(\bar{\mu}(e, \bar{u}_1(x)), \hat{v}(x)\right)\pi$$

where $e \in E\pi$, $v \in F_u(X; K, B\pi)$, $x \in X$, $\bar{u}_1(x) \in E\pi$ is any lift of $u_1(x) \in B\pi$, and $v(x) \in K$ and $\hat{v}(x) \in K(G_0, n)$ are related by the formula $v(x) = (\bar{u}_1(x), \hat{v}(x))\pi$.

Note that $\bar{\psi}$ is a fibre map which restricts to the identity on the fibre. The induced map ψ: $B\pi \to F_{u_1}(X; B\pi, *)$ between the base spaces satisfies $\psi(b, x) = \mu(b, u_1(x))$, $b \in B\pi$, $x \in X$. This means that ψ is a homotopy equivalence berween$B\pi$ and $F_{u_1}^0(X; B\pi, *)$. Hence $\bar{\psi}$ is a fibre homotopy equuivalence from $E\pi \times_\pi F_u(X; K, B\pi)$ to $F_u(X; K, *) \mid u_1$ by Dold [2].

The proof is now completed by noting that the weak homotopy equivalence of $F_u(X; K, B\pi)$ into $\prod_{i=0}^{n} K(H^{n-i}(X; u_1^*G), i)$ from Theorem 3.1 is a π-map enabling us to construct a weak homotopy equivalence

$$E\pi \times_\pi F_u(X; K, B\pi) \to E\pi \times_\pi \prod_{i=0}^{n} K\left(H^{n-i}(X; u_1^*G), i\right)$$

as claimed. □

REMARK 5.2. During the proof of Theorem 5.1 we actually established the identity

$$F_u(X; E\pi \times_\pi F, *) \mid u_1 = E\pi \times_\pi F_u(X; E\pi \times_\pi F, B\pi)$$

for any left π-space F and any map $u\colon X \to E\pi \times_\pi F$.

EXAMPLE 5.3. The classifying space BO(2) for the orthogonal group O(2) is the twisted Eilenberg-Mac Lane space $K(\mathbf{Z}, 2; \varphi)$ where $\varphi\colon \mathbf{Z}/2 \to \mathrm{Aut}(\mathbf{Z})$ is the non-trivial action.

Let u: BO(1) → BO(2) be any map. Then up to homotopy, $u_1 = 0$ or $u_1 = w_1$, the first Stiefel-Whitney class. An application of Theorem 5.1 yields

$$F_u(\mathrm{BO}(1); \mathrm{BO}(2), *) \mid 0 = \mathrm{BO}(2) + \mathrm{BO}(2),$$
$$F_u(\mathrm{BO}(1); \mathrm{BO}(2), *) \mid w_1 = \mathrm{BO}(1) \times \mathrm{BO}(1)$$

where + denotes disjoint union.

6. Spaces of lifts in $K(G,1)$-fibrations. In this section we let $p\colon Y \to B$ denote a fibration with an aspherical space $F = K(G,1)$ as fibre. G can be any, not necessarily abelian, group. We shall investigate the space $F_u(X, A; Y, B)$.

The pull-back $F \xrightarrow{i'} Y' \xrightarrow{p'} X$ of $F \xrightarrow{i} Y \xrightarrow{p} B$ along $u_1 = pu$ has a canonical section $u'\colon X \to Y'$ induced from u. Hence $i'_*\colon \pi_1(F) \to \pi_1(Y')$ is a monomorphism and a homomorphism $\varphi_u\colon \pi = \pi_1(X) \to \mathrm{Aut}(G)$ is uniquely defined $i'_*(xg) = u'_*(x) i'_*(g) u'_*(x)^{-1}$, $x \in \pi$, $g \in G$. We write xg for $\varphi_u(x)g$. Let

$$G^\pi = \{g \in G \mid \pi g = g\}$$

denote the fixpoint set of this action and let

$$Q(\pi, G) = \{f\colon \pi \to G \mid \forall x, y \in \pi\colon f(xy) = f(x) x f(y)\}$$

denote the set of crossed homomorphisms of π into G. There is an action

$$Q(\pi, G) \times G \to Q(\pi, G)$$

of G on the set of crossed homomorphisms given by $(fg)(x) = g^{-1} f(x) xg$, $f \in Q(\pi, G)$, $g \in G$, $x \in \pi$. $Q(\pi, G)/G$ denotes the set of orbits for this action.

