PACIFIC JOURNAL OF MATHEMATICS
Vol. 130, No. 1, 1987

QUOTIENTS OF THE COMPLEX BALL BY DISCRETE GROUPS

F. C. KIRWAN, R. LEE, AND S. H. WEINTRAUB

In this paper we systematically study varieties $Q(\underline{\mu})$, which are compactifications of the space Q of distinct points in $(\mathbf{P}^1)^r$ given by a sequence of "weights" $\underline{\mu}$, and which for certain $\underline{\mu}$ are also compactification of the quotient of the complex r-ball by discrete subgroups $\Gamma(\underline{\mu})$ of $\mathrm{PU}(r,1)$, as discovered by Deligne and Mostow.

We obtain a wealth of topological information about the spaces $Q(\underline{\mu})$ and their desingularizations $Q^*(\underline{\mu})$. In some cases we can completely describe them. Otherwise, we obtain computations of Betti numbers and Hodge numbers. As applications we determine the L^2-cohomology and in many cases the (ordinary) rational cohomology of the groups $\Gamma(\underline{\mu})$.

0. Introduction. In this paper we study a family of algebraic varieties which arise in two ways. The first is as quotients of the ball in $\mathbf{C}^r$ by discrete subgroups of $\mathrm{PU}(r,1)$, and the second as various compactifications of the configuration space Q of r distinct points in $\mathbf{P}^1$.

They were first discovered by Deligne and Mostow ([**DM**], [**M**]) where they arose through the investigation of generalized hypergeometric functions. We briefly recapitulate their work in §2 below.

We wish to study those varieties systematically. In this connection, we find that the second viewpoint, in terms of Mumford's geometric invariant theory [**Mu**], is more useful. Let $N = r + 3$ and let $\underline{\mu} = (\mu_1, \dots, \mu_N)$ be a sequence of positive integers (which we call "weights"). Associated to $\underline{\mu}$ is a line bundle over $(\mathbf{P}^1)^N$, and hence an r-dimensional projective variety $Q(\underline{\mu})$ obtained by taking the semistable points with respect to the linear action of PGL_2 and then forming the quotient space in the sense of geometric invariant theory (see §6). $Q(\underline{\mu})$ is a compactification of Q, and the varieties of Deligne and Mostow arise as $Q(\underline{\mu})$ for $\underline{\mu}$ satisfying certain arithmetical conditions.

These varieties $Q(\underline{\mu})$ are always rational (1.11). When $r = 2$, and when $r = 3$ and $Q(\underline{\mu})$ is nonsingular, they can be completely described (4.1). (When $r = 2$ the possibilities for $Q(\underline{\mu})$ are $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$ with k points blown up, $0 \le k \le 4$. When $r = 3$ the possibilities are more complicated.) In this case our work also determines the rational cohomology of the associated discrete subgroup $\Gamma(\underline{\mu})$ of $\mathrm{PU}(r,1)$.

In general $Q(\underline{\mu})$ is singular, and has a desingularization $Q^*(\underline{\mu})$ obtained by blowing up its singular "cusps" (1.10). The basic topological invariants of these varieties $Q^*(\underline{\mu})$ are their Betti numbers and Hodge numbers, for which we have a complete analysis (7.14). One case of particular interest to us is that of $\underline{\mu} = (1\ 1\ 1\ 1\ 1\ 1)$, where the three-fold $Q^*(\underline{\mu})$ coincides with the moduli space of curves studied extensively in [$\mathbf{LW}_1$], [$\mathbf{LW}_2$].

For the singular varieties $Q(\underline{\mu})$, it is natural to compute their intersection homology, which we do in (8.6). Since these singular varieties can be identified with Baily-Borel-Satake compactifications of the space Q their intersection homology is the same as L^2-cohomology. Thus our results give computations of L^2-cohomology, and as far as we know the first complete description of these cohomology groups for lattices in $\mathrm{PU}(r, 1)$.

We proceed as follows: In §1 we establish notation, and define and prove some basic facts about the spaces $Q^*(\underline{\mu})$. In §2 we describe more carefully the relations between these spaces and the work of [**DM**] and [**M**], and give their conditions on $\underline{\mu}$ for $\Gamma(\underline{\mu})$ to be a discrete subgroup of $\mathrm{PU}(r, 1)$. In §3 we divide the sequences of weights into equivalence classes, with equivalent sequences of weights having the same quotient. In §4 we identify the quotients, in those cases where we are able to do so. When $r = 2$, the quotients are complex surfaces, and these surfaces contain particular configurations of lines. We identify these in §5. (They are related to the configuration studied by Hirzebruch in [**H**].)

In §6 we begin our computation of the cohomology of $Q^*(\underline{\mu})$ in those cases not dealt with in §4. (Indeed, some of the descriptions of $Q^*(\underline{\mu})$ in §4 were first suggested by the computation of the cohomology of these spaces.) Section 6 explains the connection between the spaces we study and Mumford's geometric invariant theory (a connection adumbrated in §1), and §7 explains the method of [$\mathbf{K}_1$] and [$\mathbf{K}_2$] for calculating $H^*(Q^*(\underline{\mu}))$, and performs these calculations. Section 8 explains how to calculate the intersection homology of the singular variety $Q(\underline{\mu})$, though here we leave the actual calculations to the reader.

In §9 we discuss the question of the representability of homology classes of $Q^*(\underline{\mu})$ by algebraic subvarieties.

Finally, the subspace $Q_{\mathrm{st}}(\underline{\mu})$ of $Q^*(\underline{\mu})$ is the actual quotient of the ball by the group $\Gamma(\underline{\mu})$. Every point of the ball has finite stabilizer, so the rational cohomology of $Q_{\mathrm{st}}(\underline{\mu})$ is that of $\Gamma(\underline{\mu})$. In the cases we are able to, we compute that cohomology in §10.

1. Preliminaries. We shall use the language and notation of [**DM**] here, with some minor differences. Our results here are valid over any algebraically closed field k. To begin with, we have the following data:

(1.1) $N \geq 3$ an integer, $S = \{1, \ldots, N\}$

$\underline{\mu} = (\mu_s \mid s \in S)$ a non-increasing sequence of positive integers (i.e. $\mu_i \geq \mu_j$ if $i < j$) with $\sum_{s \in S} \mu_s = d$ such that $\mu_s < d/2$ for all $s \in S$.

We call a sequence $\underline{\mu} = (\mu_s)$ as in 1.1 a sequence of weights.

We let $\mathbf{P} = P^1(k)$ and $\mathbf{P}^S = \{(y_s)_{s \in S}\}$ be the space of functions from S to $\mathbf{P}$. (We may naturally identify $\mathbf{P}^S$ with $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$, where there are N factors in the product.) We let $M \subset \mathbf{P}^S$ be the subspace of injective maps from S to $\mathbf{P}$. The group $\mathrm{PGL}_2 = \mathrm{PGL}_2(k)$ acts on $\mathbf{P}^S$ by $g((y_s)_{s \in S}) = ((gy_s)_{s \in S})$, where $g \in \mathrm{PGL}_2$ acts on $\mathbf{P}$ by Möbius transformations. This action leaves M invariant, and we are interested in studying the quotient $Q = \mathrm{PGL}_2 \backslash M$. More precisely, we are interested in compactifying this quotient.

Definition 1.2. A point $y \in \mathbf{P}^S$ is called stable (resp. semi-stable) if for all $z \in \mathbf{P}$, $\sum_{y(s) = z} \mu_s < d/2$ (resp. $\leq d/2$).

The set of all stable (resp. semi-stable) points is denoted M_{st} (resp. M_{sst}). We have $M \subset M_{\mathrm{st}} \subset M_{\mathrm{sst}}$ and we set $M_{\mathrm{cusp}} = M_{\mathrm{sst}} - M_{\mathrm{st}}$. We call a point $y \in M_{\mathrm{cusp}}$ strictly semi-stable.

If S_1 is a subset of S, we will let $\underline{\mu}(S_1)$ denote $\sum_{s \in S_1} \mu_s$.

For each partition $\{S_1, S_2\}$ of S with $\underline{\mu}(S_i) = d/2$ $(i = 1, 2)$, the points y in $\mathbf{P}^S$ with $y(S_1) \cap y(S_2) = \varnothing$ and y constant on S_1 or S_2 are strictly semi-stable, and each strictly semi-stable point arises in this way, from a unique partition. While the partition only depends on the unordered pair $\{S_1, S_2\}$, we adopt the following convention:

(1.3) If $\{S_1, S_2\}$ is a partition determining a point in M_{cusp}, then $c(S_1) \leq c(S_2)$, where $c(\ )$ denotes cardinality.

On M_{sst} we define a relation $\sim$ by $y \sim y'$ if and only if

(1.4) (i) $y, y' \in M_{\mathrm{st}}$ and are in the same orbit of the action of PGL_2,
(ii) $y, y' \in M_{\mathrm{cusp}}$ and the partitions determining y and y' coincide.

We set

$$(1.5) \qquad Q_{\mathrm{sst}} = M_{\mathrm{sst}}/\sim, \quad Q_{\mathrm{st}} = M_{\mathrm{st}}/\sim, \quad Q_{\mathrm{cusp}} = M_{\mathrm{cusp}}/\sim$$

each with its quotient topology. Note $Q \subset Q_{\mathrm{st}} \subset Q_{\mathrm{sst}}$, and in fact Q is a Zariski open set in Q_{sst}. The elements of Q_{cusp} are uniquely determined by their partitions, and so Q_{cusp} is a finite set (which may be empty). We shall call $q \in Q_{\mathrm{cusp}}$ a cusp. We have:

(1.6) Q_{sst} is a projective variety, with $\dim_k Q_{\mathrm{sst}} = r = N - 3$.

(1.7) Q_{sst} is non-singular if and only if either $Q_{\text{cusp}} = \varnothing$, or else for all $q \in Q_{\text{cusp}}$, if $\{S_1, S_2\}$ is the partition determining q, then $c(S_1) = 2$. The only singular points of Q_{sst} are the cusps not satisfying this condition.

The above is all proven in **[DM]**, following **[MF]**. We shall refine 1.7 in 1.10 below.

DEFINITION 1.8. Let $\underline{\mu} = (\mu_s)_{s\in S}$ and $\underline{\mu}' = (\mu'_s)_{s\in S}$ be two sequences of weights. We call $\underline{\mu}$ and $\underline{\mu}'$ equivalent if they have the following property. For all subsets S_1 of S, $\underline{\mu}(S_1) < d/2$ (resp. $= d/2$, $> d/2$) if and only if $\underline{\mu}'(S_1) < d'/2$ (resp. $= d'/2$, $> d'/2$).

