

SCALE-INVARIANT MEASURABILITY IN ABSTRACT WIENER SPACES

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In this paper, we first prove a limit theorem for a sequence of quadratic functionals on an abstract Wiener space which generalizes a Cameron-Martin limit theorem in the Wiener space; and next we prove a version of a converse measurability theorem for the Wiener space in the setting of abstract Wiener spaces. Using these results, we discuss scale-invariant measurability and translations in an abstract Wiener space.

1. Introduction and preliminaries. Let H be a real separable infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let μ be the cylinder set measure on H defined by

$$\mu(A) = (2\pi)^{-n/2} \int_F \exp\left\{-\frac{|x|^2}{2}\right\} dx,$$

where $A = P^{-1}(F)$, F is a Borel set in the image of an n -dimensional projection P in H and dx is Lebesgue measure in PH . A norm $\|\cdot\|$ on H is called *measurable* if for every $\varepsilon > 0$ there exists a finite dimensional projection P_0 such that $\mu(\{x \in H: \|Px\| > \varepsilon\}) < \varepsilon$ whenever P is a finite dimensional projection orthogonal to P_0 . It is known (see [8]) that H is not complete with respect to $\|\cdot\|$. Let B denote the completion of H with respect to $\|\cdot\|$. Let i denote the natural injection from H into B . The adjoint operator i^* is one-to-one and maps B^* continuously onto a dense subset of H^* . By identifying H^* with H and B^* with i^*B^* , we have a triple $B^* \subset H \subset B$ and $\langle x, y \rangle = (x, y)$ for all x in H and y in B^* , where (\cdot, \cdot) denote the natural dual pairing between B and B^* . By a well-known result of Gross, $\mu \circ i^{-1}$ has a unique countably additive extension ν to the Borel σ -algebra $\mathcal{B}(B)$ of B . The triple (H, B, ν) is called an *abstract Wiener space*. For more details, see Kuo [8].

Let $C_1[a, b]$ denote the Banach space $\{x(\cdot): x \text{ is a continuous function with } x(a) = 0\}$ with the uniform norm. Let $(C_1[a, b], \mathcal{B}(C_1[a, b]), m_w)$ denote the Wiener space, where m_w is the Wiener measure on the Borel σ -algebra $\mathcal{B}(C_1[a, b])$ of $C_1[a, b]$ (see [12]), and let $C'_1[a, b] = \{x \in C_1[a, b]: x(t) = \int_a^t f(u) du, f \in L^2[a, b]\}$; then it is a real separable infinite dimensional Hilbert space with the inner product $\langle x_1, x_2 \rangle = \int_a^b Dx_1(t) \cdot Dx_2(t) dt$, where $Dx = dx/dt$.

Let $Q = (s, t)$: $a \leq s \leq b$ and $c \leq t \leq d$ and let $C_2[Q]$ denote the Banach space $\{x(\cdot, \cdot): x \text{ is a continuous function on } Q \text{ with } x(a, t) = x(s, c) = 0\}$ with the uniform norm. Let $(C_2[Q], \mathcal{B}(C_2[Q]), m_y)$ denote the Yeh-Wiener space [13], i.e. m_y is the probability measure on the Borel σ -algebra $\mathcal{B}(C_2[Q])$ of $C_2[Q]$ such that

$$m_y(W) = \prod_{j=1}^m \prod_{k=1}^n \{2\pi(s_j - s_{j-1})(t_k - t_{k-1})\}^{-1/2} \\ \int_{p_{m,n}}^{q_{m,n}} \cdots \int_{p_{1,1}}^{q_{1,1}} \exp\left\{-\sum_{j=1}^m \sum_{k=1}^n \frac{[u_{j,k} - u_{j-1,k} - u_{j,k-1} + u_{j-1,k-1}]^2}{2(s_j - s_{j-1})(t_k - t_{k-1})}\right\} \\ \times du_{11} \cdots du_{mn}.$$

where $W = \{x \in C_2[Q]: p_{j,k} \leq x(s_j, t_k) \leq q_{j,k}, \text{ for } j = 1, \dots, m \text{ and } k = 1, \dots, n\}$ and $u_{0,k} = u_{j,0} = u_{0,0} = 0$ for all j and k . Let $C'_2[Q] = \{x \in C_2[Q]: x(s, t) = \int_a^s \int_c^t f(u, v) du dv, f \in L^2[Q]\}$; then it is a real separable infinite dimensional Hilbert space with the inner product

$$\langle x_1, x_2 \rangle = \int_a^b \int_c^d D^2 x_1(s, t) \cdot D^2 x_2(s, t) ds dt$$

where $D^2 x = \partial^2 x / \partial t \partial s$. It is known [9] that the uniform norms on $C'_1[a, b]$ and $C'_2[Q]$ are measurable and that $(C'_1[a, b], C_1[a, b], m_w)$ and $(C'_2[Q], C_2[Q], m_y)$ are both examples of abstract Wiener spaces.

Let $\{e_j, j \geq 1\}$ be a complete orthonormal system in H such that the e_j 's are in B^* . For each h in H and x in B , let

$$(h, x)^{\sim} = \begin{cases} \lim_{n \rightarrow \infty} \sum_{j=1}^n \langle h, e_j \rangle (x, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(h, \cdot)^{\sim}$ is a Borel measurable functional on B and if both h and x are in H , Parseval's identity gives $(h, x)^{\sim} = \langle h, x \rangle$. (See [7].)