Let $x_0 \in X$ be the base point. To any based lift $v \in F_u(X, x_0; Y, B)$ of u_1, we can associate a crossed homomorphism $f_v \in Q(\pi, Q)$ given by $i'_* f_v(x) = v'_*(x) u'_*(x)^{-1}$, where $v'\colon X \to Y'$ is the section of p' induced from v. By some obvious modifications of the classification of based homotopy classes of based maps into an aspherical space ([**16**], Theorem 4.3, p. 225) we get

LEMMA 6.1. *For any connected CW-complex X, the map $v \to f_v$ induces a bijective correspondence between $\pi_0 F_u(X, x_0; Y, B)$ and $Q(\pi, G)$.*

Also the free vertical homotopy classes of free lifts of u_1 can be classified; cf. ([**16**], Corollary 4.4, p. 226).

LEMMA 6.2. *For any connected CW-complex X, there is a bijective correspondence between $\pi_0 F_u(X; Y, B)$ and $Q(\pi, G)/G$.*

Proof. The sets $F_u(X, x_0; Y, B)$ and $F_u(X; Y, B)$ of based and free lifts of u_1 are related by the evaluation fibration

$$F_u(X, x_0; Y, B) \to F_u(X; Y, B) \to F_u(x_0; Y, B) = F.$$

This evaluation fibration determines an action $Q(\pi, G) \times G \to Q(\pi, G)$ of the fundamental group $G = \pi_1(F)$ of its base space on the set $\pi_0 F_u(X, x_0; Y, B) = Q(\pi, G)$ of path-components of its fibre. We must show that this action coincides with the one introduced above.

Since X is connected, we may assume that the 1-skeleton X_1 is a wedge of circles. The inclusion map i_1: $X_1 \to X$ induces an injection $\bar{i}_{1*}$: $Q(\pi, G) \to Q(\pi_1(X_1), G)$ which is compatible with the G-action. Therefore, we may assume that $X = X_1$ is 1-dimensional. Furthermore, since a crossed homomorphism of $\pi_1(X_1)$ into G is uniquely determined by its value on a set of free generators, we can assume that $X = S^1$ consists of a single circle.

Let h: $(I, \dot{I}) \to (S^1, x_0)$ be the usual proclusion representing the generator $\iota \in \pi_1(S^1, x_0)$. Choose a map H: $I \times F \to Y'$ such that the diagram

$$\begin{array}{ccc} I \times F & \xrightarrow{H} & Y' \\ {\scriptstyle pr_1}\downarrow & & \downarrow {\scriptstyle p'} \\ I & \xrightarrow{h} & S^1 \end{array}$$

commutes and such that $H(t, y_0) = u'(t)$, $y_0 = u(x_0)$, $t \in I$, and $H_0 = i'$: $F \to Y'$. Then ([**9**], Theorem 1), $(H_1)_* = \iota^{-1} \in \operatorname{Aut} G$.

Consider the following diagram of maps between fibrations induced by h and H

$$\begin{array}{ccccc} F_u(S^1, x_0; Y, B) & \xrightarrow{\bar{h}} & F_{uh}(I, \dot{I}; Y, B) & \leftarrow & F_{y_0}(I, \dot{I}; F) \\ \downarrow & & & & \downarrow \\ F_u(S^1; Y, B) & \to & F_{uh}(I; Y, B) & \leftarrow & F(I; F) \\ \downarrow & & & & \downarrow \\ F_u(x_0; Y, B) & \to & F_{uh}(\dot{I}; Y, B) & \leftarrow & F(\dot{I}; F) \end{array}$$

The maps between the fibers are homeomorphisms ([**14**], p. 530) and the maps between the base spaces can be identified to

$$F \xrightarrow{\Delta} F \times F \xleftarrow{1 \times H_1} F \times F$$

where Δ is the diagonal map.

