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UNITARY EQUIVALENCE OF INVARIANT SUBSPACES IN THE POLYDISK

KEIJI IZUCHI

Invariant subspaces M and N of $H^2(T^n)$ are called unitarily equivalent if $M = \psi N$ for a unimodular function ψ on T^n. In this note, it is given a complete characterization of pairs of invariant subspaces M and N of $H^2(T^n)$ such that $M = \phi N$ for an inner function ϕ. This is a generalization of Agrawal, Clark and Douglas' results. As an application, if M is an invariant subspace of $H^2(T^n)$ and if M is unitarily equivalent to $S(f)$, an invariant subspace generated by an outer function f, then $M = \phi S(f)$ for some inner function ϕ.

It is well known that Beurling [**4**] showed that every invariant subspace M of $H^2(T)$ can be written by $M = \psi H^2(T)$ for some inner function ψ. Although it is easy to see that a Beurling-type characterization is not possible for invariant subspaces of $H^2(T^n)$, $n \geq 2$, it is very difficult to determine all invariant subspaces of $H^2(T^n)$ for $n \geq 2$. In [**3**], Ahern and Clark studied an invariant subspace which has finite codimension in $H^2(T^n)$. These invariant subspaces are not Beurling-type. Recently Agrawal, Clark and Douglas [**2**] studied pairs of invariant subspaces of $H^2(T^n)$ which are unitarily equivalent. Here two invariant subspaces M_1 and M_2 are called *unitarily equivalent* if there is a unimodular function ψ on T^n with $M_2 = \psi M_1$. In [**1**, Corollary 3], they showed that distinct invariant subspaces having finite codimensions in $H^2(T^n)$ are not unitarily equivalent. In [**9**], Rudin gives two examples of unitarily equivalent invariant subspaces of $H^2(T^2)$ answering problems posed in [**2**]. In [**6**], Nakazi gives a characterization of invariant subspaces M of $L^2(T^2)$ with $M = FH^2(T^2)$ for some unimodular function F. From the view point of the Beurling theorem, it is interesting to characterize pairs of unitarily equivalent subspaces M_1 and M_2 of $H^2(T^n)$ such that $M_2 = \psi M_1$ for some inner function ψ. In [**2**], they give some sufficient conditions of these pairs. One of these conditions is $M_2 \subset M_1$.

In §2, we shall show a theorem which contains Schneider's lemma as a corollary (Corollary 1). Also our theorem gives us a complete characterization of pairs of invariant subspaces M_1 and M_2 of $H^2(T^n)$ such that $M_2 = \psi M_1$ for some inner function ψ (Corollary 2). Of course this theorem covers Propositions 1, 2, 3, and 4 in [2]. In §3, we shall study invariant subspaces which are unitarily equivalent to the one generated by an outer function.

1. Notations and Theorems. For a positive integer n, let T^n denote the cartesian product of n unit circles. The usual Lebesgue spaces, with respect to the normalized Haar measure m_n on T^n, are denoted by $L^p(T^n)$, $1 \le p \le \infty$. Let $H^p(T^n)$ be the space of all f in $L^p(T^n)$ whose Fourier transforms

$$\hat{f}(\alpha) = \int_{T^n} f(z)\bar{z}_1^{\alpha_1}\bar{z}_2^{\alpha_2} \cdots \bar{z}_n^{\alpha_n}\, dm_n(z)$$

vanish outside $(Z_+)^n$, the n-fold product of nonnegative integers. A function ψ in $L^\infty(T^n)$ and $H^\infty(T^n)$ is called *unimodular* and *inner* if $|\psi| = 1$ a.e. dm_n, respectively. A closed subspace M of $L^2(T^n)$ is called *invariant* if $z_i M \subset M$ for every $i = 1, 2, \dots, n$. We note that if M is an invariant subspace then $H^\infty(T^n)M \subset M$. A function f in $H^2(T^n)$ is called *outer* if

$$\log|f(0)| = \int_{T^n} |f(z)|\, dm_n(z).$$

We denote by $S(f)$ the invariant subspace generated by a function f in $L^2(T^n)$. [8] is a convenient reference for the function theory in the polydisk.