Let $\{\phi_j: j \geq 1\}$ be a sequence of functions of bounded variation on Q which forms a complete orthonormal system of $L^2[Q]$. Let v be in $L^2[Q]$ and

$$v_n(s, t) = \sum_{j=1}^n \phi_j(s, t) \int_a^b \int_c^d v(s, t) \cdot \phi_j(s, t) ds dt.$$

The *Paley-Wiener-Zygmund integral* of v , $(\phi) \int_a^b \int_c^d v(s, t) \tilde{d}x(s, t)$, is defined as

$$(\phi) \int_a^b \int_c^d v(s, t) \tilde{d}x(s, t) \\ = \begin{cases} \lim_{n \rightarrow \infty} \int_a^b \int_c^d v_n(s, t) dx(s, t), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

Motivated by the well-known fact that change of scale is a pathological transformation in Wiener space, Johnson and Skoug [6] introduced the concept of scale-invariant measurability in Wiener space and examined the exact nature of this concept, and then they applied their results to clarify conceptual subtleties in the theory of the Feynman integral and in the theory of the Fourier-Feynman transform. Later, scale-invariant measurability in Yeh-Wiener space was studied by Chang [3]. In the works of [6] and [3], one notes that the keys to their discussion are a limit theorem in Wiener space due to Cameron-Martin [2] and in Yeh-Wiener space due to Skoug [11], and a converse measurability theorem for Wiener space due to Koehler [10] and for Yeh-Wiener space due to Skoug [10]. In view of this, for the extension to the abstract Wiener space setting of the results obtained in [6, 3], the question is to look for suitable versions of those two results in the setting of abstract Wiener spaces.

In §2, we prove a limit theorem for a sequence of quadratic functionals on an abstract Wiener space (H, B, ν) which is a generalization of a limit theorem in Wiener space [2] and in Yeh-Wiener space [11]. In §3, we prove a version of converse measurability theorem for Wiener space [10] and Yeh-Wiener space [11] in the setting of abstract Wiener spaces. In §4, we extend the results on scale-invariant measurability and translations obtained in [3, 6] to an abstract Wiener space.

2. A limit theorem in abstract Wiener space. In this section, we prove a limit theorem for a sequence of quadratic functionals defined on an abstract Wiener space (H, B, ν) and then we exhibit a subset of B with ν -measure one which for all real $\alpha \neq \pm 1$ is transformed into a set of ν -measure zero by the change of scale transformations $x \rightarrow \alpha x$. These are generalizations to the abstract Wiener space setting of the results first obtained by Levy [Amer. J. Math. **62** (1940), 487–550] and independently, but later, by Cameron and Martin [2] in Wiener space and extended by Skoug [11] to Yeh-Wiener space. We now begin with the following facts taken from [7].

LEMMA 2.1. *Let (H, B, ν) be an abstract Wiener space.*

(i) *For each $h (\neq 0)$ in H , the random variable $x \rightarrow (h, x)^\sim$ is Gaussian with mean zero and variance $|h|^2$.*

(ii) *$(h, \alpha x)^\sim = \alpha(h, x)^\sim$ for any real number α , $h \in H$ and $x \in B$.*

(iii) *If $\{h_1, h_2, \dots, h_n\}$ is an orthonormal set in H , then the random variables $(h_i, x)^\sim$'s are independent.*

(iv) *If $B = C_2[Q]$, $H = C_2'[Q]$, we have*

$$(h, x)^\sim = (\phi) \int_Q D^2 h(s, t) \tilde{d}x(s, t).$$

THEOREM 2.2. *Let (H, B, ν) be an abstract Wiener space. For each n , let $\{e_1^n, e_2^n, \dots, e_{d(n)}^n\}$ be an orthonormal set in H and let*

$$F_n(x) = \frac{1}{d(n)} \sum_{j=1}^{d(n)} [(e_j^n, x)^\sim]^2, \quad x \in B.$$

(i) *If $d(n) \geq n^p$ ($p > 0$) for every n , then F_n converges to 1 in $L^2(B, \nu)$ and hence in measure ν .*

(ii) *If $d(n) \geq n^p$ ($p > 1$) for every n , then F_n converges to 1 ν -almost surely.*

Proof (i) Since the random variables $(e_j^n, \cdot)^\sim$'s are independent and Gaussian with mean zero and variance one, it is easy to show that the mean of F_n is one and the variance of F_n , $\text{Var}(F_n)$, is $2/d(n)$. Hence we have

$$\int_B |F_n(x) - 1|^2 d\nu(x) = \frac{2}{d(n)} \leq \frac{2}{n^p}.$$

This implies that F_n converges to 1 in $L^2(B, \nu)$ and hence in measure ν .

(ii) For each $n = 2, 3, \dots$, let

$$E_n = \left\{ x \in B : |F_n(x) - 1| \geq \frac{\log n}{\sqrt{n^{p-1}}} \right\}.$$

Then, by Chebyshev's inequality [12], we have

$$\nu(E_n) \leq \frac{n^{p-1}}{\log^2 n} \cdot \frac{2}{d(n)} < \frac{2}{n \cdot \log^2 n}.$$

Since

$$\sum_{n=2}^{\infty} \nu(E_n) \leq \sum_{n=2}^{\infty} \frac{2}{n \cdot \log^2 n} < \infty,$$

it follows from Borel-Cantelli's Lemma [12] that $\nu(F) = 0$, where $F = \limsup E_n$. Since $\lim_{n \rightarrow \infty} (\log n) / \sqrt{n^{p-1}} = 0$, $F_n(x)$ converges to 1 ν -almost surely.

REMARK. It is worth pointing out that in Theorem 2.2, it is not important that $(e_j^n, \cdot)^\sim$ be Gaussian; in fact one can replace $(e_j^n, \cdot)^\sim$'s by any triangular sequence $X_1^n, X_2^n, \dots, X_{d(n)}^n$ of independent, identically distributed random variables on a probability space with mean zero and variance 1.

As an application of Theorem 2.2, we give the following example which shows how to obtain the corresponding result of Skoug's for Yeh-Wiener space [11].