The following is then immediate:

(1.9) If $\underline{\mu}$ and $\underline{\mu}'$ are equivalent sequences of weights, then Q_{sst} defined with respect to $\underline{\mu}'$ may be identified with Q_{sst} defined with respect to $\underline{\mu}$, with identification restricting to the identification of cusps in each space determined by the same partition.

PROPOSITION 1.10. *Let Q^*_{sst} be the non-singular variety obtained from Q_{sst} by blowing up each singular $q \in Q_{\text{cusp}}$. If $\pi\colon Q^*_{\text{sst}} \to Q_{\text{sst}}$ is the blowdown map, and q is determined by a partition $\{S_1, S_2\}$, then $\pi^{-1}(q) = \mathbf{P}^{c(S_1)-2}(k) \times \mathbf{P}^{c(S_2)-2}(k)$.*

Proof. For ease of notation, let us assume that $S_1 = \{1, \ldots, n\}$, $S_2 = \{n+1, \ldots, N\}$. Let $m \in M_{\text{cusp}}$ be the point $(0, 0, \ldots, 0, \infty, \ldots, \infty)$ (n zeroes and $N - n$ infinities) which projects to q. In order to blow up q, we must replace it by the set of lines *in* M_{st} which pass through M, modulo the action of PGL_2. Thus

$$\pi^{-1}(q) = \text{PGL}_2 \backslash \{m' = (z_1, \ldots, z_n, w_{n+1}, \ldots, w_N) \mid m' \in M_{\text{st}}\}/k^*$$

where the action of k^* on the right is the usual action

$$(z_1, \ldots, z_n, w_{n+1}, \ldots, w_N)t = (z_1 t, \ldots, z_n t, w_{n+t}, \ldots, w_N t)$$

which identifies points which are on the same line.

The condition $m' \in M_{\text{st}}$ means that not all the z_i can be zero, and not all the w_j can be ∞. Since we are interested in a neighborhood of m we can assume that the z_i are near zero and the w_i are near ∞, so that in particular $z_i \neq w_j$.

In order to analyze the quotient we may, after acting on m' by a suitable element of PGL_2, assume that $z_1 = 0$ and $w_{n+1} = \infty$. Then we must further divide out by the subgroup of PGL_2 which fixes 0 and ∞, which we may identify with k^*. Thus

$$\pi^{-1}(q) = k^* \backslash \{(0, z_2, \ldots, z_n, \infty, w_{n+2}, \ldots, w_N)\}/k^*.$$

Note, however, that the action of k^* on the left is given by

$$\lambda(0, z_2, \ldots, z_n, \infty, w_{n+2}, \ldots, w_N) = \left(0, \lambda z_2, \ldots, \lambda z_n, \infty, \lambda^{-1} w_{n+2}, \ldots, \lambda^{-1} w_N\right).$$

Since for any u, $v \in k^*$ we may simultaneously solve $\lambda t = u$, $\lambda^{-1} t = v$, it readily follows that $\pi^{-1}(q) = \mathbf{P}^{n-2}(k) \times \mathbf{P}^{N-n-2}(k)$, as claimed.

PROPOSITION 1.11. *For any sequence of weights $\underline{\mu}$ satisfying conditions* (1.1), $Q^*(\underline{\mu})$ *is rational.*

Proof. Q is a Zariski open set in $Q^*(\underline{\mu})$. Also,

$$Q = \mathrm{PGL}_2 \backslash M = \mathrm{PGL}_2 \backslash \{(z_1, \ldots, z_n\} \mid z_i \neq z_j\}$$
$$= \{(\infty, 0, 1, z_4, \ldots, z_N) \mid z_i \neq z_j, z_i \neq 0, 1, \infty\}$$

which, under the projection onto the last $N-3$ coordinates, may be identified with a Zariski open set in $(\mathbf{P}^1)^{N-3}$. Hence $Q^*(\underline{\mu})$ is birationally equivalent to $(\mathbf{P}^1)^{N-3}$, and so is rational.

2. The work of Deligne and Mostow. In this section we briefly recall the appearance of the varieties Q_{sst} in the work of **[DM]** and **[M]**. They showed that in the case $k = \mathbf{C}$ they arise as follows:

Let $\lambda_i = 2\mu_i/d$, and consider the path integrals, for $i \geq 2$ and $(z_1, \cdots, , z_N) \in M$

$$F_i(z_1, \ldots, z_N) = \int_{z_i}^{z_1} \prod_{i=2}^{N} (u - z_i)^{-\lambda_i} du \quad \text{when } z_1 = \infty$$
$$= \int_{z_i}^{z_1} \prod_{i=1}^{N} (u - z_i)^{-\lambda_i} du \quad \text{when all } z_i \text{ are finite.}$$

(Note we may pass from the first of these to the second by a Möbius transformation.) It turns out that there are $(r+1)$ linearly independent integrals among these. Of course, these integrals are not yet well-defined, as they depend on the choice of a path of integration. Thus by choosing such an independent set we obtain a multi-valued map from M to $\mathbf{C}^{r+1} - \{0\}$. The action of PGL_2 on M multiplies each of these integrals by the same factor, so we obtain a multi-valued map from $Q = \mathrm{PGL}_2 \backslash M$ to $\mathbf{P}^r$, or, more precisely, a well-defined map from the universal cover $\tilde{Q}, \tilde{f}\colon \tilde{Q} \to \mathbf{P}^r$. This map $\tilde{f}$ is equivariant under the action of $\pi_1(Q)$ by covering translations, and so gives a map $\pi_1(Q) \to \mathrm{Aut}(\mathbf{P}^{N-3}) = \mathrm{PGL}(N-2)$ with image Γ. If $\underline{\mu}$ satisfies condition INT below then Γ is discrete in $\mathrm{PU}(N-3, 1) \subset \mathrm{PGL}(N-2)$, in which case Γ has a fundamental domain for its action on a ball B^+ in $\mathbf{P}^{N-3}$ and $Q_{\mathrm{st}} = B^+/\Gamma$. In fact, the action of Γ extends to the closure $\overline{B}^+$ of B^+ (in an unusual topology) and $Q_{\text{sst}} = \overline{B}^+/\Gamma$.

In case $\underline{\mu}$ satisfies a weaker condition Σ INT below then a variation of the construction applies: Let Q' be the subset of Q consisting of the image of points $y \in \mathbf{P}^S$ with $y(i) \neq y(j)$ for $i, j \in S_1$, so that the symmetric group $\Sigma = \Sigma_n$, $n = c(S_1)$, operates freely on Q'. The quotient Q'/Σ plays the role of Q above, giving a discrete subgroup Γ_Σ of $\mathrm{PU}(N-3, 1)$. (In case $\underline{\mu}$ satisfies INT as well this construction gives a diagram

$$\begin{array}{l} Q_{\text{sst}} = \overline{B}^+/\Gamma \\ \downarrow \\ Q_{\text{sst}}/\Sigma = \overline{B}^+/\Gamma_\Sigma \end{array}$$

with $\Gamma_\Sigma/\Gamma = \Sigma_n$.) However, in this case Q_{st} will only be a V-manifold, i.e. will have finite quotient singularities.

(2.1) INT: For all $i \neq j \in S$ such that $\mu_i + \mu_j < d/2$, $1 - 2(\mu_i + \mu_j)/d$ is the reciprocal of an integer.

(2.2) Σ INT: There is a subset S_1 of S with $\mu_i = \mu_j$ for all $i \neq j \in S_1$ with $\mu_i + \mu_j < d/2$, with $1 - 2(\mu_i + \mu_j)/d$ the reciprocal of an integer if either i or j is not in S_1, and $1 - 2(\mu_i + \mu_j)/d$ the reciprocal of a half-integer if both i and j are in S_1.

(In Tables I and II below the size of the symmetric group Γ is obvious from the entries.)

Deligne and Mostow also derive the following condition for Γ to be arithmetic in $\mathrm{PU}(N-3, 1)$.

(2.3) ARITH: Assume the highest common factor of $(\mu_s)_{s \in S}$ is one. (Otherwise, first divide the μ_s by this factor.) Let $\langle x \rangle$ denote the fractional part of x, i.e. $\langle x \rangle = x - [x]$.

If d is even, then for all A with $1 < A < d/2 - 1$, $(A, d/2) = 1$,

$$\sum_{s \in S} \langle 2A\mu_s/d \rangle = 1 \text{ or } N - 1.$$

If d is odd, then for all A with $1 < A < d/2$, $(A, d) = 1$,

$$\sum_{s \in S} \langle 2A\mu_s/d \rangle = 1 \text{ or } N - 1.$$

3. Classification of the weights. Now we begin the analysis of the varieties Q_{sst}. The first step in the analysis is to divide the sequences of weights into equivalence classes. From 1.8, this is a routine (but lengthy) computation, and the results are to be found in Table I (for $N = 5$) and Table II (for $N > 5$). In both tables N denotes the number of weights, $\underline{\mu}$

TABLE I

$N.m$	$\underline{\mu}$					∞_n	∞_s	Σ	A/NA
5.1	1	1	1	1	1	–	–		
	2	2	2	2	1				
	4	3	3	3	2				NA
	4	3	3	3	3				
	5	5	5	5	4				
	6	5	5	4	4				NA
	6	5	5	5	3				NA
	8	5	5	5	5			Σ	
	8	7	7	7	7			Σ	NA
	10	7	7	7	5			Σ	NA
	10	9	9	6	6			Σ	NA
	14	9	9	9	7				NA
5.2	5	5	2	2	2	–	–		
	6	5	3	3	3			Σ	
	7	7	4	4	2				
	8	7	3	3	3				NA
	9	9	2	2	2			Σ	
	13	9	6	6	6			Σ	NA
	14	11	5	5	5				NA
	14	13	3	3	3			Σ	
	19	17	4	4	4			Σ	NA
	23	22	5	5	5			Σ	NA
	34	29	7	7	7			Σ	NA
5.3	6	3	3	3	1	–	–		
	7	4	4	4	1				
	8	5	5	5	1				
	11	8	8	8	1				
5.4	8	3	3	3	3	–	–	Σ	
	9	3	3	3	2			Σ	
	10	5	3	3	3				
	11	7	2	2	2			Σ	
	22	11	9	9	9			Σ	
	26	19	5	5	5			Σ	
5.5	8	5	5	3	3	–	–		
5.6	2	1	1	1	1	4	–		
5.7	2	2	2	1	1	3	–		
	3	3	3	2	1				
5.8	3	2	1	1	1	3	–		
	5	4	1	1	1			Σ	
5.9	3	3	2	2	2	1	–		
	7	5	4	4	4				NA
5.10	4	3	2	2	1	2	–		
5.11	5	2	2	2	1	1	–		
5.12	5	3	2	1	1	2	–	Σ	
5.13	7	6	5	3	3	1	–		NA

