ERRATA
CORRECTION TO REALIZING DIVISON ALGEBRAS

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Corollary 3.2 in [**1**] is incorrect. The proof of that corollary misuses the Double Centralizer Theorem. If the DCT is used properly with arguments in the proof of Theorem 3.1, the following weak version of Corollary 3.2 is obtained.

LEMMA 1. *If L is a left ideal of $\hat{D}$ such that $\{x \in D\colon Lx \subseteq L\}$ is the center F of D, then $QE(G(L)) \cap \mathrm{End}_F(D) = D$.*

This result is no longer sufficient to prove the principal result of [**1**], Theorem 5.1. Nonetheless, that theorem is true as stated. A correct proof uses a modified version of Lemma 4.4 in [**1**], which we now describe. It is convenient to augment some of the notation from [**1**].

As we suggested, D is a finite dimensional, non-commutative division algebra over Q with center F. The completion of F in an extension of the p-adic valuation is denoted by $\hat{F}$. Then $\hat{D} = \hat{F} \otimes_F D$ is a central simple $\hat{F}$-algebra that appears as a direct summand of $\hat{Q}_p \otimes D$. Thus, $\hat{D} = M_r(C)$ is the algebra of $r \times r$ matrices over a central $\hat{F}$ division algebra C. For the proof of Theorem 5.1, it can be assumed that $r > 1$.

The group $G = G(L)$ that p-realized D was defined in Lemma 4.4 of [**1**] by taking $L = \hat{D}e$, where $e \in \hat{D}$ is an idempotent of the form $e = I + M$ with

$$I = \begin{bmatrix} \iota & 0 \\ 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix},$$

ι the $t \times t$ identity matrix and γ a specially constructed $t \times (r-t)$ matrix. We will show that Theorem 5.1 of [**1**] can be salvaged if γ is replaced by $\alpha\gamma$ for a suitable $\alpha \in \hat{F}$. It will be convenient to denote $G(\hat{D}(I + \alpha M))$ by $G(\alpha)$. As in [**1**], K is a subfield of $\hat{F}$, finitely generated over F, such that $K \otimes_F D = M_r(B)$, where B is a central K division algebra and $\hat{F}B = C$. The matrix γ has rows that are independent over K. Write $\gamma = \sum \beta_\iota d_\iota$ with $\beta_i \in \hat{Q}_p$, $d_\iota \in D$, and let the subfield of $\hat{F}$ generated by K and $\{\beta_i\}$ be denoted as K'. Other notation and conventions used here are as in [**1**].

LEMMA 2. *Let $\alpha \in \hat{F}$ be transcendental over K'. Denote $e_\alpha = I + \alpha M$. Then:*

(a) $\dim_{\hat{F}} \hat{D}e_\alpha = tn^2/r$;
(b) *if* $x \in D$ *satisfies* $\hat{D}_\alpha x \subseteq \hat{D}e_\alpha$, *then* $x \in F$;
(c) $QE(G(\alpha)) \cap \mathrm{End}_F D = D$.

The proofs of (a) and (b) are the same as in Lemma 4.4 of [**1**], and (c) follows from (b) by Lemma 1.

To justify Theorem 5.1 of [**1**], it suffices to show that there exists $\alpha \in \hat{F}$, transcendental over K', such that $QE(G(\alpha)) \subseteq \mathrm{End}_F D$. Several more lemmas will lead to the existence of α.

LEMMA 3. *If* $G \in \Gamma_p(D)$ *satisfies* $L(G) \neq 0$, *then*:
(a) G *has no non-zero summands that are free* Z_p*-modules*;
(b) *if* $\phi \in \mathrm{End}_Q(D)$ *satisfies* $\phi(L(G)) = 0$, *then* $\phi = 0$.

Proof. (a) If $G = Z_p x \oplus H$ with $x \neq 0$, then $L(G) = d(\hat{Z}_p \otimes G) \subseteq \hat{Q}_p \otimes H$. Therefore, $(1 \otimes D)L(G) \subseteq L(G) \subseteq \hat{Q}_p \otimes H$. However, such an inclusion is impossible. In fact, if $0 \neq u \in \hat{D}$, then $(1 \otimes D)u \not\subseteq \hat{Q}_p \otimes H$. Indeed, we can write $u = \sum_{j=1}^r \alpha_j \otimes z_j$ with $r \geq 1$, $\alpha_1, \ldots, \alpha_r \in \hat{Q}_p$ linearly independent over Q, and $z_1, \ldots, z_r \in H \setminus \{0\}$. Since D is a division algebra, there exists $y \in D$ such that $yz_1 = x$; and we can find $s_j \in Q$ so that $yz_j - s_j x \in QH$ for $j \geq 2$, because $QG = Qx + QH$. Then $(1 \otimes y)u = (\alpha_1 + \sum_{j=2}^r s_j \alpha_j) \otimes x + \sum_{j=2}^r \alpha_j \otimes (yz_j - sx) \notin \hat{Q}_p H$ by the independence of $\alpha_1, \ldots, \alpha_r$.