EXAMPLE 2.3. Let $B = C_2[Q]$, $H = C_2'[Q]$ and $\nu = m_\nu$. For each n , let $a = s_0^n < s_1^n < \dots < s_{g(n)}^n = b$ and $c = t_0^n < t_1^n < \dots < t_{h(n)}^n = d$ be partitions of $[a, b]$ and $[c, d]$ respectively, and let

$$\phi_{j,k}^n(s, t) = \frac{1}{\sqrt{(s_j^n - s_{j-1}^n)(t_k^n - t_{k-1}^n)}} \int_a^s \int_c^t I_{[s_{j-1}^n, s_j^n] \times [t_{k-1}^n, t_k^n]}(u, v) du dv$$

for $j = 1, 2, \dots, g(n)$ and $k = 1, 2, \dots, h(n)$. Then it is easily shown that for each n , $\{\phi_{j,k}^n\}$ is an orthonormal set in $C_2'[Q]$, and for each $x \in C_2[Q]$, we have

$$\begin{aligned} (\phi_{j,k}^n, x)^{\sim} &= (\phi) \int_a^b \int_c^d D^2 \phi_{j,k}^n(s, t) dx(s, t) \\ &= \frac{x(s_j^n, t_k^n) - x(s_{j-1}^n, t_k^n) - x(s_j^n, t_{k-1}^n) + x(s_{j-1}^n, t_{k-1}^n)}{\sqrt{(s_j^n - s_{j-1}^n)(t_k^n - t_{k-1}^n)}}. \end{aligned}$$

Let $p > 1$, $g(n) \geq n^{p/2}$ and $h(n) \geq n^{p/2}$ for every n , and let $d(n) = g(n) \cdot h(n)$. Then $d(n) \geq n^p$ for every n , and hence by Theorem 2.2,

$$\begin{aligned} F_n(x) &= \frac{1}{d(n)} \sum_{j=1}^{g(n)} \sum_{k=1}^{h(n)} [(\phi_{j,k}^n, x)^{\sim}]^2 \\ &= \frac{1}{d(n)} \sum_{j=1}^{g(n)} \sum_{k=1}^{h(n)} \frac{[x(s_j^n, t_k^n) - x(s_{j-1}^n, t_k^n) - x(s_j^n, t_{k-1}^n) + x(s_{j-1}^n, t_{k-1}^n)]^2}{(s_j^n - s_{j-1}^n)(t_k^n - t_{k-1}^n)} \end{aligned}$$

converges to 1 ν -almost surely. In particular, if we take partitions of $[a, b]$ and $[c, d]$ such that

$$s_j^n - s_{j-1}^n = \frac{b-a}{g(n)} \quad \text{and} \quad t_k^n - t_{k-1}^n = \frac{d-c}{h(n)},$$

respectively, for all $j = 1, 2, \dots, g(n)$ and $k = 1, 2, \dots, h(n)$, then we have

$$\begin{aligned} F_n(x) &= \frac{1}{(b-a)(d-c)} \sum_{j=1}^{g(n)} \sum_{k=1}^{h(n)} [x(s_j^n, t_k^n) - x(s_{j-1}^n, t_k^n) \\ &\quad - x(s_j^n, t_{k-1}^n) + x(s_{j-1}^n, t_{k-1}^n)]^2 \end{aligned}$$

and hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{j=1}^{g(n)} \sum_{k=1}^{h(n)} \left[x(s_j^n, t_k^n) - x(s_{j-1}^n, t_k^n) - x(s_j^n, t_{k-1}^n) + x(s_{j-1}^n, t_{k-1}^n) \right]^2 \\ = (b-a)(d-c) \end{aligned}$$

which implies a result of Skoug in [11].

The following notation will be fixed throughout this paper: For each $\alpha > 0$, let

$$\begin{aligned} C_\alpha &= \left\{ x \in B : \lim_{n \rightarrow \infty} F_n(x) = \alpha^2 \right\}, \\ D &= \left\{ x \in B : \lim_{n \rightarrow \infty} F_n(x) \text{ does not exist} \right\}, \end{aligned}$$

where $\{F_n\}$ is the same as in Theorem 2.2 with $d(n) \geq n^p$ ($p > 1$) for all n . Then it follows from Lemma 2.1 that $\beta C_\alpha = C_{\alpha\beta}$ for all $\beta > 0$ and $\alpha \geq 0$. Clearly C_α ($\alpha \geq 0$) and D are all Borel sets in B , and B is the disjoint union of this family of sets.

The following results are generalizations of the results in [2] for Wiener space and in [11] for Yeh-Wiener space. The proof is an easy consequence of Theorem 2.2.

THEOREM 2.4. *Let (H, B, ν) be an abstract Wiener space. Then*

- (i) $\nu(C_\alpha) = 0$ if and only if $\alpha \neq 1$.
- (ii) $\nu(\alpha^{-1}C_\alpha) = 1$ for every $\alpha > 0$.
- (iii) $\nu(\beta^{-1}C_\alpha) = 0$ for all α, β with $\alpha \neq \beta$, $\alpha \geq 0$, $\beta > 0$.
- (iv) $\nu(D) = 0$.

3. Converse measurability theorem for abstract Wiener spaces. Let $a = s_0 < s_1 < \cdots < s_m = b$ and $c = t_0 < t_1 < \cdots < t_n = d$ be partitions of $[a, b]$ and $[c, d]$ respectively. Let F be any subset of Euclidean space R^{mn} and let

$$W = \left\{ x \in C_2[Q] : (x(s_1, t_1), \dots, x(s_m, t_n)) \in F \right\}.$$

In [10], Skoug showed that if W is $\overline{\mathcal{B}(C_2[Q])}^{m_y}$ -measurable, then F is Lebesgue measurable in R^{mn} , here and in the rest of the paper, $\overline{\mathcal{F}}^\lambda$ denotes the completion of \mathcal{F} with respect to a measure λ on a measurable space (X, \mathcal{F}) . In this section, we prove a version of this converse measurability theorem in the setting of abstract Wiener spaces.