TABLE II

N	$\underline{\mu}$						∞_n	∞_s	Σ	A/NA
6	3	3	3	3	3	1	–	–		
6	5	3	3	3	3	3	–	–	Σ	
	6	3	3	3	3	2			Σ	
6	5	5	5	3	3	3	–	–		
6	3	1	1	1	1	1	5	–		
6	3	3	3	1	1	1	3	–	Σ	
6	5	2	2	1	1	1	3	–	Σ	
6	5	3	1	1	1	1	4	–	Σ	
6	7	5	3	3	3	3	1	–		NA
6	1	1	1	1	1	1	–	10		
6	2	2	1	1	1	1	1	6		
6	3	2	2	2	2	1	–	4		
6	4	3	2	1	1	1	1	3	Σ	
6	4	4	1	1	1	1	–	6	Σ	
7	3^6	2					–	–	Σ	
7	7	7	2^5				–	–	Σ	
7	5	2	1^5				5	–	Σ	
7	2	1^6					–	15		
7	3	3	2	1^4			1	8	Σ	
7	4	2	2	1^4			2	6	Σ	
7	4	3	1^5				–	10	Σ	
8	5	1^7					7	–	Σ	
8	1^8						–	35	Σ	
8	3	2	2	1^5			–	30	Σ	
8	3	3	1^6				1	20	Σ	
8	4	2	1^6				1	15	Σ	
9	2	2	2	1^6			–	46	Σ	
9	3	2	1^7				–	42	Σ	
9	4	1^8					–	28	Σ	
10	2	2	1^8				–	98	Σ	
10	3	1^9					–	84	Σ	
11	2	1^{10}					–	210	Σ	
12	1^{12}						–	462	Σ	

the sequence of weights, ∞_n the number of non-singular cusps and ∞_s the number of singular cusps (so that Q_{st} is compact if $\infty_n + \infty_s = 0$, and Q_{sst} is non-singular if $\infty_s = 0$). Also, the column headed Σ is blank if $\underline{\mu}$ satisfies INT, but is labelled Σ if it satisfies Σ INT but not INT, and the column A/NA is blank if $\underline{\mu}$ satisfies ARITH, but is labelled NA if not.

Because of the many elements of several of the equivalence classes in Table I, we have numbered the equivalence classes 5.1–5.13 and will refer to a sequence of weights being of type $5.m$. In Table II we have used m^n to denote a sequence of n values of m.

Finally, these tables are arranged so that for a given N, the compact quotients are first, followed by the non-singular quotients. Subject to this restriction, the equivalence classes are listed in lexicographic order.

4. The structure of $Q(\underline{\mu})$. We let $Q(\underline{\mu})$ be the variety previously denoted by Q_{sst}, where the criteria for stability and semi-stability are with respect to the sequence of weights $\underline{\mu}$. We let $Q^*(\underline{\mu})$ be the non-singular variety obtained by blowing up the *singular* cusps of $Q(\underline{\mu})$.

We shall now completely analyze the structure of $Q(\underline{\mu})$ in those cases where it is non-singular, with the exception of the cases $N = 7$, $\underline{\mu} = 3^6\ 2$ or $7\ 7\ 2^5$. We remind the reader that our results here are valid over an arbitrary algebraically closed field (of any characteristic).

THEOREM 4.1.

(i) *If $\underline{\mu}$ is of type* 5.1, $Q(\underline{\mu})$ *is* $\mathbf{P}^2$ *with* 4 *points blown up.*

(ii) *If $\underline{\mu}$ is of type* 5.2 *or* 5.9, $Q(\underline{\mu})$ *is* $\mathbf{P}^2$ *with* 3 *points blown up.*

(iii) *If $\underline{\mu}$ is of type* 5.5 *or* 5.13, $Q(\underline{\mu})$ *is* $\mathbf{P}^2$ *with* 2 *points blown up.*

(iv) *If $\underline{\mu}$ is of type* 5.7, $Q(\underline{\mu})$ *is* $\mathbf{P}^1 \times \mathbf{P}^1$.

(v) *If $\underline{\mu}$ is of type* 5.3 *or* 5.10, $Q(\underline{\mu})$ *is* $\mathbf{P}^2$ *with* 1 *point blown up.*

(vi) *If $\underline{\mu}$ is of type* 5.4, 5.6, 5.8, 5.11, *or* 5.12, $Q(\underline{\mu})$ *is* $\mathbf{P}^2$.

(vii) $Q(3\ 3\ 3\ 3\ 3\ 1)$ *is a* $\mathbf{P}^1$ *bundle over* ($\mathbf{P}^2$ *with* 4 *points blown up*).

(viii) $Q(5\ 3\ 3\ 3\ 3\ 3) = Q(6\ 3\ 3\ 3\ 3\ 2)$ *is a* ($\mathbf{P}^1$ *bundle over* $\mathbf{P}^2$) *with* 4 *points blown up. This space can also be described as* $\mathbf{P}^3$ *with* 5 *points blown up.*

(ix) $Q(5\ 5\ 5\ 3\ 3\ 3)$ *is* $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ *with* 3 *points blown up.*

(x) $Q(3\ 3\ 3\ 1\ 1\ 1)$ *is* $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

(xi) $Q(7\ 5\ 3\ 3\ 3\ 3)$ *is* $Q(5\ 3\ 3\ 3\ 3\ 3)$ *with one* $\mathbf{P}^2$ *blown down.*

(xii) $Q(3\ 1\ 1\ 1\ 1\ 1)$ *is* $\mathbf{P}^3$.

(xiii) $Q(5\ 2\ 2\ 1\ 1\ 1)$ *is* $\mathbf{P}^3$.

(xiv) $Q(5\ 3\ 1\ 1\ 1\ 1)$ *is* $\mathbf{P}^3$.

(xv) $Q(5\ 2\ 1^5)$ *is* $\mathbf{P}^4$.

(xvi) $Q(5\ 1^7)$ *is* $\mathbf{P}^5$.

Proof. We shall only prove cases (vii)–(xi) here. Cases (i)–(vi), with $N = 5$, are less interesting, and (xii)–(xvi) are special cases of 4.2 below. We denote a point in $(\mathbf{P}^1)^6$ by $(z_1, \dots, z_6)$.

(vii): Let π: $(\mathbf{P}^1)^6 \to (\mathbf{P}^1)^5$ be projection on the first 5 coordinates. It is easy to check that π descends to a map

$$\bar{\pi}: Q(3\,3\,3\,3\,3\,1) \to Q(1\,1\,1\,1\,1)$$

which is a $\mathbf{P}^1$ bundle (as z_6 may be chosen arbitrarily without destroying stability), and $Q(1\,1\,1\,1\,1)$ is $\mathbf{P}^2$ with 4 points blown up by part (i).

(viii): Again consider the map $\bar{\pi}$: $Q(6\ 3\ 3\ 3\ 3\ 2) \to Q(6\ 3\ 3\ 3\ 3)$ $= Q(2\ 1\ 1\ 1\ 1)$ induced by π. Again it is easy to check that the inverse image of a stable point in $Q(2\ 1\ 1\ 1\ 1)$ is $\mathbf{P}^1$ (as z_6 may be chosen arbitrarily). There are four cusps. The situation at each is the same, so it suffices to check that cusp q determined by the partition $\{\{1,2\},\{3,4,5\}\}$

$$\bar{\pi}^{-1}(q) = \{(z_1, \dots, z_6)\,|\,\text{either } z_1 = z_2 \text{ or } z_3 = z_4 = z_5\}.$$

Let V_1 be the subset of $\bar{\pi}^{-1}(q)$ consisting of points with $z_1 = z_2$, and V_2 be the subset consisting of points with $z_3 = z_4 = z_5$. First note that

$$V_1 \cap V_2 = \mathrm{PGL}_2 \backslash \{(z, z, w, w, w, u) \in (\mathbf{P}^1)^6\,|\,z, w, u \text{ distinct}\}.$$

(The restriction on z, w, and u is due to the requirement of semi-stability.) Since under the action of PGL_2, any 3 distinct points in $\mathbf{P}^1$ are equivalent to any other 3 points, $V_1 \cap V_2 = \{(0,0,1,1,1,\infty)\}$ is a single point.

Now consider $V_2 = \mathrm{PGL}_2 \backslash \{(z_1, z_2, w, w, w, z_6)\,|\,z_i \neq w, z_1, z_2, z_6$ not all equal$\}$. (Again, the restrictions are due to the requirement of semi-stability. Henceforth, we will state this sort of restriction without comment.)

We may use the action of PGL_2 to send z_1 to 0 and w to ∞, whence we must take the quotient by the subgroup of PGL_2 fixing 0 and ∞, which we shall identify with k^*. Thus

$$V_2 = k^* \backslash \{(0, z_1, \infty, \infty, \infty, z_6)\,|\,z_2 \neq \infty, z_6 \neq \infty, (z_2, z_6) \neq (0,0)\},$$

and so $V_2 = \mathbf{P}^1$.

As for $V_1 = \mathrm{PGL}_2 \backslash \{(w, w, z_3, z_4, z_5, z_6)\,|\,z_i \neq w$, and z_3, z_4, z_5, z_6 not all equal$\}$, we may use the action of PGL_2 to send w to ∞ and z_3 to 0. Thus

$$V_1 = k^* \backslash \{(\infty, \infty, 0, z_4, z_5, z_6)\,|\,z_4 \neq \infty, z_5 \neq \infty, \\ z_6 \neq \infty, (z_4, z_5, z_6) \neq (0,0,0)\},$$

and so $V_1 = \mathbf{P}^2$.

Hence $\bar{\pi}^{-1}(q)$ is the one-point union of $\mathbf{P}^1$ and $\mathbf{P}^2$. One can check that the $\mathbf{P}^2$ can be blown down to a point, giving $\pi^{-1}(q) = \mathbf{P}^1$, and that in the neighborhood of q one still has a bundle structure, so $Q(6\ 3\ 3\ 3\ 2) = Q(5\ 3\ 3\ 3\ 3\ 3)$ is as claimed.

