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THE ORIENTED HOMOTOPY TYPE OF SPUN 3-MANIFOLDS

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We show that, bar unexpected developments in 3-manifold theory, the fundamental group and the choice of framing determine the oriented homotopy type of spun 3-manifolds.

1. The object of this note is to classify spun 3-manifolds up to oriented homotopy type. The notion of spinning was introduced by Artin [**1**] in the context of knots. The asphericity of classical knots implies that spun knots with isomorphic fundamental groups have homotopy equivalent complements. What we do is extend this to closed manifolds.

Let M^3 be a closed, oriented 3-manifold, and $\mathring{M}$ be M with an open 3-ball removed. Gordon [**4**] defines the *spin* of M to be the closed, oriented, smooth 4-manifold $s(M) = \partial(\mathring{M} \times D^2)$. Note that $s(M)$ is obtained by gluing $\mathring{M} \times S^1$ to $S^2 \times D^2$ via $\mathrm{id}_{S^2 \times S^1}$. There is one other possible choice of gluing map, the "Gluck twist" τ: $((\theta, \phi), \psi) \mapsto ((\theta + \psi, \phi), \psi)$ corresponding to $\pi_1(\mathrm{SO}(3)) \cong \mathbf{Z}_2$. The resulting manifold $s'(M) = \mathring{M} \times S^1 \cup_\tau S^2 \times D^2$ is called the *twisted spin* of M [**9**]. The two spins of M have the same fundamental group as M. In fact, they have identical 3-skeleta, but different attaching maps for the 4-cell. If M admits a circle action with fixed points (e.g. M is a lens space), then $s(M) \cong s'(M)$, but if M is aspherical $s(M) \not\simeq s'(M)$, as shown by Plotnick [**11**].

Every closed, oriented M^3 admits a (unique up to order) connected sum decomposition $M_1 \# M_2 \# \cdots \# M_n$, with prime factors M_i either aspherical, spherical, or $S^2 \times S^1$ (see e.g. [**6**]). The spherical factors are of the form Σ^3/π, with Σ^3 a homotopy 3-sphere and π a finite group acting freely on Σ^3. Consider only manifolds M^3 satisfying the condition

(1.1) All spherical factors are either homotopy 3-spheres or spherical Clifford-Klein manifolds (i.e. S^3/π, π acting linearly).

Under this assumption (no counterexamples are known!), we will prove the following

THEOREM 1.2. *If* $\pi_1(M) \cong \pi_1(M')$, *then* $s(M) \simeq s(M')$ *and* $s'(M) \simeq s'(M')$.

Here $\simeq$ stands for orientation-preserving homotopy equivalence. We use $\cong$ for orientation-preserving diffeomorphism, and $-M$ for M with reversed orientation.

2. The starting point of the proof is the following theorem of C. B. Thomas [**12**]: Two closed, oriented 3-manifolds have the same oriented homotopy type iff the prime factors pair off by orientation-preserving homotopy equivalences. In fact, if M and M' have isomorphic π_1, then $\pi_2(M) \cong \pi_2(M')$ as left $\mathbf{Z}\pi_1$-modules, and the only obstructions to an oriented homotopy equivalence are the first k-invariant of the connected sum of the aspherical factors, and the second k-invariants of the spherical factors.

Now suppose M and M' satisfy condition (1.1) and that $\pi_1(M) \cong \pi_1(M')$ but $M \not\simeq \pm M'$. It follows from Thomas' theorem and known facts about Clifford-Klein manifolds (see e.g. Orlik [**8**]) that, up to connected sum with other factors, M and M' must be a connected sum of terms of the form:

$$(2.1) \qquad M = M_1 \# M_2, \quad M' = (-M_1)\# M_2, \quad \text{with } M_i \not\simeq -M_i$$

or

$$(2.2) \quad M = L(p,q), \quad M' = L(p,q'), \quad \text{with } qq' \not\equiv \pm m^2 \ (\mathrm{mod}\, p).$$

It is clear that, in order to prove our Theorem, we have to see what happens when we spin the manifolds in (2.1) and (2.2). This will be done in the next two sections.