LEMMA 3.1. *Let X_i ($i = 1, 2$) be separable complete metric spaces and $\mathcal{B}(X_i)$ the σ -algebra of Borel sets in X_i . Let $f: X_1 \rightarrow X_2$ be a Borel measurable (i.e. $\mathcal{B}(X_1) - \mathcal{B}(X_2)$ measurable) function and let μ_1 be a finite Borel measure on X_1 and $\mu_2 = \mu_1 \circ f^{-1}$. If A is a subset of X_2 such that $f^{-1}(A)$ is in $\overline{\mathcal{B}(X_1)^{\mu_1}}$, then A is in $\overline{\mathcal{B}(X_2)^{\mu_2}}$.*

Proof Since $f^{-1}(A)$ is in $\overline{\mathcal{B}(X_1)^{\mu_1}}$, there exists a Borel subset B_1 of $f^{-1}(A)$ such that $\mu_1(B_1) = \mu_1(f^{-1}(A))$. Let $\varepsilon > 0$ be given. Since f is Borel measurable, it follows from a generalization of Lusin's Theorem [4, p. 227] that there exists a compact subset K_ε of B_1 such that $\mu_1(B_1) - \varepsilon/2 < \mu_1(K_\varepsilon)$ and f restricted to K_ε is continuous. Now let $f(K_\varepsilon) = E_\varepsilon$. Then E_ε is a compact set in X_2 and $E_\varepsilon \subset A$, and we have

$$\mu_2(E_\varepsilon) = \mu_2(f(K_\varepsilon)) = \mu_1(f^{-1}(f(K_\varepsilon))) \geq \mu_1(K_\varepsilon) \geq \mu_1(B_1) - \varepsilon/2.$$

Similarly, since $f^{-1}(A^c)$ is in $\overline{\mathcal{B}(X_1)^{\mu_1}}$, we can find a Borel subset B_2 of $f^{-1}(A^c)$ and a compact subset L_ε of A^c such that $\mu_1(B_2) = \mu_1(f^{-1}(A^c))$ and $\mu_2(L_\varepsilon) \geq \mu_1(B_2) - \varepsilon/2$. Let $L_\varepsilon^c = G_\varepsilon$. Then E_ε and G_ε are in $\mathcal{B}(X_2)$ and $E_\varepsilon \subset A \subset G_\varepsilon$, and we have

$$\begin{aligned} \mu_2(G_\varepsilon - E_\varepsilon) &= \mu_2(G_\varepsilon) - \mu_2(E_\varepsilon) = \mu_2(X_2) - \mu_2(L_\varepsilon) - \mu_2(E_\varepsilon) \\ &\leq \mu_1(X_1) - (\mu_1(B_2) - \varepsilon/2) - (\mu_1(B_1) - \varepsilon/2) = \varepsilon. \end{aligned}$$

Now letting $\varepsilon = 1/n$, $n = 1, 2, \dots$, we obtain sequences $\{E_n\}$ and $\{G_n\}$ in $\mathcal{B}(X_2)$ such that $\mu_2(G_n - E_n) \leq 1/n$. Let $F_1 = \bigcap_{n=1}^{\infty} G_n$ and $F_2 = \bigcup_{n=1}^{\infty} E_n$. Then F_1 and F_2 are in $\mathcal{B}(X_2)$, $F_2 \subset A \subset F_1$, and $\mu_2(F_1 - F_2) = 0$. Hence A is in $\overline{\mathcal{B}(X_2)^{\mu_2}}$, completing of the proof of the lemma.

REMARK. It can be shown that Lemma 3.1 is still true with the same proof, for any Hausdorff topological space X_1 with finite Radon measure μ_1 and any separable metric space X_2 .

THEOREM 3.2. *Let (H, B, ν) be an abstract Wiener space, and let $\{h_1, h_2, \dots, h_n\}$ be a linearly independent subset of H . For any subset F of Euclidean space R^n , let*

$$W = \{x \in B: ((h_1, x)^{\sim}, (h_2, x)^{\sim}, \dots, (h_n, x)^{\sim}) \in F\}.$$

Then (i) W is $\mathcal{B}(B)$ -measurable if and only if F is Borel measurable in R^n .
(ii) W is $\mathcal{B}(B)^{\nu}$ -measurable if and only if F is Lebesgue measurable in R^n .

Proof (i) Let $f(x) = ((h_1, x)^{\sim}, (h_2, x)^{\sim}, \dots, (h_n, x)^{\sim})$, $x \in B$. Then by Lemma 2.1, f is an n -dimensional Gaussian random vector on B with

mean zero and covariance $V = (v_{ij})$ where $v_{ij} = \langle h_i, h_j \rangle$ for $i, j = 1, 2, \dots, n$. Hence if F is Borel measurable in R^n , then $W = f^{-1}(F)$ is $\mathcal{B}(B)$ -measurable. Conversely, suppose that W is $\mathcal{B}(B)$ -measurable. Since $\{h_1, h_2, \dots, h_n\}$ is linearly independent in H , the matrix V is non-singular. Let's define a function $g: R^n \rightarrow B$ by $g(u) = \sum_{j=1}^n v_j h_j$ where $(v_1, v_2, \dots, v_n) = (u_1, u_2, \dots, u_n) \cdot V^{-1}$ for $u = (u_1, u_2, \dots, u_n) \in R^n$. Then it is easy to see that G is a continuous (hence Borel measurable) function, and further we have $(\sum_{i=1}^n v_i h_i, h_j)^\sim = \langle \sum_{i=1}^n v_i h_i, h_j \rangle = u_j$ for every $j = 1, 2, \dots, n$. Hence $u = (u_1, u_2, \dots, u_n) \in F$ if and only if $G(u) \in W$, equivalently $I_F(u) = I_W \circ G(u)$ for all $u \in R^n$. Therefore F is $\mathcal{B}(R^n)$ -measurable since W is $\mathcal{B}(B)$ -measurable. This completes the proof of (i).

