CONVERGENCE FOR THE SQUARE ROOT OF THE POISSON KERNEL

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Let X be a symmetric space and f an integrable function on its boundary ∂X. The 0-Poisson integral $P_0 f$ is the function on X obtained by integrating f against the square root of the Poisson kernel. We give Fatou theorems saying that the normalized function $P_0 f / P_0 1$ converges almost everywhere to f on ∂X. Many such results are known for λ-Poisson integrals $P_\lambda f$ with λ in the positive Weyl chamber. But the case $\lambda = 0$ is different, since larger regions of convergence can be used. Some of our results are general, some are given for the bidisk or $SL(3, \mathbf{R})/SO(3)$. The paper extends previous results by the author for the disk and the bidisk.

1. Introduction. Let $P = P(z, e^{i\theta})$ denote the Poisson kernel in the unit disk $\mathbf{D}$. If f is an L^1 function on $\mathbf{T} = \partial \mathbf{D}$ and $\lambda \in \mathbf{R}$, the λ-Poisson integral of f is

$$P_\lambda f(z) = \int_{\mathbf{T}} P(z, e^{i\theta})^{\lambda + 1/2} f(e^{i\theta}) \, d\theta.$$

It is an eigenfunction of the hyperbolic Laplacian in $\mathbf{D}$, with eigenvalue $4\lambda^2 - 1$. The corresponding normalized λ-Poisson integral is $\mathscr{P}_\lambda f = P_\lambda f / P_\lambda 1$, where 1 is the constant function on $\mathbf{T}$. Assume $\lambda \geq 0$. Then $\mathscr{P}_\lambda f(z) \to f(e^{i\theta})$ as $z \to e^{i\theta}$, uniformly for continuous f. This is not true when $\lambda < 0$.

Let $f \in L^1$. For $\lambda > 0$ it is known that $\mathscr{P}_\lambda f$ tends nontangentially to f a.e. in $\mathbf{T}$. This means that one lets $z \to e^{i\theta}$ satisfying a condition $|\arg z - \theta| < \text{const.}(1 - |z|)$. But when $\lambda = 0$, more is true. In fact, $\mathscr{P}_0 f(z) \to f(e^{i\theta})$ as $z \to e^{i\theta}$, $|\arg z - \theta| < \text{const.} (1 - |z|)\log(1 - |z|)^{-1}$ for a.a. θ, see Sjögren [16]. We call this weakly tangential convergence; it is false for $\mathscr{P}_\lambda f$, $\lambda > 0$. J. Taylor has found a simple description of it in terms of the hyperbolic metric.

In this paper, we shall give some convergence results for $\mathscr{P}_0 f$ in symmetric spaces, most of which are not valid for other $\mathscr{P}_\lambda f$. For rank 1 spaces, the generalization of weakly tangential convergence is rather straightforward, see Theorem 5.4 below. A metric description of the approach region appearing here is also given. Korányi and Picardello [11]

have an analogous result for homogeneous trees, which in many respects behave like rank 1 symmetric spaces.

For the bidisk $\mathbf{D}^2$, Sjögren [14] contains a result of this kind, which we now recall. Poisson integrals $P_\lambda f$ and $\mathscr{P}_\lambda f$, $f \in L^1(\mathbf{T}^2)$, are here defined as in $\mathbf{D}$, by means of the kernel

$$P(z_1, e^{i\theta_1})^{\lambda+1/2} P(z_2, e^{i\theta_2})^{\lambda+1/2}, \quad (z_1, z_2) \in \mathbf{D}^2, \quad (e^{i\theta_1}, e^{i\theta_2}) \in \mathbf{T}^2.$$

In $\mathbf{D}^2$ one defines restricted (nontangential) convergence by letting $z = (z_1, z_2) \to (e^{i\theta_1}, e^{i\theta_2}) \in \mathbf{T}^2$ in such a way that each z_j tends nontangentially to $e^{i\theta_j}$ and

$$(1.1) \qquad\qquad 1 - |z_1| \sim 1 - |z_2|.$$

Here and in the sequel, $F \sim G$ means $C^{-1} \le F/G \le C$, and C denotes many different (large) constants. As is well known, restricted nontangential convergence holds a.e. when $f \in L^1$ for the standard Poisson integral $P_{1/2}f$, and also for $\mathscr{P}_\lambda f$, $\lambda > 0$. Theorem 2 of [14] says that for $\mathscr{P}_0 f$ one can replace (1.1) by the weaker condition

$$(1.2) \qquad\qquad \log(1 - |z_1|) \sim \log(1 - |z_2|).$$

This is called weakly restricted (nontangential) convergence.

