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DENSITY OF THE POLYNOMIALS IN BERGMAN SPACES

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Let G be a bounded simply connected domain in the complex plane. Using a result of Hedberg, we show that the polynomials are dense in Bergman space $L_a^2(G)$ if G is the image of the unit disk U under a weak-star generator of H^∞. We also show that density of the polynomials in $L_a^2(G)$ implies density of the polynomials in $H^2(G)$. As a consequence, we obtain new examples of cyclic analytic Toeplitz operators on $H^2(U)$ and composition operators with dense range on $H^2(U)$. As an additional consequence, we show that if the polynomials are dense in $L_a^2(G)$ and φ maps U univalently onto G, then φ is univalent almost everywhere on the unit circle C.

1. Introduction. Let Ω be an open, nonempty subset of the complex plane, and let dA be two-dimensional Lebesgue measure. The Bergman space of Ω, $L_a^2(\Omega)$, is the Hilbert space of those functions f which are analytic on Ω and which satisfy

$$\|f\|^2 = \int_\Omega |f|^2\, dA < \infty.$$

Let H^∞ denote the algebra of functions which are bounded and analytic on the open unit disk U.

For any domain G in the plane, define the Carathéodory hull of G, G^*, to be the complement of the closure of the unbounded component of the complement of the closure of G. If G is a component of its Carathéodory hull G^*, then G is said to be a Carathéodory domain. Carathéodory domains are simply connected (cf. for example [**14**, Lemma 2.13]). An old theorem (1934) of Farrell [**6**] and Markusevic [**8**] states that if G is a bounded Carathéodory domain, then the polynomials are dense in $L_a^2(G)$. In 1953, S. N. Mergeljan remarked in his survey article on polynomial approximation that the Carathéodory domains apparently form the largest class of domains G with a purely topological definition such that the polynomials are dense in $L_a^2(G)$ (cf. [**9**, p. 121]). We show that there is a larger class of such domains.

THEOREM 3.1. *If $G = \varphi(U)$ where φ is a weak-star generator of H^∞, then the polynomials are dense in $L_a^2(G)$.*

The function $\varphi \in H^\infty$ is a weak-star generator of H^∞ provided the polynomials in φ are weak-star dense in H^∞. Here, it will be convenient to view H^∞ as the dual of a quotient space of $L^1(U)$. (The predual of H^∞ is, in fact, unique [**2**, Theorem 1].) In [**16**] Sarason characterizes the images of weak-star generators of H^∞ using the concept of relative hulls. Proposition 4 of [**16**] shows that relative hulls have a topological description based on the notion of a crosscut. Theorem 3.1 extends the result of Farrell and Markusevic since if φ maps U univalently onto a bounded Carathéodory domain, then φ is a weak-star generator of H^∞ (of order 1) [**16**]. Moreover, there are many weak-star generators of H^∞ which map U onto non-Carathéodory domains [**16**, **17**].

Weak-star generators of H^∞ are univalent on U and univalent almost everywhere on the unit circle C [**15**, Propositions 2 and 3]. We show that for the polynomials to be dense in $L_a^2(G)$, it is necessary that the univalent map of U onto G be univalent a.e. on C. This gives another way to see that if, for example, G contains slits (which are not too close together) then the polynomials are not dense in $L_a^2(G)$. Here, and for the remainder of this paper, we use the letter G to denote a bounded simply connected domain in the plane.

2. Preliminaries. Let A^2 denote the Bergman space of the unit disk; that is, let $A^2 = L_a^2(U)$. For $f \in H^\infty$, define $B_f\colon A^2 \to A^2$ by

$$(B_f g)(z) = f(z)g(z).$$

Similarly, define $T_f\colon H^2(U) \to H^2(U)$ by $(T_f h)(z) = f(z)h(z)$; here, $H^2(U)$ denotes the Hardy space of U. For any operator A, let $\operatorname{Lat} A$ represent the lattice of invariant subspaces of A (subspace $\equiv$ closed subspace). $\operatorname{Lat} T_\varphi = \operatorname{Lat} T_z$ if and only if φ is a weak-star generator of H^∞ [**15**, Proposition 1].