The fibre $F_{y_0}(I, \dot{I}; F)$ of the fibration to the right is the loop space ΩF of F and the associated action of $\pi_1(F(\dot{I}; F), y_0) = G \times G$ on $\pi_0 F_{y_0}(I, \dot{I}; F) = \pi_0(\Omega F) = G$ is given by $g_1 \cdot (h_0, h_1) = h_0^{-1} g_1 h_1$ for all $g_1, h_0, h_1 \in G$. Hence the corresponding action of $\pi_1(F_u(x_0; Y, B), y_0) = G$ on $\pi_0 F_u(S^1, x_0; Y, B) = Q(\pi_1(S^1), G) = G$ is given by $g_1 \cdot g = g^{-1} g_1 \iota g$, $g \in G$. Taking into account the identifications made, this means that

$$(fg)(z) = g^{-1} f(z) z g$$

for all $f \in Q(\pi_1(S^1), G)$, $g \in G$, $z \in \pi_1(S^1)$. □

Finally, we compute the higher homotopy groups of $F_u(X, x_0; Y, B)$ and $F_u(X; Y, B)$. More generally, let (X, A) be a finite relative CW-complex where both X and A are 0-connected. Assume that (X, A) has a CW-decomposition with 0-skeleton $X_0 = A$ if $A \neq \varnothing$ and $X_0 = \{x_0\}$ if $A = \varnothing$.

THEOREM 6.3. (1) *If $A \neq \varnothing$, each component of $F_u(X, A; Y, B)$ is weakly contractible.*

(2) *If $A = \varnothing$, each component of $F_u(X; Y, B)$ is an aspherical space. The fundamental group $\pi_1(F_u(X; Y, B), u)$ of the component containing u is isomorphic to the fixpoint set G^{π}.*

Proof. We proceed as in ([**8**], Theorem 2). Let X_q be the q-skeleton of a CW-decomposition of (X, A) such that $X_0 = A$ if $A \neq \varnothing$ and $X_0 = \{x_0\}$ if $A = \varnothing$. The inclusion maps $i_q \colon X_{q-1} \to X_q$ induce a tower of fibrations

$$F_u(X,A;Y,B) \to F_u(X_q,A;Y,B) \overset{\bar{i}_q}{\to} F_u(X_{q-1},A;Y,B)$$

$$\to \cdots \to F_u(X_2,A;Y,B) \overset{\bar{i}_2}{\to} F_u(X_1,A;Y,B) \overset{\bar{i}_1}{\to} F_u(X_0,A;Y,B).$$

The fibre $F_u(X_q, X_{q-1}; Y, B)$ of $\bar{i}_q$ can be identified to a product of a number of copies of the q-fold loop space $\Omega^q F$. The number of factors equals the number of q-cells in (X, A). Since $F = K(G,1)$ is aspherical, it follows that $F_u(X, A; Y, B)$ and $F_u(X_1, A; Y, B)$ are weakly homotopy equivalent. Moreover, if $A \neq \varnothing$,

$$F_u(X_1,A;Y,B) \simeq \Omega F \times \cdots \times \Omega F \simeq G \times \cdots \times G$$

is just a discrete set of points.

If $A = \varnothing$, we consider the evaluation fibration

$$F_u(X,x_0;Y,B) \to F_u(X;Y,B) \to F_u(x_0;Y,B) = F$$

with the discrete fibre $F_u(X, x_0; Y, B) = F_u(X_1, x_0; Y, B)$. In the associated homotopy sequence

$$1 \to \pi_1(F_u(X;Y,B),u) \to G \overset{\partial}{\to} Q(\pi,G) \to \pi_0 F_u(X;Y,B) \to *$$

one has $\partial g = 1g$ for all $g \in G$. Hence

$$\pi_1(F_u(X;Y,B),u) \cong \operatorname{kern}\partial = \{g \in G \mid 1g = g\} = G^{\pi}. \qquad \square$$

If $p = \mathrm{pr}_1$: $B \times K(G,1) \to B$ is the trivial $K(G,1)$-fibration over B and $u = (b_0, u)$: $X \to \{b_0\} \times K(G,1) \subset B \times K(G,1)$ a continuous map, the action of π on G is given by $xg = u_*(x)gu_*(x)^{-1}$. Thus the fixpoint set G^{π} is the centralizer of $u_*(\pi_1(X))$ in G. In this way we recover the theorem of Gottlieb **[6]**.

If G is abelian, the fibration p: $Y \to B$ determines a system of local coefficients, also denote by G, in B. The pull-back u_1^*G in X is given by φ_u: $\pi \to \mathrm{Aut}(G)$. Since $Q(\pi, G) \cong H^1(X, x_0; u_1^*G)$, $Q(\pi, G)/G \cong H^1(X; u_1^*G)$, and $G^{\pi} = H^0(X; u_1^*G)$, 6.1–6.3 reduce to Theorem 3.1 for $n = 1$ in this case.

Although suppressed in the used notation, the group G^{π} in general depends on the choice of u. Thus the components of $F_u(X; Y, B)$ may represent more than just one (weak) homotopy type.

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