Another natural generalization to $\mathbf{D}^2$ of the result in $\mathbf{D}$ would be to use weakly tangential convergence in each z_j. Then we again have a.e. convergence of $\mathscr{P}_0 f$, $f \in L^1$, provided we use the restriction condition (1.1). This is Theorem 4.1 below. It is easy to see that (1.1) cannot be replaced by (1.2) here. As in $\mathbf{D}$, one would like to describe Theorem 4.1 in terms of the bihyperbolic distance. However, the natural attempt to do this fails, as verified in §4.

In §5, we consider a general Riemannian symmetric space $X = G/K$ of the noncompact type. It has a boundary K/M. The (standard) notation used here will be explained below. Several modes of convergence at K/M are used, which we briefly recall. See further Korányi [10]. Admissible convergence corresponds to approaching a boundary point kM by means of points $k \exp(H) \cdot x$, with $H \to +\infty$ in $\mathfrak{a}$ and x staying in a compact subset of X. Letting instead $H \to +\infty$ along the ray $\{tH_0 : t > 0\}$ for any fixed H_0 in the positive Weyl chamber $\mathfrak{a}_+$, one defines restricted convergence. Weakly restricted convergence is defined by letting $H \to +\infty$ staying in a strict subcone of $\mathfrak{a}_+$. This corresponds to (1.2) in $\mathbf{D}^2$. In Theorem 5.1, we show that the normalized 0-Poisson integral $\mathscr{P}_0 f$ converges weakly restrictedly a.e. to f for $f \in L^1(K/M)$. When f is continuous, we obtain admissible convergence for $f \in L^1(K/M)$, uniformly in K/M, which has also been proved by van den Ban and Schlichtkrull [2].

Theorem 5.1 was known previously only for rank $X = 1$, see Michelson [12]. The rank 1 result Theorem 5.4 was mentioned above.

The last two sections of this paper deal with the rank 2 space $SL(3, \mathbf{R})/SO(3)$. Theorem 6.1 gives a.e. admissible convergence of $\mathscr{P}_0 f$ for $f \in L^p$, $p > 1$. The proof of admissible convergence of $\mathscr{P}_\lambda f$, $\lambda \in \mathfrak{a}_+$, in a general symmetric space from Sjögren [17] does not apply to $\mathscr{P}_0 f$, and we use a method which is specific to $SL(3, \mathbf{R})/SO(3)$.

Finally, Theorem 7.1 is an $SL(3, \mathbf{R})/SO(3)$ analog of the weakly tangential convergence result in the bidisk. With the $\overline{N}$ realization of the boundary, we approach $n_1 \in \overline{N}$ by means of points

$$n_1 \exp(tH_0)n \cdot x, \qquad t \to +\infty.$$

Here $H_0 \in \mathfrak{a}_+$ is fixed and $x \in X$ stays in a compact set. Further, n satisfies a size condition $|n| \le Ct^q$, where $q = q(H_0)$ is positive, which makes this convergence stronger than restricted convergence. We show that $\mathscr{P}_0 f$ converges to f a.e. in this sense, when $f \in L^1$. The proof is rather calculatory and relies on the explicit expression for the Poisson kernel. As a byproduct of the proof, we obtain a restricted convergence result for $\mathscr{P}_\lambda f$ with $\lambda \neq 0$ on the boundary of $\mathfrak{a}_+$.

The preparatory §2 describes the necessary symmetric space theory and notation. Section 3 contains our main lemma and also an estimate for the spherical function corresponding to $\lambda = 0$ previously obtained by van den Ban.

The author has profited from a valuable conversation with T. Lyons.

2. Preliminaries about symmetric spaces. Let X be a Riemannian symmetric space of the noncompact type, written as $X = G/K$ in the standard way. The action of G on X is written $g \cdot x$, and $o = eK \in X$. Take an Iwasawa decomposition KAN. Thus any $g \in G$ can be written $g = k(g) \exp(H(g))n(g)$, with $H(g)$ in the Lie algebra $\mathfrak{a}$ of A. As usual, the nilpotent group $\overline{N}$ is the image of N under the Cartan involution, and M is the centralizer of A in K.