To state our theorem, we use the following notations. Let H_k and $\mathscr{H}_k$ denote the closure in $L^2(T^n)$ of the algebra generated by

$$\{1, z_i;\ i = 1, 2, \dots, n\} \cup \{\bar{z}_k\} \quad \text{and}$$
$$\{1, z_i, \bar{z}_i\colon i = 1, 2, \dots, n\} \setminus \{\bar{z}_k\},$$

respectively. Let L_k^p denote the closure in $L^p(T^n)$, weak*-closure if $p = \infty$, of the algebra generated by

$$\{1, z_i, \bar{z}_i\colon i = 1, 2, \dots, n\} \setminus \{z_k, \bar{z}_k\}.$$

Then H_k and $\mathscr{H}_k$ are invariant subspaces, $\bigcap_{k=1}^n \mathscr{H}_k = H^2(T^n)$, and $\mathscr{H}_k$ coincides with the closed linear span of $\{z_k^m L_k^2;\ m = 0, 1, 2, \dots\}$.

For an invariant subspace M (generally not closed), let $(M)_k$ denote the closure of $L_k^\infty M$ in $L^2(T^n)$. Then $(M)_k$ is an invariant subspace and $L_k^\infty (M)_k = (M)_k$. We note $(H^2(T^n))_k = \mathscr{H}_k$. A closed subspace N of $L^2(T^n)$ is called *reducing* if $z_i N = N$ for every $i = 1, 2, \dots, n$. If N is reducing, then $L^\infty(T^n)N = N$, hence $N = \chi_U L^2(T^n)$, where χ_U is a characteristic function for a Borel subset U of T^n. We note that $\mathscr{H}_k$ does not contain any reducing subspaces.

Our main results are

THEOREM 1. *Let M_1 be an invariant subspace of $H^2(T^n)$ and $\phi \in L^\infty(T^n)$. Let M_2 denote the closure of ϕM_1 in $L^2(T^n)$. Then $\phi \in H^\infty(T^n)$ if and only if $(M_2)_k \subset (M_1)_k$ for every $k = 1, 2, \ldots, n$.*

COROLLARY 7. *Let $f \in H^2(T^n)$ be an outer function, and M be an invariant subspace of $H^2(T^n)$ which is unitarily equivalent to $S(f)$. Then $M = \psi S(f)$ for some inner function ψ.*

2. Proof of Theorem 1 and its applications. The following lemma is a corollary of the Merrill and Lal theorem **[5]** (see Remark after Lemma 1). In this case, we can prove it directly. For the sake of completeness, we give its proof.

LEMMA 1. *Let M be an invariant subspace of $H^2(T^n)$. Then for each $k = 1, 2, \ldots, n$, $(M)_k = F_k \mathscr{H}_k$ for a unimodular function F_k in $\mathscr{H}_k$.*

Proof. Let fix k. Since $M \subset H^2(T^n)$, $(M)_k \subset \mathscr{H}_k$. Hence $\bigcap_{i=1}^\infty z_k^i (M)_k = \{0\}$. Put

$$N = (M)_k \ominus z_k (M)_k.$$

Then $N \neq \{0\}$. Since $L_k^\infty (M)_k = (M)_k$, $L_k^\infty N = N$. Thus we have

$$(M)_k = N \oplus z_k N \oplus z_k^2 N \oplus \cdots . \tag{1}$$

Let $g \in N$. Since $g \perp g z_k^i$ for $i = 1, 2, \ldots,$ we get

$$\int_{T^n} |g|^2 z_k^i \, dm_n = 0$$

for every nonzero integer i. This implies $|g| \in L_k^2$. Since $|f| > 0$ a.e. dm_n for $f \in H^2(T^n)$, by (1) there exists g_0 in N such that $|g_0| > 0$ a.e. dm_n. Put $g_0 = F|g_0|$, where F is unimodular. Since $L_k^\infty N = N$, $N \supset L_k^\infty g_0 = F L_k^\infty |g_0|$. Since $L_k^\infty |g_0|$ is dense in L_k^2, we have $F L_k^2 \subset N$.

To show $F L_k^2 = N$, let $g \in N$. Since $F \in N$,

$$F z_k^i \perp g z_k^j$$

for every $i, j \geq 0$ with $i \neq j$. Hence

$$\int_{T^n} \overline{F} g z_k^p \, dm_n = 0$$

for every nonzero integer p. Thus $\bar{F}g \in L_k^2$, so that $g \in FL_k^2$. Consequently $FL_k^2 = N$. By (1),

$$(M)_k = FL_k^2 \oplus Fz_k L_k^2 \oplus Fz_k^2 L_k^2 \oplus \cdots = F\mathscr{H}_k.$$

Since $F \in (M)_k \subset \mathscr{H}_k$, this completes the proof.