For the second description, consider the subset $U \subset Q(6\ 3\ 3\ 3\ 3\ 2)$ of points with $z_i \neq z_1$ for $i > 1$. Then

$$\begin{aligned} U &= \mathrm{PGL}_2 \backslash \{(z_1, \dots, z_6) \mid z_i \neq z_1 \text{ for } i > 1, \text{ at most 3 of } z_2, \dots, z_6 \text{ equal}\} \\ &= k^* \backslash \{(\infty, 0, z_3, \dots, z_6) \mid \text{at most 2 of } z_3, \dots, z_6 \text{ equal to } 0, \\ &\qquad\qquad\qquad \text{at most 3 of } z_3, \dots, z_6 \text{ equal}\} \\ &= \mathbf{P}^3 - ([0,0,0,1] \cup [0,0,1,0] \cup [0,1,0,0] \cup [1,0,0,0] \cup [1,1,1,1]). \end{aligned}$$

The space $Q(6\ 3\ 3\ 3\ 3\ 2)$ is covered by U and sets V_{1j}, $j = 2, \dots, 6$, where V_{1j} denotes the set of points where $z_1 = z_j$. Clearly all the V_{1j} have the same structure, so it suffices to consider V_{12}:

$$\begin{aligned} V_{12} &= \mathrm{PGL}_2 \backslash \{(w, w, z_3, z_4, z_5, z_6) \mid z_i \neq w, \\ &\qquad\qquad \text{and } z_3, z_4, z_5, z_6 \text{ not all equal}\} \\ &= k^* \backslash \{(\infty, \infty, 0, z_4, z_5, z_6) \mid z_i \neq \infty, (z_4, z_5, z_6) \neq (0,0,0)\} = \mathbf{P}^2. \end{aligned}$$

One can check that a neighborhood of V_{1j} has the structure of a blow-up, so $Q(6\ 3\ 3\ 3\ 3\ 2)$ is as claimed.

(ix): Let $U \subset Q(5\ 5\ 5\ 3\ 3\ 3)$ be the subset of points with z_1, z_2, and z_3 distinct. Then

$$\begin{aligned} U &= \mathrm{PGL}_2 \backslash \{(z_1, \dots, z_6) \mid z_1 \neq z_2 \neq z_3 \neq z_1, \\ &\qquad\qquad (z_4, z_5, z_6) \neq (z_i, z_i, z_i)\ i = 1, 2, \text{or } 3\} \\ &= \{(0, 1, \infty, z_4, z_5, z_6) \mid (z_4, z_5, z_6) \neq (0,0,0), (1,1,1), (\infty, \infty, \infty)\} \\ &= \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 - ((0,0,0) \cup (1,1,1) \cup (\infty, \infty, \infty)). \end{aligned}$$

The space $Q(5\ 5\ 5\ 3\ 3\ 3)$ is covered by U and V_{12}, V_{13}, V_{23}, where V_{ij} denotes the set of points where $z_i = z_j$. Clearly, V_{12}, V_{13}, and V_{23} all have the same structure, so it suffices to consider V_{12}:

$$\begin{aligned} V_{12} &= \mathrm{PGL}_2 \backslash \{(w, w, z_3, z_4, z_5, z_6) \mid z_i \neq w, \\ &\qquad\qquad \text{and } z_3, z_4, z_5, z_6 \text{ not all equal}\}/\mathrm{PGL}_2 \\ &= k^* \backslash \{(\infty, \infty, 0, z_4, z_5, z_6) \mid z_i \neq \infty, (z_4, z_5, z_6) \neq (0,0,0)\} = \mathbf{P}^2. \end{aligned}$$

One can again check that a neighborhood V_{12} has the structure of a blow-up, so $Q(5\ 5\ 5\ 3\ 3\ 3)$ is as claimed.

(x): Let U be the subset of $Q(3\ 3\ 3\ 1\ 1\ 1)$ with z_1, z_2, and z_3 distinct. Using the action of PGL_2, we may let

$$U = \{(0, 1, \infty, z_4, z_5, z_6) \mid z_i \in \mathbf{P}^1\}/\sim$$

where $\sim$ is the relation of 1.4, which we must consider in this case as $Q(3\ 3\ 3\ 1\ 1\ 1)$ has cusps.

This relation implies that points with $z_4 = z_5 = z_6 = 0$ must be identified to a point (and also $= 1$ or $= \infty$), but this has no effect as such a condition already defines a single point. Hence we see that

$$U = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1.$$

Now we must consider points not in U. Let V_{23} be the points with $z_2 = z_3$. Again, such points are strictly semi-stable, and so define a cusp determined by the partition $\{\{2,3\},\{1,4,5,6\}\}$, which is the same as the cusp defined by $z_4 = z_5 = z_6 = 0$ considered above, and similarly for points with $z_1 = z_3$ or $z_1 = z_2$. Thus $Q(3\ 3\ 3\ 1\ 1\ 1)$ is as claimed.

(xi): Consider the identity map i: $(\mathbf{P}^1)^6 \to (\mathbf{P}^1)^6$. The map i descends to a map $\bar{i}$: $Q(5\ 3\ 3\ 3\ 3\ 3) \to Q(7\ 5\ 3\ 3\ 3\ 3)$ which is *not* the identity. However, it is easy to check that the inverse image of any stable point of $Q(7\ 5\ 3\ 3\ 3\ 3)$ is a single point of $Q(5\ 3\ 3\ 3\ 3\ 3)$.

This leaves us to determine $\bar{i}^{-1}(q)$, where q is the (unique) cusp of $Q(7\ 5\ 3\ 3\ 3\ 3)$, determined by the partition $\{\{1,2\},\{3,4,5,6\}\}$. Then

$$\begin{aligned}\bar{i}^{-1}(q) &= \mathrm{PGL}_2 \backslash \{(w, w, z_3, z_4, z_5, z_6) \mid z_i \neq w, \text{not all } z_i \text{ equal}\} \\ &= k^* \backslash \{(\infty, \infty, 0, z_4, z_5, z_6) \mid z_i \neq \infty, (z_4, z_5, z_6) \neq (0,0,0)\} \\ &= \mathbf{P}^2\end{aligned}$$

so $Q(5\ 3\ 3\ 3\ 3\ 3)$ is obtained from $Q(7\ 5\ 3\ 3\ 3\ 3)$ by blowing up a point, and the result follows.

THEOREM 4.2. *Let μ be a sequence of weights such that $\mu_1 + \mu_N \geq d/2$ (and hence $\mu_1 + \mu_i \geq d/2$ for all $i = 2, \ldots, N$). Then $Q(\underline{\mu}) = \mathbf{P}^{N-3}$.*

Proof. Assume first that $\mu_1 + \mu_2 > d/2$. Then $z_1 \neq z_2$, so using the action of PGL_2 we may let $z_1 = \infty$, $z_2 = 0$, whence

$$Q(\underline{\mu}) = k^* \backslash \{(\infty, 0, z_3, \ldots, z_N)\}/\sim .$$

If $\mu_1 + \mu_N > d/2$ then there are no strictly semi-stable points, so the relation $\sim$ of 1.4 (ii) does nothing, and stability requires $z_i \neq \infty$, $i = 3, \ldots, N$, and $(z_3, \ldots, z_N) \neq (0, \ldots, 0)$ so $Q(\underline{\mu})$ is $\mathbf{P}^{N-3}$.

If $\mu_1 + \mu_i = d/2$ for $i = N$ (and similarly for other values as well) we have a non-singular cusp determined by the partition $\{\{1, N\}, \{2, \ldots, N-1\}\}$. While this cusp can be represented by points with $z_1 = z_N$, it can also be represented by points with $z_2 = \cdots = z_{N-1}$ and distinct from z_1 and z_N, which we may take distinct from each other, i.e. by a point $\{(\infty, 0, \ldots, 0, z_N)\}$. These points have already been considered in the previous paragraph, and they are already equivalent under the action of k^* to a single point, so this case changes nothing.

Finally, if $\mu_1 + \mu_2 = d/2$, we have a cusp determined by the partition $\{\{1, 2\}, \{3, \ldots, N\}\}$, so it has a representative of the form $\{(\infty, 0, z, \ldots, z)\}$, and the argument is as above.

5. Configurations of lines in surfaces. We have determined the structure of all of the surfaces in 4.1. There are natural configurations of projective lines on these surfaces: On the surface $Q(\underline{\mu})$ there are projective lines $\Delta_{ij} = \{z_i = z_j\}$ whenever $\mu_i + \mu_j < d/2$. (Note that if $Q(\underline{\mu})$ contains Δ_{ij} and Δ_{kl} with i, j, k, l distinct, they intersect in a double point, while if $Q(\underline{\mu})$ contains Δ_{ij}, Δ_{jk}, and Δ_{ik} with i, j, k distinct, they intersect in a triple point.) In Figure I (i)–(vi) we shall draw these configurations. It is routine to verify that they are correct.

Although each case (i)–(vi) of 4.10 may involve several equivalence classes of sequences of weights, the configurations do not depend on the equivalence classes, but only on the appropriate case of 4.1. What differs among the equivalence classes is the number and location of the cusps, which are always triple points.