PROPOSITION 2.1. *If φ is a weak-star generator of H^∞, then* $\operatorname{Lat} B_\varphi = \operatorname{Lat} B_z$.

Proof. That $\operatorname{Lat} B_z \subset \operatorname{Lat} B_\varphi$ for any $\varphi \in H^\infty$ is well known (cf. for example [**19**, Theorem 12]). If φ is a weak-star generator of H^∞ then there is a net $\{p_\alpha\}$ of polynomials such that $p_\alpha(\varphi) \to z$ weak-star in H^∞. This means that for any $f \in L^1(U)$

$$\int_U (p_\alpha(\varphi) - z) f \, dA \to 0.$$

It follows that $B_{p_\alpha(\varphi)} \to B_z$ in the weak operator topology. Hence, $\operatorname{Lat} B_\varphi \subset \operatorname{Lat} B_z$.

John Conway and Robert Olin have pointed out to the author that the converse of Proposition 2.1 is true. If $\operatorname{Lat} B_\varphi = \operatorname{Lat} B_z$, then by the reflexivity of subnormal operators, there is a net $\{p_\alpha\}$ of polynomials such that $B_{p_\alpha(\varphi)} \to B_z$ in the weak operator topology. That $p_\alpha(\varphi) \to z$ weak-star follows from Theorems 1 and 2 of [**12**].

The proof of the following theorem appears in [**19**, pp. 112–114].

THEOREM 2.2 (*Hedberg*). *If R is a simply connected domain of finite area, then $H^\infty(R)$ is dense in $L_a^2(R)$.*

Via a change of variables, Hedberg's result is equivalent to the following (cf. [**19**, Proposition 41]).

COROLLARY 2.3. *If f maps U univalently onto a domain of finite area, then the derivative of f is cyclic for B_z.*

The vector $g \in A^2$ is cyclic for B_f: $A^2 \to A^2$ provided g is not contained in any proper invariant subspace of B_f. Alternatively, g is cyclic for B_f if $\{p(f)g\colon p \text{ is a polynomial}\}$ is dense in A^2.

3. Results. One may combine Corollary 2.3 and Proposition 2.1 to obtain a simple proof of Theorem 3.1.

THEOREM 3.1. *If $G = \varphi(U)$ where φ is a weak-star generator of H^∞, then the polynomials are dense in $L_a^2(G)$.*

Proof. Since weak-star generators are univalent, φ' is cyclic for B_z; and since $\operatorname{Lat} B_\varphi = \operatorname{Lat} B_z$, φ' is cyclic for B_φ.

Now, let $g \in L_a^2(G)$ be arbitrary. Since $g(\varphi)\varphi' \in L_a^2(U) = A^2$ and since φ' is cyclic for B_φ, there is a sequence $\{p_n\}$ of polynomials such that

$$\int_U \left| p_n(\varphi)\varphi' - g(\varphi)\varphi' \right|^2 dA \to 0.$$

Changing variables, we have $\int_G |p_n - g|^2\, dA \to 0$. Hence, the polynomials are dense in $L_a^2(G)$.

Robert Olin has related to the author another argument which yields Theorem 3.1. The author wishes to thank Prof. Olin for his permission to give that argument here. Let S be a bounded open set in the plane and let $\tilde{K}$ be its Sarason hull. ($\tilde{K}$ is the Sarason hull of μ where μ is two-dimensional Lebesgue measure on S. See [18].) The polynomials are weak-star dense in $H^\infty(S)$ if and only if every function in $H^\infty(S)$ extends to a function in $H^\infty(\operatorname{int}\tilde{K})$ [**18**, Corollary 3]. The following proposition holds since if G is the image of a weak-star generator of H^∞, then G is the interior of its Sarason hull $\tilde{K}$ (cf. [**11**, Lemma 1]).