REMARK. In [5], they showed the following (in more general form); if M is an invariant subspace of $L^2(T^n)$ with $z_i M = M$ for $i = 1, 2, \ldots, n$ with $i \neq k$, then $M = \chi_U F\mathscr{H}_k \oplus \chi_V L^2(T^n)$, where F is unimodular. In this case, it is easy to see that $M = F\mathscr{H}_k$ if and only if M has no reducing subspaces and there is a function f in M with $|f| > 0$ a.e. dm_n. This fact is essentially pointed out, for the case $n = 2$, by Nakazi (see [6, Theorem 6]). Using this fact, we can also prove Lemma 1.

Proof of Theorem 1. Let M_1 be an invariant subspace of $H^2(T^n)$, $\phi \in L^\infty(T^n)$ and M_2 be the closure of ϕM_1 in $L^2(T^n)$. By Lemma 1, $(M_1)_k = F_k \mathscr{H}_k$ for some unimodular function F_k for $k = 1, 2, \ldots, n$.

First suppose that $(M_1)_k \supset (M_2)_k$ for $k = 1, 2, \ldots, n$. Then

$$F_k \mathscr{H}_k = (M_1)_k \supset (M_2)_k \supset \phi (M_1)_k = \phi F_k \mathscr{H}_k.$$

Hence $\phi \mathscr{H}_k \subset \mathscr{H}_k$, so that $\phi \in \bigcap_{k=1}^n \mathscr{H}_k = H^2(T^n)$. Thus $\phi \in H^\infty(T^n)$.

Next suppose $\phi \in H^\infty(T^n)$. We note that $(M_2)_k$ coincides with the closure of $\phi(M_1)_k$ in $L^2(T^n)$. Since $\phi \mathscr{H}_k \subset \mathscr{H}_k$, we have

$$\phi(M_1)_k = \phi F_k \mathscr{H}_k \subset F\mathscr{H}_k = (M_1)_k.$$

Thus $(M_2)_k \subset (M_1)_k$. This completes the proof.

The following corollary is proved in [2, Proposition 3] using an idea of Schneider [10]. We can prove this as an application of our theorem.

COROLLARY 1. *Let* $\phi \in L^\infty(T^n)$ *and* $f \in H^2(T^n)$ *such that* $f \neq 0$ *and* $\phi^m f \in H^2(T^n)$ *for* $m = 1, 2, \ldots$. *Then* $\phi \in H^\infty(T^n)$.

Proof. Let M_1 denote the invariant subspace of $H^2(T^n)$ generated by $\{\phi^m f;\ m = 1, 2, \ldots\}$. Let M_2 denote the closure of ϕM_1 in $L^2(T^n)$. Then $M_2 \subset M_1 \subset H^2(T^n)$, so that $(M_2)_k \subset (M_1)_k$ for $k = 1, 2, \ldots, n$. By Theorem 1, $\phi \in H^\infty(T^n)$.

The following is a direct corollary of our theorem. This answers the question posed in the introduction.

COROLLARY 2. *Let* M_1 *and* M_2 *be unitarily equivalent invariant subspaces of* $H^2(T^n)$. *Put* $M_2 = \psi M_1$, *where* ψ *is unimodular. Then* ψ *is inner if and only if* $(M_1)_k \supset (M_2)_k$ *for every* $k = 1, 2, \ldots, n$.

COROLLARY 3. *Let* M_1 *and* M_2 *be invariant subspaces of* $H^2(T^n)$ *such that* $(M_1)_k = (M_2)_k$ *for* $k = 1, 2, \ldots, n$. *Then* M_1 *is unitarily equivalent to* M_2 *if and only if* $M_1 = M_2$.

Proof. Suppose that $M_2 = \psi M_1$ and ψ is unimodular. By Corollary 2, ψ and $\bar{\psi}$ are inner. Hence ψ is constant, so that $M_1 = M_2$.

COROLLARY 4. *Let* M_1 *be an invariant subspace of* $H^2(T^n)$ *such that* $(M_1)_k = \mathscr{H}_k$ *for* $k = 1, 2, \ldots, n$. *If* M_2 *is an invariant subspace of* $H^2(T^n)$ *with* $M_2 = \psi M_1$, *where* ψ *is unimodular, then* ψ *is inner*.