(b) The structure theorem for torsion free $\hat{Z}_p$-modules of finite rank implies that $\hat{G} = \hat{Z}_p \otimes G = L(G) \oplus M$, where M is a free $\hat{Z}_p$-module of finite rank. Consequently, $\phi(\hat{G}) = \phi(M)$ is a finitely generated, torsion free $\hat{Z}_p$-module. Therefore, $\phi(G)$ is also a finitely generated torsion free module, hence free. It follows that $\phi(G)$ is isomorphic to a direct summand of G, so that $\phi(G) = 0$ by (a). Finally, $\phi = 0$, because $QG = D$.

The following notation will be useful. For $\phi \in \mathrm{End}_Q(D)$ and $f \in \hat{F}$, define

$$N_\phi(f) = \{a \in \hat{Q}_p \colon \phi \in QE(G(af))\}.$$

LEMMA 4. *The following conditions are equivalent*:
(a) $N_\phi(f) = \hat{Q}_p$;
(b) $|N_\phi(f)| \geq 3$;
(c) *for all* $d \in D$,
  (i) $\phi(dI)I = \phi(dI)$,
  (ii) $\phi(dfM)fM = 0$, *and*
  (iii) $\phi(dI)fM + \phi(dfM)I = \phi(dfM)$.

Proof. By Proposition 2.8 of [**1**], $\phi \in QE(G(af))$ if and only if $\phi(\hat{D}(I + afM)) \subseteq \hat{D}(I + afM)$. Since ϕ is $\hat{Q}_p$-linear and $I + afM$ is idempotent, this condition is equivalent to $\phi(d(I+afM))(I+afM) = \phi(d(I+afM))$ for all $d \in D$. That is,

$$(*)\quad \phi(dI)I + a[\phi(dI)fM + \phi(dfM)I] + a^2\phi(dfM)fM = \phi(dI) + a\phi(fM).$$

If $(*)$ holds for three values of a, then (c) is satisfied; and (c) clearly implies $(*)$ for all choices of $a \in \hat{Q}_p$.

LEMMA 5. *If $\phi \in \mathrm{End}_Q(D)$ and $f \in \hat{F}$ are such that $N_\phi(1) = N_\phi(f) = \hat{Q}_p$, then $\phi f = f\phi$ (considering f as the left translation $\lambda(f)$).*

Proof. Let $d \in D$ be arbitrary. By Lemma 4, $N_\phi(1) = \hat{Q}_p$ yields $\phi(dfI)M + \phi(dfM)I = \phi(dfM)$, and $N_\phi(f) = \hat{Q}_p$ implies $\phi(dI)fM + \phi(dfM)I = \phi(dfM)$. Thus, $[\phi(dfI) - f\phi(dI)]IM = [\phi(dfI) - \phi(dI)f]M = 0$, since f centralizes $\hat{D}$. Consequently, $[\phi(dfI) - f\phi(dI)]I = 0$ because the rows of γ are linearly independent over $K \otimes_F D$. By Lemma 4(c)(i), $(\phi f - f\phi)dI = 0$. Since $d \in D$ is arbitrary, $(\phi f - f\phi)\hat{D}I = 0$; and $\phi f = f\phi$ by Lemma 3.

PROPOSITION. *There exists $\alpha \in \hat{F}$, transcendental over K', such that $QE(G(\alpha)) = D$.*

Proof. To simplify notation, write Φ for $\mathrm{End}_Q(D)\setminus \mathrm{End}_F(D)$. Choose $f \in F$ so that $F = Q(f)$. If $\phi \in \mathrm{End}_Q(D)$ satisfies $\phi f = f\phi$, then $\phi \in \mathrm{End}_F(D)$. Hence, by Lemmas 4 and 5, $\phi \in \Phi$ implies $|N_\phi(1)| \leq 2$ or $|N_\phi(f)| \leq 2$. Assume that $b \in \hat{Q}_p$ is such that $|N_\phi(b+f)| \leq 3$. By Lemma 4,

$$\phi(dI)I = \phi(dI), \quad \phi(dM)M = 0,$$

and

$$b(\phi(dI)M + \phi(dM)I - \phi(dM)) + (\phi(dI)fM + \phi(dfM)I - \phi(dfM)) = 0$$

for all $d \in D$. If also $b \neq c \in \hat{Q}_p$ and $|N_\phi(c+f)| \leq 3$, then $\phi \in \mathrm{End}_F(D)$ by Lemmas 4 and 5. Hence, if $\phi \in \Phi$ and $|N_\phi(b+f)| \geq 3$, then $|N_\phi(c+f)| \leq 2$ for all $c \neq b$ in $\hat{Q}_p$. Since $\mathrm{End}_\phi(D)$ is countable and $\hat{Q}_p$ is uncountable, there exists $c \in \hat{Q}_p$ such that $|N_\phi(c+f)| \leq 2$ for all $\phi \in \Phi$. The countability of K' then implies the existence of $a \in \hat{Q}_p$, transcendental over $K'(c)$, such that $a \notin N_\phi(c+f)$ for all $\phi \in \Phi$. By the definition of $N_\phi(c+f)$, this means that $\Phi \cap QE(G(a(c+f))) = \varnothing$. Hence, $QE(G(\alpha)) \subseteq \mathrm{End}_F(D)$, where $\alpha = a(c+f)$ is transcendental over K'. By Lemma 2, $QE(G(\alpha)) = D$.

REFERENCES

[1] R. S. Pierce and C. Vinsonhaler, *Realizing central division algebras*, Pacific J. Math., **109** (1983), 165–177.