We number the cases in Figure I below to correspond to the cases of 4.1. We denote the line Δ_{ij} by ij. Thus a triple point is denoted ijk. The cusp situation is as follows (with the same numbering):

(5.1)

(i) $\underline{\mu}$ of type 5.1—no cusps

(ii) $\underline{\mu}$ of type 5.2—no cusps

$\underline{\mu}$ of type 5.9—1 cusp, the triple point 345

(iii) $\underline{\mu}$ of type 5.5—no cusps

$\underline{\mu}$ of type 5.13—1 cusp, the triple point 245

(iv) $\underline{\mu}$ of type 5.7—3 cusps, the triple points 145, 245, and 345

(v) $\underline{\mu}$ of type 5.3—no cusps

$\underline{\mu}$ of type 5.10—2 cusps, the triple points 245 and 345

(vi) $\underline{\mu}$ of type 5.4—no cusps

$\underline{\mu}$ of type 5.6—4 cusps, all of the triple points

$\underline{\mu}$ of type 5.8—3 cusps, the triple points 234, 235, 245

$\underline{\mu}$ of type 5.11—1 cusp, the triple point 234

$\underline{\mu}$ of type 5.12—2 cusps, the triple points 234 and 235

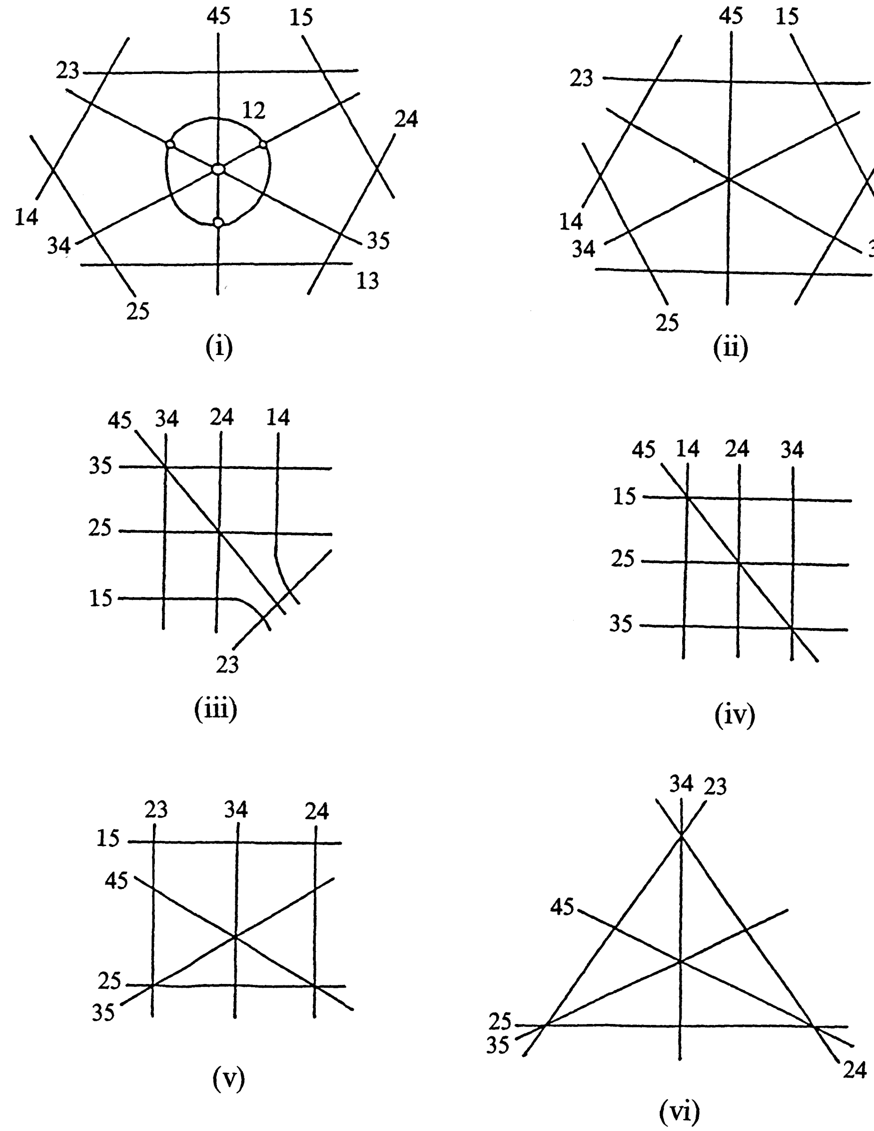

FIGURE I (i)–(vi)

(In (i) open circles denote no intersection.)

Diagrams such as those in Figure I are not complete without specifying the self-intersection of the lines therein. Rather than labelling the lines in the diagrams, we do that here (with the same numbering):

(5.2)

(i) all lines -1

(ii) 34, 35, 45 are 0; all others are -1

(iii) 45 is 1; 14, 15, 23 are −1; all others are 0
(iv) 45 is 2; all others are 0
(v) 15 is −1; 23, 24 34 are 0; all others are 1
(vi) all lines 1.

REMARK 5.3. From these diagrams it is easy to see the blow-ups (vi) → (v) → (iii) → (i) → (i) (as well as (iv) → (iii))—compare 4.1.

REMARK 5.4. The configuration (vi) of lines in $\mathbf{P}^2$ is the configuration $A_1(6)$ of Hirzebruch **[H]**. This configuration was used in **[H]** to produce examples of surfaces with $c_1^2 = 3c_2$. If the technique of **[H]** is applied to configurations (i)—(v), exactly the same surfaces are obtained.

6. Geometric invariant theory. It was observed in **[DM]** 4.6 that the quotient $Q(\bar{\mu})$ has the following description in terms of Mumford's geometric invariant theory **[MF]**.

For each $s \in S$ let $\pi_s\colon \mathbf{P}^S \to \mathbf{P}$ be the projection of $\mathbf{P}^S$ onto the sth factor. Let L_s be the pullback via π_s of the tangent bundle $\mathcal{O}(2)$ on $\mathbf{P}$ and let

$$L_{\underline{\mu}} = \bigotimes_{s \in S} L_s^{\otimes \mu_s}.$$

This line bundle $L_{\underline{\mu}}$ admits a natural PGL_2 action. The stable and semi-stable points of $\mathbf{P}^S$ defined at 1.2 are the same as those defined in **[MF]** for this action, and $Q(\underline{\mu})$ is the same as Mumford's projective "quotient" variety.

The associated variety $Q^*(\underline{\mu})$ can also be described as a quotient as follows.

For each partition $\{S_1, S_2\}$ of S with $\mu(S_1) = \mu(S_2) = d/2$, let $Z_{\{S_1,S_2\}}$ be the subvariety of M_{sst} consisting of all $\gamma \in M_{\mathrm{sst}}$ such that γ is constant on both S_1 and S_2. Then the $Z_{\{S_1,S_2\}}$ are disjoint nonsingular closed subvarieties of M_{sst}. Let Z be their union and let $\rho\colon M^* \to M_{\mathrm{sst}}$ be the blow-up of M_{sst} along Z. Let M^*_{sst} be the complement in M^* of the proper transform of M_{cusp} in M^*. The action of PGL_2 on M_{sst} induces an action on M^* which preserves M^*_{sst}.

LEMMA 6.1. *The stabilizer in* PGL_2 *of* $\gamma \in M^*_{\mathrm{sst}}$ *has order* 2 *if* $\gamma \in \rho^{-1}(Z)$ *and has order* 1 *otherwise.*

Proof. The only element of PGL_2 which fixes three distinct points of $\mathbf{P}$ is the identity. Hence PGL_2 acts freely on $M_{\mathrm{sst}} - Z$ and hence on $M^*_{\mathrm{sst}} - \rho^{-1}(Z)$. The stabilizer of any $\gamma \in Z$ is $\mathbf{C}^*$, and in an appropriate coordinate system this acts on the normal to Z at γ as

$$(u_1, \dots, u_{n-1}, v_1, \dots, v_{N-n-1}) \to (tu_1, \dots, tu_{n-1}, t^{-1}v_1, \dots, t^{-1}v_{N-n-1})$$

where $\gamma \in Z_{\{S_1, S_2\}}$ and $n = c(S_1)$. In the associated projective representation the stabilizer of a point (u, v) is $\{\pm 1\}$ unless $u = 0$ or $v = 0$. In the latter cases (u, v) represents a point in the proper transform of M_{cusp}. Therefore the stabilizer in PGL_2 of each $\gamma \in \rho^{-1}(Z) \cap M^*_{\text{sst}}$ has order 2.

LEMMA 6.2. *The quotient* $\text{PGL}_2 \setminus M^*_{\text{sst}}$ *is the geometric invariant theoretic quotient of* M^*_{sst} *by* PGL_2 *and is a projective variety. It is non-singular and is naturally isomorphic to the blow-up* $\tilde{Q}^*(\underline{\mu})$ *of* $Q^*(\underline{\mu})$ *at the nonsingular cusps.*

Proof. This is a special case of [$\mathbf{K}_2$, 6.9].

7. The computation of $H^*(Q^*(\underline{\mu}))$. In this section we shall consider only the case $k = \mathbf{C}$. Our aim is to compute the cohomology (except for the 2-torsion) of the complex variety $Q^*(\underline{\mu})$.

In the last section we gave a description of $Q^*(\underline{\mu})$ in terms of geometric invariant theory, and this means that we can use the general procedure described in [$\mathbf{K}_1$] to investigate its cohomology. For the convenience of the reader, instead of describing the procedure of [$\mathbf{K}_1$] in general and then applying it to our special case, we shall analyze the special case directly without overt reference to [$\mathbf{K}_1$].

Let $\mathbf{F}$ be any field of characteristic different from 2. For any topological space A let $P(A)$ denote the Poincaré series of A with coefficients in $\mathbf{F}$. That is,

$$P(A) = \sum_{i \geq 0} t^i \dim_{\mathbf{F}} H^i(A; \mathbf{F}).$$

If a group G acts on A let $P^G(A)$ denote the equivariant Poincaré series

$$P^G(A) = \sum_{i \geq 0} t^i \dim_{\mathbf{F}} H^i_G(A; \mathbf{F})$$

where H^i_G denotes equivariant cohomology (cf. [**AB**] §1 or [$\mathbf{K}_1$] §2).

Throughout this section the group PGL_2 *will be denoted by* G, *and all cohomology will be with coefficients in* $\mathbf{F}$.

By Lemma 6.2 the blow-up $\tilde{Q}^*(\underline{\mu})$ of $Q^*(\underline{\mu})$ at all the nonsingular cusps is isomorphic to $G \setminus M^*_{\text{sst}}$. Therefore

$$P(Q^*(\underline{\mu})) = P(G \setminus M^*_{\text{sst}}) - m(t^2 + \cdots + t^{2(N-4)}) \tag{7.1}$$

where m is the number of nonsingular cusps. By Lemma 6.1 the order of the stabilizer in G of every point of M^*_{sst} is either 1 or 2. This implies that

$$(7.2) \qquad P(G \setminus M^*_{\mathrm{sst}}) = P^G(M^*_{\mathrm{sst}})$$

since the characteristic of $\mathbf{F}$ is different from 2 (cf. [$\mathbf{K}_1$], 5.6 and 8.14).

Recall that M^* is the blow-up of M_{sst} along Z and that M^*_{sst} is the complement in M^* of the proper transform of M_{cusp}. A point $\gamma \in M_{\mathrm{sst}}$ belongs to M_{cusp} if and only if there exists some $S_1 \subseteq S$ with $\mu(S_1) = d/2$ such that γ is constant on S_1. The proper transform of M_{cusp} in M^* is the disjoint union of the proper transforms, $T(S_1)$ say, of the sets

$$\{\gamma \mid \gamma \text{ constant on } S_1\}$$

for $S_1 \subseteq S$ with $\mu(S_1) = d/2$.