Proof. Since $M_2 \subset H^2(T^n)$,

$$(M_2)_k \subset \left(H^2(T^n)\right)_k = \mathscr{H}_k = (M_1)_k.$$

By Corollary 2, ψ is inner.

An invariant subspace M of $H^2(T^n)$ has *full range* if the closed linear span of $\{\bar{z}_k^m M;\ m = 1, 2, \ldots\}$ coincides with H_k for $k = 1, 2, \ldots, n$ (see [**2**, p. 5]).

By the following lemma, we can consider that Corollary 4 is a generalization of both Propositions 1 and 2 in [**2**].

LEMMA 2. *Let* M *be one of the following invariant subspaces of* $H^2(T^n)$.

(1) M *has full range*.

(2) M *contains a nonzero function independent of* z_k *for each* $k = 1, 2, \ldots, n$.

Then $(M)_k = \mathscr{H}_k$ *for* $k = 1, 2, \ldots, n$.

Proof. (1) Suppose that M has full range. Then by the definitions, $H_i \subset (M)_k$ for $i \neq k$. Since $\mathscr{H}_k$ coincides with the linear span of $\{H_i;\ i = 1, 2, \ldots, n$ and $i \neq k\}$, we get $\mathscr{H}_k \subset (M)_k$, so that $\mathscr{H}_k = (M)_k$.

(2) Suppose that $f_k \in M$ is a nonzero function independent of z_k. Then

$$\begin{aligned}(M)_k &= \text{the closure of } L_k^\infty M \text{ in } L^2(T^n)\\ &\supset \text{the closure of } L_k^\infty f_k \text{ in } L^2(T^n) = L_k^2,\end{aligned}$$

the last equality follows from $|f_k| > 0$ a.e. dm. Since $z_k(M)_k \subset (M)_k$, we get $\mathscr{H}_k \subset (M)_k$, so that $\mathscr{H}_k = (M)_k$.

The following example shows that Corollary 4 is not covered by the work of Agrawal, Clark and Douglas [**2**].

EXAMPLE. For cases $n \geq 3$, there is an invariant subspace M of $H^2(T^n)$ such that

(a) M does not contain a function independent of z_k,

(b) M does not have full range, and

(c) $(M)_k = \mathscr{H}_k$ for $k = 1, 2, \ldots, n$.

We shall show the existence of M as above for $n = 3$. Let $\{\psi_i\}_{i=0}^{\infty}$ be a sequence of nonconstant inner functions in $H^\infty(T)$ satisfying the following conditions.

(i) $\psi_i H^2(T) \subsetneqq \psi_{i+1} H^2(T)$ for every i, and

(ii) $\bigcup_{i=0}^{\infty} \psi_i H^2(T)$ is dense in $H^2(T)$.

Let M denote the invariant subspace of $H^2(T^3)$ generated by

$$\bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} z_1^i z_2^j \psi_j(z_3) H^2(T^3).$$

Then every nonzero function in M is not independent of z_3. Hence M satisfies (a). By (i), $\psi_0(z_3)H^2(T^3) \subsetneqq H^2(T^3)$. Hence by the definition of M, the linear span of $\{\bar{z}_1^m M;\ m = 1, 2, \ldots\}$ does not contain $H^2(T^3)$, because it does not contain nonconstant functions. Thus M satisfies (b). By (ii), $(M)_3 = \mathscr{H}_3$. Since the linear span of $\{z_3^k \psi_j(z_3);\ k \text{ is an integer}\}$ coincides with $L^2(T)$, we have $(M)_k = \mathscr{H}_k$ for $k = 1, 2$. Thus M satisfies (c).

COROLLARY 5 [**2**, Proposition 4]. *Let M and M_1 be invariant subspaces of $H^2(T^n)$ such that $M \supset M_1$ and M_1 has finite codimension in M. If M_2 is an invariant subspace of M with $M_2 = \psi M_1$, where ψ is unimodular, then ψ is inner.*

Proof. Since $M \ominus M_1$ has finite dimension, it is easy to see $(M)_k = (M_1)_k$ for $k = 1, 2, \ldots, n$. Since $M \supset M_2$, $(M_2)_k \subset (M)_k = (M_1)_k$. By Corollary 2, ψ is inner.