3. We first study the behavior of spinning with respect to connected sum and change in orientation. This was done by Gordon [**4**] for untwisted spins. A direct argument can be given to prove a similar result for twisted spins. Instead, we will prove an equivariant version using Fintushel's classification of circle actions on 4-manifolds in terms of their "weighted orbit spaces" [**2**].

Given M^3, the two spins $s(M) = \mathring{M} \times S^1 \cup_{\mathrm{id}} S^2 \times D^2$ and $s'(M) = \mathring{M} \times S^1 \cup_{\tau} S^2 \times D^2$ admit effective circle actions: translation on the second factor of $\mathring{M} \times S^1$ extends to $S^2 \times D^2$ via $t \cdot ((\theta,\phi),(r,\psi)) = ((\theta,\phi),(r,\psi + t))$, resp. $= ((\theta - t,\phi),(r,\psi + t))$ [**9**]. The weighted orbit spaces are $\mathring{M}$ (resp. M), with fixed point sets $S^2 \times \{0\}$ (resp. $S^0 \times \{0\}$) labelled 0 (resp. ± 1).

LEMMA 3.1. *There are equivariant diffeomorphisms* $s(M_1 \# M_2) \cong s(M_1)\# s(M_2)$ *and* $s'(M_1 \# M_2) \cong s'(M_1)\# s'(M_2)$.

Proof. Taking equivariant connected sum about suitable fixed points, one finds circle actions on $s(M_1)\# s(M_2)$ and $s'(M_1)\# s'(M_2)$ with orbit spaces $\mathring{M}_1 \natural \mathring{M}_2$ (resp. $M_1\# M_2$) and fixed point sets S^2 (resp. S^0), labelled 0 (resp. ± 1). These are precisely the weighted orbit spaces of $s(M_1\# M_2)$ and $s'(M_1\# M_2)$. The lemma follows from **[2]**, Theorem 9.7. □

LEMMA 3.2. *There are equivariant diffeomorphisms* $s(-M) \cong s(M)$ *and* $s'(-M) \cong s'(M)$.

Proof. Certainly $s(-M) \cong -s(M)$ and $s'(-M) \cong -s'(M)$. It remains to show that $s(M)$ and $s'(M)$ admit orientation-reversing diffeomorphisms. For that, start with the map $(x, \psi) \to (x, -\psi)$ on $\mathring{M} \times S^1$ and extend it to $S^2 \times D^2$ by mapping $((\theta, \phi), (r, \psi))$ to $((\theta, \phi), (r, -\psi))$, respectively to $((\theta + 2\psi, \phi), (r, -\psi))$. □

Before proceeding with the proof, we pause for a few remarks. Recall that in general the two spins of M are not even homotopy equivalent. The next proposition shows that they are stably diffeomorphic.

PROPOSITION 3.3. *There is an equivariant diffeomorphism* $s(M)\#\mathbf{CP}^2 \cong s'(M)\#\mathbf{CP}^2$.

Proof. Define an S^1-action on $\mathbf{CP}^2$ by $t \cdot (z_0 : z_1 : z_2) = (tz_0 : z_1 : z_2)$. The orbit space is D^3 and the fixed point set consists of a sphere labelled 1, and a point labelled -1. Form the equivariant connected sums: $s(M)\#\mathbf{CP}^2$ along the fixed S^2's and $s'(M)\#\mathbf{CP}^2$ along fixed points with opposite labels. The resulting S^1-manifolds have the same orbit data: orbit space $\mathring{M}$ and fixed point set a sphere labelled 1, and a point labelled -1. Hence they are equivariantly diffeomorphic. □