(ii) Suppose that F is a Lebesgue measurable set in R^n . Then there exist a Borel set G and a subset N_1 of a Borel null set N in R^n . Let f be as in the proof of (i). Then since f is Borel measurable, it follows that $f^{-1}(G)$ and $f^{-1}(N)$ are in $\mathcal{B}(B)$. Since $\nu \circ f^{-1}(N) = 0$, $f^{-1}(N_1)$ is in $\mathcal{B}(B)^\nu$ and hence $W = f^{-1}(F) = f^{-1}(G) \cup f^{-1}(N_1)$ is in $\mathcal{B}(B)$. Conversely, suppose that W is in $\mathcal{B}(B)^\nu$. Let $\lambda = \nu \circ f^{-1}$. Then λ is a Borel probability measure on R^n . By Lemma 3.1, F is $\mathcal{B}(R^n)^\lambda$ -measurable and hence Lebesgue measurable because λ is clearly equivalent to Lebesgue measure on R^n . This completes the proof of (ii).

As an application of Theorem 3.2, we give the following example which shows how to obtain the corresponding result of Skoug's for Yeh-Wiener space [10].

EXAMPLE 3.3. Let $B = C_2[Q]$, $H = C_2'[Q]$ and $\nu = m_y$. Let $a = s_0 < s_1 < s_2 < \dots < s_m = b$ and $c = t_0 < t_1 < \dots < t_n = d$ be partitions of $[a, b]$ and $[c, d]$, respectively. Let

$$h_{ij}(s, t) = \int_a^s \int_c^t I_{[a, s_i] \times [c, t_j]}(u, v) du dv,$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Then $\{h_{ij}\}$ is clearly a linearly independent set in $C_2'[Q]$, and for any $x \in C_2[Q]$,

$$\begin{aligned} (h_{ij}, x)^\sim &= (\phi) \int_a^b \int_c^d D^2 h_{ij}(s, t) \tilde{d}x(s, t) \\ &= (\phi) \int_a^{s_i} \int_c^{t_j} \tilde{d}x(s, t) = x(s_i, t_j). \end{aligned}$$

Hence by Theorem 3.2, for any subset F of R^{mn}

$$\begin{aligned} W &= \{x \in C_2[Q] : ((h_{11}, x)^\sim, \dots, (h_{mn}, x)^\sim) \in F\} \\ &= \{x \in C_2[Q] : (x(s_1, t_1), \dots, x(s_m, t_n)) \in F\} \end{aligned}$$

is in $\overline{\mathcal{B}(C_2[Q])}^{m_\nu}$ if and only if F is Lebesgue measurable in R^{mn} , and furthermore, W is in $\mathcal{B}(C_2[Q])$ if and only if F is Borel measurable in R^{mn} .

COROLLARY 3.4. *Let (H, B, ν) be an abstract Wiener space, and let $\{h_1, h_2, \dots, h_n\}$ be a linearly independent set in H . Let G be a complex-valued function defined on R^n . Then the functional*

$$F(x) = G((h_1, x)^\sim, \dots, (h_n, x)^\sim)$$

on B is $\overline{\mathcal{B}(B)}^\nu$ -measurable ($\mathcal{B}(B)$ -measurable) if and only if G is Lebesgue (Borel, resp.) measurable in R^n .

4. Scale-invariant measurability and translations in abstract Wiener space. Let (H, B, ν) be an abstract Wiener space. For any $\alpha > 0$, let $T_\alpha: B \rightarrow B$ be the transformation defined by $T_\alpha(x) = \alpha x$ and ν_α the Borel measure on B defined by $\nu \circ T_\alpha^{-1}$. Then $\nu_1 = \nu$ and $\nu_\alpha(A) = \nu_1(\alpha^{-1}A)$ for every A in $\mathcal{B}(B)$. Let $\mathcal{S}_\alpha = \overline{\mathcal{B}(B)}^{\nu_\alpha}$, $\mathcal{S} = \bigcap_{\alpha > 0} \mathcal{S}_\alpha$, $\mathcal{N}_\alpha = \{A \in \mathcal{S}_\alpha: \nu_\alpha(A) = 0\}$ and $\mathcal{N} = \bigcap_{\alpha > 0} \mathcal{N}_\alpha$. Every set in $\mathcal{S}(\mathcal{N})$ is called scale-invariant measurable (scale-invariant null, resp.) set. In this section, we extend the results of [6] on scale-invariant measurable sets, translations and scale-invariant measurable functions in Wiener space to abstract Wiener space.

Now, extensions to the abstract Wiener space setting of the results contained in Propositions 2 to 4 and Theorem 5 of [6] can be shown to hold using exactly the same arguments as given in [6]. For brevity we omit these statements and refer the reader to [6]. Further, the proof of the following two results in the abstract Wiener space setting can also be given using similar techniques as used in [6]. However, for completeness, we include the statements and proof of these results.