PROPOSITION 3.2. *If $G = \varphi(U)$ where φ is a weak-star generator of H^∞, then the polynomials are weak-star dense in $H^\infty(G)$.*

Here, $H^\infty(G)$ is viewed as the dual of a quotient space of $L^1(G)$. If G is the image of a weak-star generator, then it follows easily from Proposition 3.2 that $H^\infty(G)$ is contained in *weak* closure of the polynomials in $L^2_a(G)$. Since for a convex subset of a Banach space weak closure is equivalent to norm closure, $H^\infty(G)$ is contained in the norm closure of the polynomials in $L^2_a(G)$. Theorem 3.1 now follows from Hedberg's result (Theorem 2.2).

There is no problem extending Hedberg's result to a bounded open set each of whose components is simply connected. Hence, Olin's argument provides a generalization of Theorem 3.1: If S is a bounded open subset of the plane with Sarason hull $\tilde{K}$, then the polynomials are dense in $L^2_a(S)$ if each $f \in L^2_a(S)$ extends to a function $f \in L^2_a(\operatorname{int}\tilde{K})$. One way to see that the components of $\operatorname{int}\tilde{K}$ are simply connected is to combine Lemma 7.1 of [**18**] with Theorem 5.1 of [**7**]. This generalization of Theorem 3.1 provides an extension of a result of Sinanjan [**20**] who showed that the polynomials are dense in $L^2_a(S)$ if S is a bounded Carathéodory set. (S is a Carathéodory set provided it is the union of some of the components of S^*.) Rubel and Shields have actually shown that for a bounded Carathéodory set S the polynomials are weak-star sequentially dense in $H^\infty(S)$; that is, each $f \in H^\infty(S)$ is the pointwise limit of a uniformly bounded sequence of polynomials (cf. [**14**, Theorem 3.2]). Note that it's a simple matter to combine the result of Rubel and Shields with Hedberg's result to obtain Sinanjan's result.

We turn now to a proposition which yields a necessary condition for the polynomials to be dense in $L^2_a(G)$ and which provides new examples of cyclic analytic Toeplitz operators on $H^2(U)$ and composition operators with dense range on $H^2(U)$. For any positive Borel measurable function w on G, let $L^2_a(G, w\,dA)$ represent the weighted Bergman space consisting of those analytic functions f on G which satisfy

$$\|f\|_w^2 = \int_G |f|^2 w\,dA < \infty.$$

PROPOSITION 3.3. *Let φ map U univalently onto G. The polynomials are dense in $L^2_a(G, (1 - |\varphi^{-1}(z)|^2)\,dA)$ if and only if the polynomials in φ are dense in $H^2(U)$.*

Proof. Recall that for $f = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^2(U)$,

$$\|f\|_{H^2(U)}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta$$

where for almost every θ, $f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$. By considering the Taylor expansion of $f \in H^2(U)$, one may easily verify that $\| \ \|_{H^2(U)}$ is equivalent to $\| \ \|$ where

$$\|f\|^2 = |f(0)|^2 + \int_U |f'|^2 (1 - |z|^2) \, dA.$$

Now if the polynomials are dense in $L_a^2(G, (1 - |\varphi^{-1}(z)|^2) \, dA)$, then $\{ p(\varphi)\varphi' \colon p \text{ is a polynomial} \}$ is dense in $L_a^2(U, (1 - |z|^2) \, dA)$. Integrating, we see that $\{ p(\varphi) \colon p \text{ is a polynomial} \}$ is dense in $H^2(U)$. The converse follows by differentiating.

This simple proposition has several interesting consequences. The following consequence seems to have been overlooked in the literature. It shows that, for example, a result of Caughran (cf. [**4**, Theorem 1]) is actually an easy consequence of the result of Farrell and Markusevic. Recall that a function f belongs to the Hardy space $H^2(G)$ provided $|f|^2$ has a harmonic majorant on G. One defines the norm of $f \in H^2(G)$ by $\|f\|_{H^2(G)} = [u(z_0)]^{1/2}$, where z_0 is a fixed point in G and u is the least harmonic majorant of $|f|^2$. If φ maps U univalently onto G with $\varphi(0) = z_0$ then the correspondence $f \leftrightarrow f(\varphi)$ is an isometric isomorphism between $H^2(G)$ and $H^2(U)$ (normed as in the proof of Proposition 3.3) (cf. for example [**5**, Chapter 10]).