COROLLARY 6. *Let M_1 and M_2 be invariant subspaces of $H^2(T^n)$. Suppose that both of $M_1 \ominus M_2$ and $M_2 \ominus M_1$ have finite dimensions. Then M_1 and M_2 are unitarily equivalent if and only if $M_1 = M_2$.*

Proof. Let M denote the invariant subspace generated by M_1 and M_2. Then M_1 and M_2 have finite codimensions in M. Put $M_2 = \psi M_1$ for some unimodular function ψ. By Corollary 5, ψ is constant, so that $M_1 = M_2$.

3. Outer functions. Rudin [7] showed the following.

(i) If $S(f) = H^2(T^n)$ and $f \in H^2(T^n)$, then f is outer.

(ii) There is an outer function f such that $S(f) \neq H^2(T^n)$.

If M is an invariant subspace of $H^2(T^n)$ such that M is unitarily equivalent to $H^2(T^n)$, then $M = \psi H^2(T^n)$ for some inner function ψ [**2**, Corollary 1]. In this section, we shall show that the above assertion is true if $H^2(T^n)$ is replaced by $S(f)$ for outer functions f.

THEOREM 2. *Let $f \in H^2(T^n)$ be an outer function. Then $(S(f))_k = \mathscr{H}_k$ for every $k = 1, 2, \ldots, n$.*

By Corollary 4, we get

COROLLARY 7. *Let $f \in H^2(T^n)$ be an outer function and let M be an invariant subspace of $H^2(T^n)$. If M is unitarily equivalent to $S(f)$, then $M = \psi S(f)$ for some inner function ψ.*

Proof of Theorem 2. Let $f \in H^2(T^n)$ be an outer function. Without loss of generality, we may assume $k = n$. By Lemma 1, $(S(f))_n = F_n \mathscr{H}_n$ for some unimodular function F_n in $\mathscr{H}_n$. We shall show that F_n is independent of z_n. We can write $f = F_n h$, where $h \in \mathscr{H}_n$. Write

$$z = (z', z_n) \in T^n, \quad \text{where } z' \in T^{n-1}.$$

Since f, F_n and h are contained in $\mathscr{H}_n$, there is a Borel subset E of T^{n-1} with $m_{n-1}(E) = 1$ such that for every fixed $z' \in E$,

$$f(z', z_n),\ F_n(z', z_n),\ h(z', z_n) \in H^2(T) \tag{2}$$

and $F_n(z', z_n)$ is inner. Since $f(z', 0) \in H^2(T^{n-1})$,

$$\begin{aligned}\log|f(0)| &= \log\left|\int_{T^{n-1}} f(z', 0)\, dm_{n-1}(z')\right| \\ &\le \int_{T^{n-1}} \log|f(z', 0)|\, dm_{n-1}(z') \quad \text{by [\mathbf{8}, p. 47]}.\end{aligned}$$

Hence, by our assumption,

$$\int_{T^{n-1}} \left\{\log|f(z', 0)| - \int_T \log|f(z', z_n)|\, dm_1(z_n)\right\} dm_{n,-1}(z') \ge 0.$$

Since $\log|f(z', 0)| \le \int_T \log|f(z', z_n)|\, dm_1(z_n)$ for $z' \in E$,

$$\log|f(z', 0)| = \int_T \log|f(z', z_n)|\, dm_1(z_n) \quad \text{a.e. } z' \in E.$$

Thus $f(z', z_n)$ is outer for a.e. $z' \in E$. Since $f = F_n h$, for a.e. fixed $z' \in E$, we have

$$f(z', z_n) = F_n(z', z_n) h(z', z_n) \quad \text{a.e. } z_n \in T.$$

By (2), an inner function $F_n(z', z_n)$ is constant for a.e. $z' \in E$. Then for nonzero integers i,

$$\int_{T^n} F_n(z) z_n^i \, dm_n(z) = \int_{T^{n-1}} dm_{n-1}(z') \int_T F_n(z', z_n) z_n^i \, dm_1(z_n) = 0.$$

This implies that $F_n(z)$ is independent of z_n. Hence F_n is invertible in $\mathscr{H}_n$, so that we get $(S(f))_n = \mathscr{H}_n$. This completes the proof.

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KANAGAWA UNIVERSITY
YOKOHAMA 221, JAPAN