PROPOSITION 4.1. *For every $\alpha > 0$, $\mathcal{B}(B) \subsetneq \mathcal{S} \subsetneq \mathcal{S}_\alpha$.*

Proof. The proof of the containments, $\mathcal{B}(B) \subset \mathcal{S} \subset \mathcal{S}_\alpha$ is clear from the fact that $\mathcal{B} \subset \mathcal{S}_\alpha$ for every $\alpha > 0$ and the equality $\mathcal{S} = \bigcap_{\alpha > 0} \mathcal{S}_\alpha$. The proof of $\mathcal{B}(B) \neq \mathcal{S}$ can be easily shown by using Theorem 3.2. To show that $\mathcal{S} \neq \mathcal{S}_\alpha$, let $\alpha, \beta > 0$ with $\alpha \neq \beta$. Then by using Theorem 3.2, we can find a subset W of B such that W is not in \mathcal{S}_1 . Clearly $W \cap C_\alpha$ is not in \mathcal{S}_α . Now let $A = \beta^{-1}(W \cap C_\alpha)$. Then A is in \mathcal{N}_α and hence A is in \mathcal{S}_α . But $\beta A = W \cap C_\alpha$ is not in \mathcal{S}_α . Thus A is in \mathcal{S}_α , but not in \mathcal{S}_β and so not in \mathcal{S} .

THEOREM 4.2. *Let f be any function from $(0, \infty)$ to $[0, 1]$. Then there exists A in \mathcal{S} such that $\nu_1(\alpha A) = f(\alpha)$ for all $\alpha > 0$.*

Proof. We first show that for each $\alpha > 0$, there exists $A_\alpha \subset C_\alpha$ such that A_α is in \mathcal{S}_α and $\nu_\alpha(A_\alpha) = f(\alpha^{-1})$. To show this, let $f(\alpha^{-1}) = p$ and $h (\neq 0) \in H$. Then there exists a real number a_p such that

$$\frac{1}{\sqrt{2\pi|h|}} \int_{-\infty}^{a_p} \exp\left\{-\frac{u^2}{2|h|}\right\} du = p.$$

Let $W = \{x \in B: (h, x)^\sim \leq a_p\}$. Then W is in $\mathcal{B}(B)$ and hence in \mathcal{S}_1 and $\nu_1(W) = p$. Let $A_1 = W \cap C_1$. Then A_1 is in \mathcal{S}_1 and $\nu_1(A_1) = p$. Let $A_\alpha = \alpha A_1$. Then A_α is in \mathcal{S}_α , $A_\alpha \subset C_\alpha$ and $\nu_\alpha(A_\alpha) = \nu_\alpha(\alpha A_1) = \nu_1(A_1) = p = f(\alpha^{-1})$. Next, we show that the set $A = \bigcup_{\alpha > 0} A_\alpha$ is the desired set. A is clearly in \mathcal{S} and $\nu_1(\alpha A) = \nu_{\alpha^{-1}}(A) = \nu_{\alpha^{-1}}(A_{\alpha^{-1}}) = f(\alpha)$, completing the proof of the theorem.

Before we study translations in abstract Wiener spaces, we collect a few more definitions and notation, which will be needed in this section. Let E be a real separable Banach space with the topological dual E^* of E . Let (\cdot, \cdot) denote the natural dual pairing between E and E^* . For two Borel probability measures λ_1 and λ_2 on E , the *convolution* $\lambda_1 * \lambda_2$ of λ_1 and λ_2 is defined by

$$\lambda_1 * \lambda_2(G) = \lambda_1 \times \lambda_2(\{(x, y) \in E \times E: x + y \in G\})$$

for every Borel set G in E . The *characteristic functional* of a Borel probability measure λ on E is defined on E^* by

$$\hat{\lambda}(y) = \int_E e^{i(x, y)} d\lambda(x), \quad y \in E^*.$$

It is well-known that $\widehat{\lambda_1 * \lambda_2}(\cdot) = \hat{\lambda}_1(\cdot) \cdot \hat{\lambda}_2(\cdot)$ and that $\lambda_1 = \lambda_2$ if and only if $\hat{\lambda}_1(\cdot) = \hat{\lambda}_2(\cdot)$.

LEMMA 4.3. *Let p and q be positive real numbers.*

(i) $\nu_p * \nu_q = \nu_{\sqrt{p^2+q^2}}$ on $\mathcal{B}(B)$, $\overline{\mathcal{B}(B)}^{\nu_p * \nu_q} = \mathcal{S}_{\sqrt{p^2+q^2}}$.

(ii) *If $T: (B \times B, \mathcal{B}(B) \times \mathcal{B}(B), \nu_p \times \nu_q) \rightarrow (B, \mathcal{B}(B))$ is the transformation defined by $T(x, y) = x + y$, then for any $A \subset B$, $T^{-1}(A)$ is in $\overline{\mathcal{S}_p \times \mathcal{S}_q}^{\nu_p * \nu_q}$ if and only if A is in $\mathcal{S}_{\sqrt{p^2+q^2}}$. Further, we have $(\nu_p \times \nu_q) \circ T^{-1} = \nu_p * \nu_q = \nu_{\sqrt{p^2+q^2}}$ on $\mathcal{S}_{\sqrt{p^2+q^2}}$.*

(iii) *If $T_p \times T_q: (B \times B, \mathcal{B}(B) \times \mathcal{B}(B), \nu_1 \times \nu_1) \rightarrow (B \times B, \mathcal{B}(B) \times \mathcal{B}(B))$ is the transformation defined by $T_p \times T_q(x, y) = (px, qy)$, then for any $A \subset B \times B$, $(T_p \times T_q)^{-1}(A)$ is in $\overline{\mathcal{S}_1 \times \mathcal{S}_1}^{\nu_1 \times \nu_1}$ if and only if A is in $\overline{\mathcal{S}_p \times \mathcal{S}_q}^{\nu_p \times \nu_q}$. Further, we have $(\nu_1 \times \nu_1) \circ (T_p \times T_q)^{-1} = \nu_p \times \nu_q$ on $\overline{\mathcal{S}_p \times \mathcal{S}_q}^{\nu_p \times \nu_q}$.*