COROLLARY 3.4. *Let φ map U univalently onto G. Density of the polynomials in $L_a^2(G)$ (or in $L_a^2(G, (1 - |\varphi^{-1}(z)|^2) \, dA)$) implies density of the polynomials in $H^2(G)$.*

Proof. It's easy to see via a change of variables that density of the polynomials in $L_a^2(G)$ implies density of the polynomials in $L_a^2(G, (1 - |\varphi^{-1}(z)|^2) \, dA)$. By Proposition 3.3, $\{ p(\varphi) \colon p \text{ is a polynomial} \}$ is dense in $H^2(U)$, but this is equivalent to density of the polynomials in the Hardy space $H^2(G)$.

COROLLARY 3.5. *If φ maps U univalently onto G and if the polynomials are dense in $L_a^2(G)$ (or in $L_a^2(G, (1 - |\varphi^{-1}(z)|^2) \, dA)$), then φ is univalent almost everywhere on the unit circle C.*

Proof. By Proposition 3.3, $\{p(\varphi): p \text{ is a polynomial}\}$ is dense in $H^2(U)$; in particular, there is a sequence $\{p_n\}$ of polynomials such that $\|p_n(\varphi) - z\|_{H^2(U)} \to 0$. Choose a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ such that $p_{n_j}(\varphi(z)) \to z$ a.e. on C. Off the set of measure zero on which $p_{n_j}(\varphi(z))$ may not go to z, φ must be univalent.

The following two corollaries are immediate consequences of Proposition 3.3.

COROLLARY 3.6. *If φ maps U univalently to G and if the polynomials are dense in $L_a^2(G, (1 - |\varphi^{-1}(z)|^2)\,dA)$, then the analytic Toeplitz operator T_φ: $H^2(U) \to H^2(U)$ is cyclic with cyclic vector* 1.

If φ is a weak-star generator of H^∞, then it's easy to see (for example, by using $\operatorname{Lat} T_z = \operatorname{Lat} T_\varphi$) that T_φ is cyclic with cyclic vector 1. John Akeroyd [**1**] has produced examples of cyclic analytic Toeplitz operators whose symbols are not weak-star generators. In fact, he has shown that if φ maps U univalently onto a crescent bounded by two internally tangent circles, then T_φ is cyclic with cyclic vector 1. That a crescent is not the image of a weak-star generator follows from [**16**, Corollary 2]. Corollary 3.6 above provides further examples of cyclic analytic Toeplitz operators. There are bounded simply connected domains G such that the polynomials are dense in $L_a^2(G)$ (hence in $L_a^2(G, (1 - |\varphi^{-1}(z)|^2)\,dA)$), but G is not the image of a weak-star generator. For example, Mergeljan and Tamadjan [**10**] (cf. also [**3**]) have shown that if sufficiently many slits are put in the unit disk, one can obtain a domain G such that the polynomials are dense in $L_a^2(G)$. Once again, Corollary 2 of [**16**] shows that the disk with these slits is not the image of a weak-star generator of H^∞.

COROLLARY 3.7. *Let φ map U univalently onto $G \subset U$ and define C_φ: $H^2(U) \to H^2(U)$ by $(C_\varphi f)(z) = f(\varphi(z))$. If the polynomials are dense in $L_a^2(G, (1 - |\varphi^{-1}(z)|^2)\,dA)$ then C_φ has dense range.*

Corollary 3.7 extends a result of Roan [**13**] by providing additional examples of composition operators with dense range.

REMARK. It's easy to see that if C_φ has dense range or if T_φ is cyclic, then φ is univalent on U and univalent a.e. on C.

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