Each $T(S_1)$ is nonsingular and has codimension $c(S_1) - 1$ in M^*. Let B be the Borel subgroup of $G = \mathrm{PGL}_2$ represented by upper triangular matrices. Then

$$T(S_1) \cong G \times_B Y(S_1)$$

where $Y(S_1)$ is the proper transform of the set of $\gamma \in M_{\mathrm{sst}}$ such that $\gamma(s) = \infty$ for each $s \in S_1$. This implies (see [AB] §13) that

$$(7.3) \qquad H^*_G(T(S_1)) \cong H^*_B(Y(S_1)).$$

However, B is homotopy equivalent to its maximal reductive subgroup $\mathbf{C}^*$, so

$$(7.4) \qquad H^*_B(Y(S_1)) \cong H^*_{\mathbf{C}^*}(Y(S_1)).$$

Moreover, $Y(S_1)$ retracts onto its intersection $Z(S_1)$ with the exceptional divisor $\rho^{-1}(Z)$ in M^*, and $Z(S_1)$ is isomorphic to projective space of dimension $N - c(S_1) - 2$ and is fixed by $\mathbf{C}^*$. Therefore

$$(7.5) \qquad H^*_{\mathbf{C}^*}(Y(S_1)) \cong H^*_{\mathbf{C}^*}(Z(S^1)) \cong H^*(Z(S_1)) \times H^*(B\mathbf{C}^*).$$

The representation of $\mathbf{C}^*$ on the normal to $Y(S_1)$ (and hence a fortiori on the normal to $T(S_1)$) at any point of $Z(S_1)$ is scalar multiplication by the character $t \to t^2$.

The subvarieties $\{T(S_1) \mid S_1 \subseteq S, \mu(S_1) = d/2\}$ together with M^*_{sst} form a smooth stratification of M^*. Since

$$(7.6) \qquad H^*_G(T(S_1)) \cong H^*_{\mathbf{C}^*}(Z(S_1))$$

the fact that $\mathbf{C}^*$ fixes $Z(S_1)$ pointwise and that its representation on the normal to $T(S_1)$ at any point of $Z(S_1)$ is primitive for any odd prime implies that this stratification is equivariantly perfect over the field $\mathbf{F}$ of characteristic $\neq 2$ ([**AB**] 1.9 and 13.4). In other words its equivariant Morse inequalities are all equalities, or equivalently

$$(7.7) \qquad P^G(M^*) = P^G(M^*_{\text{sst}}) + \sum t^{2(c(S_1)-1)} P^G(T(S_1))$$

where the sum is over all subsets $S_1 \subseteq S$ satisfying $\mu(S_1) = d/2$. By (7.5) and (7.6) this implies that

$$(7.8) \quad P^G(M^*_{\text{sst}}) = P^G(M^*) - \sum t^{2(c(S_1)-1)} P(\mathbf{P}^{N-c(S_1)-2})(1-t^2)^{-1}.$$

Now M^* is the blow-up of M_{sst} along the disjoint union of the subvarieties

$$Z_{\{S_1,S_2\}} = \{\gamma \in M_{\text{sst}} \mid \gamma \text{ constant on } S_1 \text{ and } S_2\}$$

for all partitions $\{S_1, S_2\}$ of S such that $\mu(S_1) = d/2$. Each $Z_{\{S_1,S_2\}}$ is isomorphic to $\mathbf{C}^* \setminus G$ and so its equivariant Poincaré series is $P(B\mathbf{C}^*) = (1-t^2)^{-1}$. Therefore

$$(7.9) \quad P^G(M^*) = P^G(M_{\text{sst}}) + \sum (t^2 + \cdots + t^{2(N-3)})(1-t^2)^{-1}$$

where the sum is over all partitions $\{S_1, S_2\}$ with $\mu(S_1) = d/2$.

It is easy to check that

$$(1-t^2)^{-1}\left(P(\mathbf{P}^{N-3}) - t^{2(c(S_1)-1)}P(\mathbf{P}^{N-c(S_1)-2}) - t^{2(c(S_1)-1)}P(\mathbf{P}^{N-c(S_2)-2})\right)$$
$$= P(\mathbf{P}^{c(S_1)-2} \times \mathbf{P}^{c(S_2)-2}).$$

Therefore from (7.8) and (7.9) we obtain

$$(7.10) \quad P^G(M^*_{\text{sst}}) = P^G(M_{\text{sst}}) + \sum \left[P(\mathbf{P}^{c(S_1)-2} \times \mathbf{P}^{c(S_2)-2}) - (1-t^2)^{-1}\right]$$

where the sum is over all partitions $\{S_1, S_2\}$ of S with $\mu(S_1) = \mu(S_2) = d/2$.

Now for each $S_1 \subseteq S$ with $\mu(S_1) > d/2$, let $T(S_1)$ be the set of all $\gamma \in \mathbf{P}^S$ such that there exists $x \in \mathbf{P}$ satisfying $\gamma(s) = x$ for all $s \in S_1$. Then

$$T(S_1) \cong G \times_B Y(S_1)$$

where

$$Y(S_1) = \{\gamma \in \mathbf{P}^S \mid \gamma(s) = \infty \text{ iff } s \in S_1\} \cong \mathbf{C}^{N-c(S_1)}$$

Thus $T(S_1)$ is nonsingular of codimension $c(S_1) - 1$ in $\mathbf{P}^S$ and

$$P^G(T(S_1)) = P^B(Y(S_1)) = P^{\mathbf{C}^*}(Y(S_1)).$$

Moreover, $Y(S_1)$ retracts onto the point γ given by $\gamma(s) = \infty$ if $s \in S_1$ and $\gamma(s) = 0$ otherwise. This point is fixed by $\mathbf{C}^*$. The representation of $\mathbf{C}^*$ on the normal to $T(S_1)$ at γ is primitive (this time for all primes).

Thus the argument used above shows that M_{sst} together with the subvarieties $\{T(S_1) \mid S_1 \subsetneq S, \mu(S_1) > d/2\}$ form a smooth stratification of $\mathbf{P}^S$ which is equivariantly perfect. Hence

$$(7.11) \qquad P^G(M_{\text{sst}}) = P^G(\mathbf{P}^S) - \sum t^{2(c(S_1)-1)}(1-t^2)^{-1}$$

where the sum is over all $S_1 \subsetneq S$ with $\mu(S_1) > d/2$.

Finally by Proposition 5.8 of [$\mathbf{K}_2$] and the fact that PGL_2 is homotopy equivalent to its maximal compact subgroup we have

$$(7.12) \qquad P^G(\mathbf{P}^S) = P(\mathbf{P}^S)P(BG) = (1-t^2)^N(1-t^4)^{-1}.$$

From (7.1), (7.2), (7.10), (7.11) and (7.12) we obtain the formula

$$(7.13) \quad P\big(Q^*(\underline{\mu})\big) = (1+t^2)^N(1-t^4)^{-1} - \sum_{u,n} t^{2(c(S_1)-1)}(1-t^2)^{-1} + \sum_{s}\Big[P(\mathbf{P}^{c(S_1)-2} \times \mathbf{P}^{c(S_2)-2}) - (1-t^2)^{-1}\Big]$$

where the first sum is over the partitions $\{S_1, S_2\}$ of S which define an unstable point ($\underline{\mu}(S_1) > d/2$) or a nonsingular cusp ($\mu(S_1) = d/2$ and $c(S_1) = 2$), and the last sum is over the partitions $\{S_1, S_2\}$ defining a singular cusp ($\underline{\mu}(S_1) = d/2$ and $c(S_1) > 2$).

The reader may verify that the right hand side of (7.13) is a polynomial of degree $2(N-3)$ in t which satisfies Poincaré duality.

THEOREM 7.14. *Let $\underline{\mu}$ be of any sequence of weights satisfying* 1.1. *Then*

(i) $H^i(Q^*(\underline{\mu}); \mathbf{Z})$ *has no odd torsion, for any i;*

(ii) $H^i(Q^*(\underline{\mu}); \mathbf{Z}[\frac{1}{2}]) = 0$ *for i odd;*

(iii) $H^{2i}(Q^*(\underline{\mu}); \mathbf{Z}[\frac{1}{2}])$ *has rank b_{2i} for any i where b_{2i} is the coefficient of t^{2i} in the right hand side of* 7.13 (*see Table* III);

(iv) *In the Hodge decomposition of $H^i(Q^*(\underline{\mu}); \mathbf{C})$, the (p,q)-cohomology $H^{p,q} = 0$ unless $p = q$.*

Proof. The fact that the formula 7.13 is valid for any field of characteristic different from 2 implies that $H^*(Q^*(\underline{\mu}); \mathbf{Z})$ has no odd torsion. (ii) and (iii) follow immediately from 7.13, while (iv) follows from the last remark of §14 of [$\mathbf{K}_1$].

Using formula 7.13 it is then routine to compute the Poincaré polynomials, and hence the even betti numbers b_{2i}.

The answers are to be found in Table III, for all cases not covered by Theorem 4.1. (Recall that $Q(3^6\ 2)$ and $Q(7\ 7\ 2^5)$ are compact and hence non-singular; in all other cases $Q(\underline{\mu})$ is singular.)

TABLE III

$\underline{\mu}$						b_{2i}, $i = 0, \ldots, N-3$									
1	1	1	1	1	1	1	16	16	1						
2	2	1	1	1	1	1	11	11	1						
3	2	2	2	2	1	1	10	10	1						
4	3	2	1	1	1	1	7	7	1						
4	4	1	1	1	1	1	11	11	1						
3^6	2					1	7	22	7	1					
7	7	2^5				1	6	16	6	1					
2	1^6					1	22	37	22	1					
3	3	2	1^4			1	14	24	14	1					
4	2	2	1^4			1	11	17	11	1					
4	3	1^5				1	16	26	16	1					
1^8						1	43	99	99	43	1				
3	2	2	1^5			1	28	58	58	28	1				
3	3	1^6				1	27	62	62	27	1				
4	2	1^6				1	22	37	37	22	1				
2	2	2	1^6			1	55	128	178	128	55	1			
3	2	1^7				1	51	114	149	114	51	1			
4	1^8					1	37	65	65	65	37	1			
2	2	1^8				1	108	242	396	396	242	108	1		
3	1^9					1	94	214	308	308	214	94	1		
2	1^{10}					1	221	496	806	1016	806	476	221	1	
1^{12}						1	474	991	1618	2410	2410	1618	991	474	1

In the case $\underline{\mu} = 1 \cdots 1$, the betti numbers were computed in [$\mathbf{K}_1$] in case N odd (when there are no cusps) and in [$\mathbf{K}_2$] in case N even. The answer is

$$N \text{ odd: } b_{2i} = \sum_0^i \binom{N-1}{k}, \qquad i \leq (N-3)/2,$$

$$N \text{ even: } b_{2i} = \sum_{k=0}^i \binom{N-1}{k} + \frac{1}{2}\binom{N}{N/2}(i), \qquad i \leq (N-4)/2.$$

As a practical matter, in using 7.13, it is only necessary to compute $P(Q^*(\underline{\mu}))$ up through dimension $N-3$, as Poincaré duality then yields the remaining terms.