Proof. (i) Since $\hat{\nu}_p(y) = \exp\{-(p^2/2)|y|^2\}$ for every $y \in B^*$ (see [8, p. 78]), it follows that

$$\overline{\nu_p * \nu_q}(y) = \hat{\nu}_p(y) \cdot \hat{\nu}_q(y) = \exp\left\{-\frac{1}{2}(p^2 + q^2)|y|^2\right\} = \hat{\nu}_{\sqrt{p^2+q^2}}(y)$$

for every $y \in B^*$. Hence $\nu_p * \nu_q = \nu_{\sqrt{p^2+q^2}}$ on $\mathcal{B}(B)$, which implies that $\overline{\mathcal{B}(B)}^{\nu_p * \nu_q} = \overline{\mathcal{B}(B)}^{\nu_{\sqrt{p^2+q^2}}} = \mathcal{S}_{\sqrt{p^2+q^2}}$.

(ii) Suppose $T^{-1}(A)$ is in $\overline{S_p \times \mathcal{S}_q^{\nu_p \times \nu_q}}$. Since T is Borel measurable and $(\nu_p \times \nu_q) \circ T^{-1} = \nu_p * \nu_q$ on $\mathcal{B}(B)$, it follows from Lemma 3.1 that A is in $\overline{\mathcal{B}(B)}^{\nu_p * \nu_q} = \mathcal{S}_{\sqrt{p^2+q^2}}$. Conversely, suppose A is in $\mathcal{S}_{\sqrt{p^2+q^2}}$. Then $A = G \cup N$ where G is in $\mathcal{B}(B)$ and $N \subset M$, M is a $\nu_p * \nu_q$ -null set in $\mathcal{B}(B)$. Thus $(\nu_1 \times \nu_1) \circ T^{-1}(M) = \nu_p * \nu_q(M) = \nu_{\sqrt{p^2+q^2}}(M) = 0$, $T^{-1}(N) \subset T^{-1}(M)$ and $T^{-1}(A) = T^{-1}(G) \cup T^{-1}(N)$. Since T is Borel measurable and $\mathcal{B}(B \times B) = \mathcal{B}(B) \times \mathcal{B}(B)$, $T^{-1}(A)$ is in $\overline{\mathcal{B}(B) \times \mathcal{B}(B)}^{\nu_p \times \nu_q} = \overline{S_p \times \mathcal{S}_q^{\nu_p \times \nu_q}}$ and

$$\begin{aligned} (\nu_p \times \nu_q)T^{-1}(A) &= (\nu_p \times \nu_q)T^{-1}(G) + (\nu_p \times \nu_q)T^{-1}(N) \\ &= \nu_p * \nu_q(G) + \nu_p * \nu_q(N) = \nu_p * \nu_q(A) \\ &= \nu_{\sqrt{p^2+q^2}}(G) + \nu_{\sqrt{p^2+q^2}}(N) = \nu_{\sqrt{p^2+q^2}}(A). \end{aligned}$$

This completes the proof of (ii).

(iii) Let A be a subset of $B \times B$. Since $T_p \times T_q$ is Borel measurable and $(\nu_1 \times \nu_1) \circ (T_p \times T_q)^{-1} = \nu_p \times \nu_q$ on $\mathcal{B}(B \times B) = \mathcal{B}(B) \times \mathcal{B}(B)$, it follows from Lemma 3.1 that if $(T_p \times T_q)^{-1}(A)$ is in $\overline{\mathcal{S}_1 \times \mathcal{S}_1^{\nu_1 \times \nu_1}}$, then A is in $\overline{\mathcal{B}(B) \times \mathcal{B}(B)}^{\nu_p \times \nu_q} = \overline{S_p \times \mathcal{S}_q^{\nu_p \times \nu_q}}$. Since $(T_p \times T_q)^{-1} = T_{p^{-1}} \times T_{q^{-1}}$ is also a Borel measurable transformation from $(B \times B, \mathcal{B}(B) \times \mathcal{B}(B))$, $\nu_p \times \nu_q$ to $(B \times B, \mathcal{B}(B) \times \mathcal{B}(B))$ and $(\nu_p \times \nu_q) \circ (T_{p^{-1}} \times T_{q^{-1}})^{-1} = \nu_1 \times \nu_1$ on $\mathcal{B}(B) \times \mathcal{B}(B)$, by Lemma 3.1 again, it follows that if

$$(T_{p^{-1}} \times T_{q^{-1}})^{-1}[(T_p \times T_q)^{-1}(A)] = A$$

is in $\overline{\mathcal{S}_p \times \mathcal{S}_q^{\nu_p \times \nu_q}}$, then $(T_p \times T_q)^{-1}(A)$ is in $\overline{\mathcal{S}_1 \times \mathcal{S}_1^{\nu_1 \times \nu_1}}$, proving the first part of (iii). The second part of (iii) can be proved by the similar argument as in the proof of second part of (ii).

The following result follows immediately from Lemma 4.3, (ii) and the Fubini theorem.

PROPOSITION 4.4. *Let p and q be positive real numbers. Let A be in $\mathcal{S}_{\sqrt{p^2+q^2}}$. Then*

(i) $A - y$ is in \mathcal{S}_p for ν_q -a.s. y and $\nu_p(A - y)$ is a \mathcal{S}_q -measurable function of y .

(ii) $A - x$ is in \mathcal{S}_q for ν_p -a.s. y and $\nu_q(A - x)$ is a \mathcal{S}_p -measurable function of x .