REMARK 3.4. A straightforward Mayer-Vietoris sequence shows $H_2(s(M)) \cong H_2(s'(M)) \cong H_1(M) \oplus H_2(M)$. Thus, if M is a homology 3-sphere, the spins of M are homology 4-spheres **[9]**. Lemma 3.1 says that s and s' are homomorphisms from the monoid of oriented homology 3-spheres to the monoid of oriented homology 4-spheres. These homomorphisms are *not* injective. Indeed, if $\Sigma = \Sigma(p, q, r)$ is a Brieskorn homology sphere, then $\Sigma \not\cong -\Sigma$ by [7], but $s(\Sigma) \cong s(-\Sigma)$ and $s'(\Sigma) \cong s'(-\Sigma)$ by 3.2. Even if we ignore orientations s and s' fail to be injective: $(-\Sigma)\#\Sigma \not\cong \pm(\Sigma\#\Sigma)$ by Thomas' theorem, but $s((-\Sigma)\#\Sigma) \cong s(\pm(\Sigma\#\Sigma))$ and $s'((-\Sigma)\#\Sigma) \cong s'(\pm(\Sigma\#\Sigma))$ by 3.1 and 3.2.

4. We now study the effect of spinning on spherical 3-manifolds. As we are mainly interested in lens spaces, for which the framing is irrelevant, we will consider only untwisted spins.

Let $M^3 = \Sigma^3/\pi$ be a spherical manifold. The punctured manifold $\mathring{M}$ has $\pi_1(\mathring{M}) = \pi$ and $\pi_2(\mathring{M}) = \mathbf{Z}\pi/N$, where $N = \sum_{g\in\pi} g$ is the norm element. The spin of M has $\pi_1(s(M)) = \pi$ and $\pi_2(s(M)) \cong I\pi \oplus \pi_2(\mathring{M})$, where $I = I\pi$ is the augmentation ideal of $\mathbf{Z}\pi$. If we let $I^* = \operatorname{Hom}_{\mathbf{Z}}(I, \mathbf{Z})$, with left π-action given by $gx^*(y) = x^*(g^{-1}y)$, then $\mathbf{Z}\pi/N \cong I^*$. Therefore, as $\mathbf{Z}\pi$-modules,

$$\pi_2(s(M)) \cong I \oplus I^*.$$

The equivariant intersection form on $\pi_2(s(M))$ corresponds to the canonical hyperbolic form on $I \oplus I^*$ (see **[11]** for details). As for the k-invariant of $s(M)$, note that the inclusion map $\mathring{M} \to s(M)$ induces id: $\pi_1(\mathring{M}) \to \pi_1(s(M))$ and the inclusion $\pi_2(\mathring{M}) \hookrightarrow I\pi \oplus \pi_2(\mathring{M})$. The induced map $H^3(\pi_1(\mathring{M}); \pi_2(\mathring{M})) \to H^3(\pi_1(s(M)); \pi_2(s(M)))$ sends $k(\mathring{M})$ to $k(s(M))$. Hence $k(s(M)) = (0, k(\mathring{M})) \in H^3(\pi; I) \oplus H^3(\pi; I^*)$.

LEMMA 4.1. *Let M and M' be spherical 3-manifolds with $\mathring{M} \simeq \mathring{M}'$. Then $s(M) \simeq s(M')$.*

Proof. Let f: $\mathring{M} \to \mathring{M}'$ be the given homotopy equivalence. It induces an isomorphism α: $\pi \to \pi$ and an α-isomorphism β^*: $I^* \to I^*$. Let β: $I \to I$ be the α-isomorphism dual to β^*. Define an α-isomorphism $\bar{\beta}$: $\pi_2(s(M)) \to \pi_2(s(M'))$ by

$$\bar{\beta} = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta^* \end{pmatrix} : I \oplus I^* \to I \oplus I^*.$$

Clearly $\bar{\beta}$ is an isometry of the hyperbolic form on $I \oplus I^*$. Moreover,

$$\bar{\beta}_*(k(s(M))) = (0, f_*(k(\mathring{M}))) = \left(0, \alpha^*\left(k(\mathring{M}')\right)\right) = \alpha^*\left(k\left(s(M')\right)\right),$$

showing that $\bar{\beta}$ preserves k-invariants. The lemma now follows from the following result of Hambleton and Kreck:

THEOREM **[5]**. *The homotopy type of a closed, oriented 4-manifold with π_1 a finite group having periodic cohomology of period dividing 4 is determined by (π_1, π_2, k) and the equivariant intersection form on π_2.* □

Now let $L = L(p,q)$ and $L' = L(p,q')$ be two lens spaces with isomorphic fundamental group. Clearly, $s(L)$ and $s(L')$ are equivariantly diffeomorphic if and only if L is homeomorphic to L'. Whether $s(L)$ is diffeomorphic to $s(L')$ seems an interesting question.[1] A partial answer is provided by the following corollary to Lemma 4.1.

COROLLARY 4.2. $s(L(p,q)) \simeq s(L(p,q'))$. □

As noted in [5], standard surgery techniques now yield

COROLLARY 4.3. *If p is odd, $s(L(p,q))$ is homeomorphic to $s(L(p,q'))$.* □

The homotopy equivalence in Corollary 4.2 can actually be defined "by hand", without using [5]. Here is a sketch of the construction. Start with the usual cell decomposition $e_0 \cup e_1 \cup e_2$ for $\mathring{L} = \mathring{L}(p,q)$, with lifts $\tilde{e}_i$ in the universal cover $\tilde{\mathring{L}}$. Then $\pi_1 = \mathbf{Z}_p$, generated by $g = [e_1]$ and $\pi_2 = \mathbf{ZZ}_p/N$, generated by the boundary sphere $(g^s - 1)\tilde{e}_2$, where $qs \equiv 1 \pmod p$. Let $f\colon \mathring{L} \to \mathring{L}'$ be the homotopy equivalence gotten by deform retracting the punctured lens spaces to their 2-skeleta. The lift $\tilde{f}\colon \tilde{\mathring{L}} \to \tilde{\mathring{L}}'$ takes $(g^s - 1)\tilde{e}_2$ to $x(g^{s'} - 1)\tilde{e}_2$, for some x in $\mathbf{ZZ}_p$. (Under the isomorphism $\mathbf{ZZ}_p/N \cong (I\mathbf{Z}_p)^*$, the map $\cdot x$ corresponds to β^*.) Similarly, $\tilde{f}^{-1}$ takes $(g^{s'} - 1)\tilde{e}_2$ to $y(g^s - 1)\tilde{e}_2$, for some $y = \sum_{i=0}^{p-1} n_i g^i \in \mathbf{ZZ}_p$. Let $\bar{y} = \sum_{i=0}^{p-1} n_i g^{-i}$ be the conjugate of y. (The map $\cdot\bar{y}\colon I\mathbf{Z}_p \to I\mathbf{Z}_p$ is β^{-1}.) Define an extension $f\colon \mathring{L} \times S^1 \to \mathring{L}' \times S^1$ by sending S^1 to $\bar{y}S^1$. In general f is *not* a homotopy equivalence. But notice that

$$\tilde{f}\big((g^s - 1)\tilde{e}_2 \times S^1\big) = \bar{y}(g^s - 1)\tilde{e}_2 \times S^1 = g^{s-s'}\big(g^{s'} - 1\big)\tilde{e}_2 \times S^1,$$

i.e. $\tilde{f}$ maps the lifts of $\partial(\mathring{L} \times S^1)$ to lifts of $\partial(\mathring{L}' \times S^1)$. Hence, up to homotopy, f preserves boundaries and thus can be extended via id: $S^2 \times D^2 \to S^2 \times D^2$ to a map $f\colon s(L) \to s(L')$.

The same construction, using $f^{-1}\colon \mathring{L}' \to \mathring{L}$ and sending S^1 to $\bar{x}S^1$, yields a map $f^{-1}\colon s(L') \to s(L)$. It is easy to check that $f_{\#}^{-1}$ is a chain homotopy inverse of $f_{\#}$ on the chain complexes of the universal covers. By Whitehead's theorem, f is a homotopy equivalence.