REMARK 7.15. By (1.11), $Q^*(\underline{\mu})$ is rational. It is then a classical fact that $Q^*(\underline{\mu})$ is simply-connected and it is shown in [**AM**] that $H^3(Q^*(\underline{\mu}); \mathbf{Z})$ is torsion-free (and hence, when $Q^*(\underline{\mu})$ is a three-fold $H^i(Q^*(\underline{\mu}); \mathrm{Z})$ is torsion-free for all i.)

REMARK 7.16. If a finite group G acts on a space X and if $\mathbf{F}$ is a field of characteristic prime to the order of G, then $H^*(X/G; \mathbf{F}) = H^*(X; \mathbf{F})^G$, the subspace of elements fixed by G. Thus the formulas of this section may be used to compute the cohomology of $Q^*_\Sigma(\underline{\mu})$ if the Poincaré polynomials therein are replaced by the Poincaré polynomials of the fixed cohomology under the action of Σ, and the summations in (7.7)–(7.13) over various subsets of $\{1, \dots, N\}$ are taken over various Σ-orbits instead. We leave these computations to the reader, except for the following ones (which are the only complicated cases in which $Q_\Sigma(\underline{\mu})$ is a V-manifold):

(7.17) If $\underline{\mu} = 3^6\ 2$ (and $\Sigma = \Sigma_6$) or $\underline{\mu} = 7\ 7\ 2^5$ (and $\Sigma = \Sigma_5$) then the even betti numbers of $Q_\Sigma(\underline{\mu})$ are $b_{2i} = 1, 2, 3, 2, 1$ for $i = 0, \dots, 4$.

8. Intersection homology. It is also possible to calculate the dimension of the rational intersection homology (with respect to the middle perversity) groups of the singular variety $Q(\underline{\mu})$. Of course this is the same as calculating the dimension of the corresponding rational intersection cohomology groups $IH^i(Q(\underline{\mu}))$.

Throughout this section, all (co)homology, singular or intersection, is to be taken with rational coefficients.

Let E be the exceptional divisor of the blow-up $\pi\colon Q^*(\underline{\mu}) \to Q(\underline{\mu})$. Let $\tilde{U}$ be an open neighborhood of E in $Q^*(\underline{\mu})$ which is isomorphic to the normal bundle to E in $Q^*(\underline{\mu})$. Let $U = \pi(\tilde{U})$ which we may assume to be a union of disjoint open contractible neighborhoods of the singular cusps in $Q(\underline{\mu})$.

Since $\tilde{U}$ is nonsingular we have

$$IH^i(\tilde{U}) \cong H^i(\tilde{U}) \cong H^i(E) \quad \text{for all } i.$$

Since U has isolated singularities and dimension $N-3$ we have

$$IH^i(U) \cong H^i(U) = 0 \quad \text{for } i > N-3$$

and

$$IH^{N-3}(U) \cong \operatorname{Im}\big(H^{N-3}(U - Q_{\text{s cusp}}) \to H^{N-3}(U)\big) = 0$$

where $Q_{\text{s cusp}}$ is the set of singular cusps in $Q(\underline{\mu})$ (**[CGM**, 2.28]).

By [**CGM**, 5.1] there are natural embeddings

$$IH_i\big(Q(\underline{\mu})\big) \to IH_i\big(Q^*(\underline{\mu})\big)$$

for $i \le N-3$. These induce surjective maps

$$IH^i\big(Q^*(\underline{\mu})\big) \to H^i\big(Q(\underline{\mu})\big)$$

for $i \le N-3$, and hence by Poincaré duality embeddings

$$IH^i\big(Q(\underline{\mu})\big) \to IH^i\big(Q^*(\underline{\mu})\big)$$

for $i \ge N-3$.

Since $\tilde{U} \cap \rho^{-1}(Q_{st})$ is isomorphic to $U \cap Q_{st}$ and is nonsingular we have Mayer-Vietoris sequences for ordinary and intersection cohomology as follows for $i \geq N-3$:

$$\begin{array}{ccccccccc}
\to & H^{i-1}(U\cap Q_{st}) & \to & H^i(Q^*(\underline{\mu})) & \to & H^i(\tilde{U})+H^i(Q_{st}) & \to & H^i(U\cap Q_{st}) & \to \\
 & \| & & \uparrow & & \| & & \| & \\
\to & IH^{i-1}(U\cap Q_{st}) & \to & IH^i(Q(\underline{\mu})) & \to & IH^i(U)+IH^i(Q_{st}) & \to & IH^i(U\cap Q_{st}) & \to \\
 & & & & & \| & & & \\
 & & & & & 0 & & &
\end{array}$$

From this it follows that

$$(8.1)\quad \begin{aligned} \dim IH^i(Q(\underline{\mu})) &= \dim H^i(Q^*(\underline{\mu})) - \dim H^i(\tilde{U}) + \dim IH^i(U) \\ &= \dim H^i(Q^*(\underline{\mu})) - \dim H^i(E) \end{aligned}$$

for $i \geq N-3$.

By 1.10 the exceptional divisor E is the disjoint union over partitions $\{S_1, S_2\}$ of S with $\underline{\mu}(S_1) = d/2$ and $c(S_1) > 2$ of $\mathbf{P}^{c(S_1)-2} \times \mathbf{P}^{c(S_2)-2}$. Therefore for $i \geq N-3$ the dimension of $IH^i(Q(\underline{\mu}))$ is equal to the coefficient of t^i in the series

$$\begin{aligned} &P(Q^*(\underline{\mu})) - \sum_s P(\mathbf{P}^{c(S_1)-2} \times \mathbf{P}^{c(S_2)-2}) \\ &\quad = (1+t^2)^N(1-t^4)^{-1} - \sum_{u,n} t^{2(c(S_1)-1)}(1-t^2)^{-1} - \sum_s (1-t^2)^{-1} \end{aligned}$$

with the same conventions as before for the sums Σ_s and $\Sigma_{u,n}$. If i is odd this coefficient is zero, and if i is even it is

$$(8.2)\quad \begin{aligned} \sum_{j\leq i/2} \binom{N-1}{j} &- c\{\text{partitions } \{S_1,S_2\} \text{ of } S \mid \underline{\mu}(S_1) > d/2, \\ &\qquad\qquad c(S_1) \leq i/2+1\} \\ &- c\{\text{partitions } \{S_1,S_2\} \text{ of } S \mid \underline{\mu}(S_1) = d/2\}. \end{aligned}$$

It follows by Poincaré duality that if $i \leq N-3$ then

$$(8.3)\quad \begin{aligned} \dim IH^i(Q(\underline{\mu})) &= \dim IH^{2(N-3)-i}(Q(\underline{\mu})) \\ &= \sum_{j\leq N-3-i/2} \binom{N-1}{j} - c\{\text{partitions } \{S_1,S_2\} \text{ of } S \mid \underline{\mu}(S_1) > d/2, \\ &\qquad\qquad c(S_1) \leq N-i/2-2\} \\ &\qquad - c\{\text{partitions } \{S_1,S_2\} \text{ of } S \mid \underline{\mu}(S) = d/2\}. \end{aligned}$$

Alternatively using Poincaré duality for $Q^*(\underline{\mu})$ and E we have the formula

$$(8.4)\qquad \dim IH^i\big(Q(\underline{\mu})\big) = \dim H^i\big(Q^*(\underline{\mu})\big) - \dim H^{i-2}(E)$$

for $i \leq N-3$. From this we see that if $i \leq N-3$ then the dimension of $IH^i(Q(\underline{\mu}))$ is equal to coefficient of t^i in the series

$$\begin{aligned} P\big(Q^*(\underline{\mu})\big) &- \sum_S t^2 P(\mathbf{P}^{c(S_1)-2} \times \mathbf{P}^{c(S_2)-2}) \\ &= (1+t^2)^N(1-t^4)^{-1} - \sum_{u,n} t^{2(c(S_1)-1)}(1-t^2)^{-1} \\ &\quad + \sum_S \left[(1-t^2)P(\mathbf{P}^{c(S_1)-2} \times \mathbf{P}^{c(S_2)-2}) - (1-t^2)^{-1}\right]. \end{aligned}$$

When $i \leq N-3$ is even this coefficient is given by

$$(8.5)\qquad \sum_{j \leq i/2} \binom{N-1}{j} - c\{\text{partitions } \{S_1, S_2\} \text{ of } S \mid \underline{\mu}(S_1) \geq d/2,$$
$$c(S_1) \leq i/2 + 1\}.$$

The reader can check that the two formulas 8.3 and 8.5 do indeed agree.

Thus we have proved

THEOREM 8.6. *The dimension of the ith intersection homology group of $Q(\underline{\mu})$ is 0 if i is odd and if i is even it is*

$$\sum_{j \leq \min(i/2,\, N-3-i/2)} \binom{N-1}{j} - c\{\textit{partitions } \{S_1, S_2\} \textit{ of } S \mid \underline{\mu}(S_1) \geq d/2,$$
$$c(S_1) \leq \min(i/2, N-3-i/2) + 1\}.$$

REMARK 8.7. This computation is a special case of a general procedure for computing the intersection Betti numbers of the geometric invariant theoretic quotient of a nonsingular complex projective variety by a reductive group action described in [$\mathbf{K}_3$].

9. Algebraic cycles in $Q^*(\underline{\mu})$. Let us fix $\underline{\mu}$, set $Q^* = Q^*(\underline{\mu})$, and let $D = Q^* - Q$. Then D is a union of components of complex dimension $N-4$.

THEOREM 9.1. *Let $\mathbf{F}$ be a field of characteristic not equal to 2. Then the inclusion $D \to Q^*$ induces epimorphisms on homology with coefficients in $\mathbf{F}$ in dimension less than or equal to $N-4$ and in dimension $2(N-4)$ ($=$ codimension 2).*

Proof. We have the exact sequence of the pair

$$H_{i+1}(Q^*, D) \to H_i(D) \to H_i(Q^*) \to H_i(Q^*, D)$$

and the equality $H_i(Q^*, D) = H^{2(N-3)-i}(Q)$ from Alexander duality.