(iii)

$$\begin{aligned} \nu_p * \nu_q(A) &= \nu_p \times \nu_q(\{(x, y): x + y \text{ is in } A\}) \\ &= \int_B \nu_p(A - y) d\nu_q(y) = \int_B \nu_q(A - x) d\nu_p(x). \end{aligned}$$

The extensions to the abstract Wiener space setting of the results contained in Corollaries 11 to 15 of [6] now can be easily proved by using Proposition 4.4. For brevity, we omit the statements of these corollaries and their proofs. Now we give a proof of the following result which is the extension of Theorem 9 in [6] to abstract Wiener space. We point out that the present proof is different in spirit from the proof of corresponding result given in [6]: The proof in [6] uses, for instance, a result of Bearman concerning rotations in Wiener space; our proof depends on Lemma 3.1 and on some properties of convolution of measures.

THEOREM 4.5. *Let p and q be positive numbers and f a complex-valued function on B . Then the following assertions are equivalent:*

- (a) $f(\sqrt{p^2 + q^2} z)$ is a \mathcal{S}_1 -measurable function of z .
- (b) $f(z)$ is a $\mathcal{S}_{\sqrt{p^2 + q^2}}$ -measurable function of z .
- (c) $f(x + y)$ is a $\mathcal{S}_p \times \mathcal{S}_q$ -measurable function of x and y .
- (d) $f(px + qy)$ is a $\mathcal{S}_1 \times \mathcal{S}_1$ -measurable function of x and y .

If any one (and hence all) of (a)–(d) holds, then

$$\begin{aligned} \int_B f(\sqrt{p^2 + q^2} z) d\nu_1(z) &\stackrel{*}{=} \int_B f(z) d\nu_{\sqrt{p^2 + q^2}}(z) \\ &\stackrel{*}{=} \int_{B \times B} f(x + y) d(\nu_p \times \nu_q)(x, y) \\ &\stackrel{*}{=} \int_{B \times B} f(px + qy) d(\nu_1 \times \nu_1)(x, y) \end{aligned}$$

where $\stackrel{*}{=}$ means that the existence of one side implies that of the other with the equality.

Proof. (i) The equivalence of (a) and (b): Define a transformation $T_{\sqrt{p^2 + q^2}}: (B, \mathcal{B}(B), \nu_1) \rightarrow (B, \mathcal{B}(B), \nu_{\sqrt{p^2 + q^2}})$ by $T_{\sqrt{p^2 + q^2}}(z) = \sqrt{p^2 + q^2} z$. Then $T_{\sqrt{p^2 + q^2}}$ and $T_{\sqrt{p^2 + q^2}}^{-1}$ are Borel measurable and $\nu_1 \circ T_{\sqrt{p^2 + q^2}}^{-1} = \nu_{\sqrt{p^2 + q^2}}$, $\nu_{\sqrt{p^2 + q^2}} \circ T_{\sqrt{p^2 + q^2}} = \nu_1$. Hence by Lemma 3.1, for any Borel set G in the complex plane C , $T_{\sqrt{p^2 + q^2}}^{-1}(f^{-1}(G))$ is in $\overline{\mathcal{B}(B)}^{\nu_1} = \mathcal{S}_1$ if

and only if $f^{-1}(G)$ is in $\overline{\mathcal{B}(B)}^{\nu\sqrt{p^2+q^2}} = \mathcal{S}_{\sqrt{p^2+q^2}}$. Therefore, (a) and (b) are equivalent and the corresponding integration formula follows from the fact that $\nu_1 \circ T^{\frac{1}{\sqrt{p^2+q^2}}} = \nu\sqrt{p^2+q^2}$ on $\mathcal{S}_{\sqrt{p^2+q^2}}$ and the change of variable theorem ([5], p. 163).

(ii) The equivalence (b) and (c): Let G be a Borel set in C and T the transformation defined in Lemma 4.3, (ii). Then by Lemma 4.3, (ii), $f^{-1}(G)$ is in $\mathcal{S}_{\sqrt{p^2+q^2}}$ if and only if $T^{-1}(f^{-1}(G)) = (f \circ T)^{-1}(G)$ is in $\overline{\mathcal{S}_p \times \mathcal{S}_q}^{\nu_p \times \nu_q}$. Hence (b) and (c) are equivalent and the corresponding integration formula follows from the fact that $\nu_p \times \nu_q \circ T^{-1} = \nu\sqrt{p^2+q^2}$ on $\mathcal{S}_{\sqrt{p^2+q^2}}$ and the change of variable theorem.

(iii) The equivalence of (c) and (d): Let G be a Borel set in C and T the transformation defined in Lemma 4.3, (ii). Then by Lemma 4.3, (iii), $(T_p \times T_q)^{-1}(T^{-1}(f^{-1}(G))) = (f \circ T \circ T_p \times T_q)^{-1}(G)$ is in $\overline{\mathcal{S}_1 \times \mathcal{S}_1}^{\nu_1 \times \nu_1}$ if and only if $T^{-1}(f^{-1}(G)) = (f \circ T)^{-1}(G)$ is in $\overline{\mathcal{S}_p \times \mathcal{S}_q}^{\nu_p \times \nu_q}$. Hence (c) and (d) are equivalent and the corresponding integration formula follows from the fact that $\nu_1 \times \nu_1 \circ (T_p \times T_q)^{-1} = \nu_p \times \nu_q$ and the change of variable theorem.

Let $\mathcal{B}(\mathcal{F})$ ($\mathcal{S}(\mathcal{F})$, $\mathcal{S}_\alpha(\mathcal{F})$, $\alpha > 0$) denote the class of all complex-valued functions which are defined on a Borel measurable (scale-invariant measurable, \mathcal{S}_α -measurable, resp.) subset of B and which are measurable with respect to the σ -algebra $\mathcal{B}(B)$ (\mathcal{S} , \mathcal{S}_α resp.). The extensions to the abstract Wiener space setting of all the results contained in §4 of [6] can be shown to hold by using the same arguments as given in [6]. For brevity, we omit these statements and refer the reader to [6].

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