5. Proof of Theorem 1.2. Let M and M' be 3-manifolds with isomorphic fundamental groups and satisfying condition (1.1). Take prime

[1] (*Added in proof.*) The answer is yes, they are diffeomorphic. This follows from P. S. Pao, *The topological structure of 4-manifolds with effective torus actions* (I), Trans. Amer. Math. Soc., **227** (1977), 279–317.

decompositions $M = M_1 \sharp \cdots \sharp M_n$ and $M' = M'_1 \sharp \cdots \sharp M'_n$. After reordering the factors if necessary, the Kurosh subgroup theorem provides isomorphism $\pi_1(M_i) \cong \pi_1(M'_i)$, $1 \le i \le n$. The factors M_i and M'_i are both $S^2 \times S^1$, aspherical or spherical. If $M_i = S^2 \times S^1$, then clearly $s(M_i) \cong s(M'_i)$. If M_i is aspherical, then $M_i \simeq \pm M'_i$, and so, by Lemma 3.2, $s(M_i) \simeq s(M'_i)$. Finally, if M_i is spherical, the discussion in §2 and Corollary 4.2 show $s(M_i) \simeq s(M'_i)$.

Now piece together the orientation-preserving homotopy equivalences $s(M_i) \simeq s(M'_i)$ (see [12], Lemma 2.9) to get $s(M_1) \sharp \cdots \sharp s(M_n) \simeq s(M'_1) \sharp \cdots \sharp s(M'_n)$. By Lemma 3.1, $s(M) \simeq s(M')$. This proves the Theorem in the untwisted case. The proof for twisted spins is identical. $\square$

The above proof raises the question: if $\pi_1(M) \cong \pi_1(M')$, is $s(M) \cong s(M')$? If the aspherical pieces M_i and M'_i are Haken, then indeed $M_i \cong \pm M'_i$ and so $s(M_i) \cong s(M'_i)$. If the spherical pieces are geometric, one still has to answer the question in §4, namely is $s(L(p,q)) \cong s(L(p,q'))$?

6. We conclude with an application to knot theory. We will need the following theorem of Goldsmith and Kauffman ("Fox's conjecture") [3]. Let $K = (S^n, S^{n-2})$, $n \ge 3$, be a smooth knot. Let $(a,b) \in \mathbf{Z} \times \mathbf{Z} - \{(0,0)\}$ and $k = \text{g.c.d.}(a,b)$. Then the b-fold cyclic branched cover $M_b(S^{n+1}, K^a)$ of the a-twist spin $K^a = (S^{n+1}, S^{n-1})$ is diffeomorphic to one of the two spins of the k-fold cyclic branched cover $M_k(S^n, K)$ of K. In particular, $s(M_k(S^n, K)) \cong M_k(S^{n+1}, K^0)$, where K^0 is the spin of K.

Every lens space $L(p,q)$ is the 2-fold branched cover of a unique 2-bridge link $B_{p,q}$. If p is odd, $B_{p,q}$ is a knot and we can form the 2-twist spin $B^2_{p,q}$, with exterior $X_{p,q}$ fibered over S^1, with fiber $\mathring{L}(p,q)$. Clearly $X_{p,q} \simeq X_{p,q'}$ for every q, q'. But, as shown in [10], $X_{p,q} \simeq X_{p,q'} (\operatorname{rel} \partial)$ if and only if $L(p,q) \simeq L(p,q')$. On the other hand, $M_{2k}(S^4, B^2_{p,q}) \cong s(L(p,q))$ and $M_{2k+1}(S^4, B^2_{p,q}) \cong S^4$ by the above result of Goldsmith and Kauffman. These facts together with Corollary 4.3 imply

COROLLARY 6.1. *There are* (*arbitrarily many*) *knots in* S^4 *whose exteriors are homotopy equivalent but not homotopy equivalent* $(\operatorname{rel} \partial)$, *yet all of whose finite cyclic branched covers are homeomorphic.* $\square$

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