By the argument of [**LW**$_2$, 4.15] $H^i(Q)$ vanishes for $i > N-3$, yielding the first part of theorem, and $H^1(Q)$ has rank $\binom{N-2}{2} - 1$.

For the second part of the theorem, the relevant part of the exact sequence is

$$H^1(Q) \to H_{2(N-4)}(D) \to H_{2(N-4)}(Q^*)$$

so we need only show

$$\begin{aligned} \operatorname{rank}\big(H_{2(N-4)}(D)\big) &= \operatorname{rank}\big(H_{2(N-4)}(Q^*)\big) + \operatorname{rank}\big(H^1(Q)\big) \\ &= \operatorname{rank}\big(H^2(Q^*)\big) + \binom{N-2}{2} - 1 \end{aligned}$$

by Poincaré duality.

Let us examine the terms in the expression for $P(Q^*)$ up through dimension 2. We see

$$P(Q^*) = (1-t^2)^{-1}\left[(1+t^2)^{N-1} - \sum_{u,n}{}' t^2 + \sum_s \left[(1+t^2) - 1\right]\right] + \cdots$$

where $\Sigma'_{u,n}$ is taken over all unstable or semi-stable *pairs* (i.e. all $\{i, j\}$ with $\mu_i + \mu_j \geq d/2$) and Σ_s is taken over all singular cusps (determined by subsets S_1 of S with $\mu(S_1) = d/2$, $c(S_1) \geq 3$).

This gives the expression for the second betti number

$$b_2 = 1 + (N-1) - c_1 + c_2$$

where c_1 is the number of summands in $\Sigma'_{u,n}$ and c_2 the number of summands in Σ_s.

Then there are $\binom{N}{2} - c_1$ pairs $\{i, j\}$ with $\mu_i + \mu_j < d/2$. Thus D has $\binom{N}{2} - c_1 + c_2$ components, so we obtain

$$\binom{N}{2} - c_1 + c_2 = b_2 - N + \binom{N}{2} = b_2 + \binom{N-2}{2} - 1$$

as required.

CONJECTURE 9.2. The inclusion $D \to Q^*$ (or $D \to Q^*_\Sigma$) induces epimorphisms in homology in *all* dimensions $\leq 2(N-4)$.

10. Group cohomology. In this section we consider the rational cohomology of the groups $\Gamma(\underline{\mu})$ in the case where $Q(\underline{\mu})$ is either compact or has only nonsingular cusps, and in the additional case $\underline{\mu} = 1\,1\,1\,1\,1\,1$, where we use an entirely different method.

In any case, $H^*(\Gamma(\underline{\mu})) = H^*(Q_{st}(\underline{\mu}))$. However, if $Q(\underline{\mu})$ is compact then $H^*(\Gamma(\underline{\mu})) = H^*(Q(\underline{\mu}))$, whereas if $Q(\underline{\mu})$ has ∞_n nonsingular cusps, $H^*(\Gamma(\underline{\mu})) = H^*(Q(\underline{\mu}) - \infty_n \text{ points})$.

The cohomology then follows immediately from Table I, and Theorem 4.1, in all cases except $\underline{\mu} = 3^6\ 2$ or $7\ 7\ 2^5$, and these last two are covered by Table III. Similarly, when $Q(\underline{\mu})$ is non-singular, we may compute the rational cohomology of $\Gamma_\Sigma(\underline{\mu})$ by using (7.15).

We now specialize to the case of $\underline{\mu} = 1\ 1\ 1\ 1\ 1\ 1$. Then the space $Q^*(\underline{\mu})$ has been extensively studied under a different guise—as the Igusa compactification of the Siegel space of degree two and level two [**G**], [**LW$_1$**], [**LW$_2$**]. We shall use the work of [**LW$_2$**] to get some finer information, and in particular to determine the homology of Q_{st} (1 1 1 1 1 1).

THEOREM 10.1. *Let* **F** *be a field of characteristic not equal to* 2. *Then rank* $H^i(Q_{st}(1\ 1\ 1\ 1\ 1\ 1); \mathbf{F}) = 1, 0, 6, 5, 1, 9$ *for* $i = 0, \dots, 5$ *and is* 0 *for* $i > 5$.

Proof. Let $D = Q^* - Q$, and $C = Q^* - Q_{st}$. We consider the exact sequence of the pair (Q^*, C). Let us recall the results of [**LW$_2$**]:

In [**LW$_2$**] C was called the union of the Humbert surfaces and D was called the union of the boundary and Humbert surfaces. The group Σ_6, the symmetric group on six elements, acts on Q^* by permuting the coordinates. If we let $\{S_1, S_2\}$ be partitions of $\{1, \dots, 6\}$ with $\underline{\mu}(S_i) = c(S_i) = 3$, the components of C are indexed by such partitions. Letting $\Delta_0 = \{\{1,2,3\},\{4,5,6\}\}$ and denoting the corresponding component of C by C_0, then using the results of [**LW$_2$**] we have, in the interesting dimensions 2 and 4:

$$H_4(C) \to H_4(Q^*), \qquad H_2(C) \to H_2(Q^*)$$

$$\mathrm{Ind}_P(\tau) \to \mathrm{Ind}_P(\tau) + F, \qquad \mathrm{Ind}_P(\tau + \sigma) \to \mathrm{Ind}_P(\tau) = \tau + F.$$

Here τ denotes the trivial representation, Ind_P the induced representation from P (the stabilizer of Δ_0) to Σ_6, and σ the one-dimensional representation in which the permutation (14)(23)(56) is multiplication by -1. (This gives the action of P on $H_2(C_0)$—this element switches the two factors in $C_0 = \mathbf{P}^1 \times \mathbf{P}^1$. The representation F is the representation of the same name in [**LW$_2$**], 3.1.5.)

We first show that the inclusion of each of the representations $\mathrm{Ind}_P(\tau)$ is a monomorphism. Consider the map on H_4. For each partition $\Delta = \{\{i, j, k\}, \{i', j', k'\}\}$ let t_Δ be the intersection of the component C_Δ with the union of the components Δ_{ij} and $\Delta_{i',j'}$, so t_0 represents the component of $H_2(C_0)$ acted trivially on by P. The intersection number of the component C_D with t_Δ is the sum of the self-intersections of $(C_\Delta \cap \Delta_{ij})$ in Δ_{ij} and $(C_\Delta \cap \Delta_{i'j'})$ in $\Delta_{i'j'}$, which is $-1 + -1 = -2$ by [**LW**$_2$], 2.3.4. Since the different components of C are disjoint, this gives a pairing of the two copies of $\mathrm{Ind}_P(\tau)$ which has determinant $(-2)^{10}$ and is hence nonsingular.

Hence it remains to determine the map $\mathrm{Ind}_P(\sigma) \to \tau + F$. An easy calculation (using Frobenius reciprocity) shows $\mathrm{Ind}_P(\sigma)$ does not contain τ, so it remains to determine whether the map to F has non-zero image (as it will then be onto F, since F is irreducible). As F is self-dual we instead evaluate the image against F. A typical element of F is $f = 3(\{1,2\} + \{1,3\} + \{1,4\} + \{1,5\} + \{1,6\}) - \Sigma\{i,j\}$. If s_0 represents a generator of the representation σ, we may take $s_0 = (C_0 \cap \Delta_{12}) - (C_0 \cap \Delta_{45})$. Taking the intersection of s_0 with the sum of the components Δ_{ij} indexed by f, we find the answer is -2. (Note $(C_0 \cap \Delta_{12}) \cap \Delta_{12} = 0$ as this is the self-intersection of $(C_0 \cap \Delta_{12})$ in $C_0 = \mathbf{P}^1 \times \mathbf{P}^1$, but $(C_0 \cap \Delta_{45}) \cap \Delta_{12} = 1$ as this is a transverse intersection—in fact the intersection of the two generators of $H_2(C_0)$. Similarly for Δ_{13}, and the intersection is empty for Δ_{14}, Δ_{15}, and Δ_{16}.) The theorem then follows readily.

REMARK 10.2. By comparison with [**LW**$_2$] we may identify $H^i(Q_{st}(1\ 1\ 1\ 1\ 1\ 1))$ as representation spaces of Σ_6, under the obvious action. For $i = 2,3,4,5$ they are respectively the representations with Young diagrams [6] + [51], [51], [6], [42].

REFERENCES

[AM] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc., **25** (1972), 75–95.

[AB] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. London, A **308** (1982), 523–615.

[CGM] J. Cheeger, M. Goresky and R. MacPherson, *L^2-cohomology and intersection homology of singular algebraic varieties*, in *Seminar on Differential Geometry*, S.-T. Yau, ed., Princeton Univ. Press, (1982), 303–340.

[DM] P. Deligne and G. D. Mostow, *Monodromy of hypergeometric functions and non-integral lattice monodromy*, Publ. Math. I.H.E.S., **63** (1986), 5–89.

[G] G. van der Geer, *On the geometry of a Siegel modular threefold*, Math. Ann., **260** (1982), 317–350.

[H] F. Hirzebruch, *Arrangements of lines and algebraic surfaces*, in *Arithmetic and Geometry* vol. II, M. Artin and J. Tate eds., Birkhauser, (1983), 113–140.

[K_1] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes series, Princeton University Press, 1985.

[K_2] ______, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers*, Ann. Math., **122** (1985), 41–85.

[K_3] ______, *Rational intersection homology of quotients of nonsingular varieties*, to appear.

[LW_1] R. Lee and S. H. Weintraub, *Cohomology of a Siegel modular variety of degree two*, in *Group Actions on Manifolds*, R. Schultz ed., Amer. Math. Soc., Providence, (1985), 433–488.

[LW_2] ______, *Cohomology of* $Sp_4(\mathbf{Z})$ *and related groups and spaces*, Topology, **24** (1985), 391–410.

[M] D. Mostow, *Generalized Picard lattices arising from half-integral conditions*, Publ. Math. I.H.E.S., **63** (1986), 91–106.

[MF] D. Mumford and J. Fogarty, *Geometric Invariant Theory*, Springer-Verlag, 1982.

Received February 12, 1986.

MATHEMATICAL INSTITUTE
OXFORD UNIVERSITY
24–29 ST. GILES
OXFORD, OX1 3LB
ENGLAND

DEPARTMENT OF MATHEMATICS
YALE UNIVERSITY
BOX 2155 YALE STATION
NEW HAVEN, CT 06520
USA

MATHEMATISCHES INSTITUT
UNIVERSITÄT GÖTTINGEN
BUNSENSTRASSE 3-5
D-3400 GÖTTINGEN
FEDERAL REPUBLIC OF GERMANY