The (maximal distinguished) boundary of X is K/M, and $n \to k(n)M$ gives a diffeomorphism of $\overline{N}$ onto almost all of K/M. Here we refer to the normalized invariant measure dkM in K/M. Let $\mathfrak{a}_+ \subset \mathfrak{a}$ be the positive Weyl chamber. By means of the Killing form $\langle \cdot , \cdot \rangle$, we identify $\mathfrak{a}$ with its dual. Then the (restricted) roots are in $\mathfrak{a}$, and $\rho \in \mathfrak{a}_+$ denotes the half-sum of the positive roots, counted with multiplicity. Let Σ^+ be the set of positive roots and Σ_0 the set of roots α for which $\alpha/2$ is not a root. Put $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$. The Weyl group W consists of linear orthogonal maps

$\mathfrak{a} \to \mathfrak{a}$ which permute Σ_0. For $s \in W$, we write

$$(-1)^s = (-1)^{\text{card}(s(\Sigma_0^+)\setminus\Sigma_0^+)}.$$

This expression has the same value for s and s^{-1}.

Let H vary in $\mathfrak{a}$. We shall say that $H \to +\infty$ in $\mathfrak{a}$ if $\langle \alpha, H \rangle \to +\infty$ for all $\alpha \in \Sigma^+$, and similarly we speak of large H in $\mathfrak{a}$ or $\mathfrak{a}_+$. A function $\varphi \colon \mathfrak{a} \to \mathbf{R}$ is called increasing if $H - H' \in \mathfrak{a}_+$ implies $\varphi(H) \geq \varphi(H')$. If $H \in \mathfrak{a}$ and $n \in \overline{N}$, we write n^H for $\exp(H) n \exp(-H)$. In terms of canonical coordinates in $\overline{N}$, (see [17], Sec. 5), the map $n \to n^H$ is an anisotropic dilation $(x_j) \to (e^{-\langle \alpha_j, H \rangle} x_j)$, where $\alpha_j \in \Sigma^+$. The same applies to N, except that there is no minus sign in front of $\langle \alpha_j, H \rangle$.

For $\lambda \in \mathfrak{a}$, the λ-Poisson integral of $f \in L^1(K/M)$ is

$$P_\lambda f(g) = P_\lambda f(g \cdot o) = \int_{K/M} f(kM) e^{-\langle \lambda + \rho, H(g^{-1}k) \rangle} \, dkM, \qquad g \in G.$$

Considered as a function in X, it is a joint eigenfunction of all G-invariant differential operators in X. Since the Laplacian of X is G-invariant, we have Harnack's inequality: $P_\lambda f(x) \sim P_\lambda f(x')$ if x and x' stay in a compact set in X, for $f \geq 0$. By G-invariance similarly

$$(2.1) \qquad\qquad P_\lambda f(g \cdot x) \sim P_\lambda f(g \cdot x'),$$

uniformly for $g \in G$.

When $f \equiv 1$, we get the spherical function $P_\lambda 1$, which is biinvariant, i.e., defined on $K \backslash G / K$. The normalized λ-Poisson integral of f is $\mathscr{P}_\lambda f = P_\lambda f / P_\lambda 1$. In the case $\lambda = 0$, we set for $H \in \mathfrak{a}$

$$\psi(H) = e^{\langle \rho, H \rangle} P_0 1(\exp H).$$

This function grows polynomially for large H.

The integral defining $P_\lambda f$ can be transformed to $\overline{N}$. For $g \cdot o = n_1 a_1 \cdot o$, $n_1 \in \overline{N}$, $a_1 \in A$, this gives

$$(2.2) \quad P_\lambda f(n_1 a_1) = \int_{\overline{N}} f(k(n)M) e^{\langle \lambda - \rho, H(n) \rangle} e^{-\langle \lambda + \rho, H(a_1^{-1} n_1^{-1} n) \rangle} \, dn.$$

Of course, dn is Haar measure in $\overline{N}$. We often replace $f(k(n)M)$ by $f(n)$ here, working with functions in $\overline{N}$.

Let $a_1 = \exp H$, $H \in \mathfrak{a}_+$. To estimate $P_0 1(n_1 \exp H)$, we write $n_1 = kan \in KAN$. If n_1 stays in a compact subset of $\overline{N}$, also k, a, and n must stay in compact sets. Then $n_1 \exp H = k \exp(H) an^{-H}$, and n^{-H} stays bounded in N for large H. Since $P_0 1$ is biinvariant, (2.1) gives

$$(2.3) \qquad P_0 1(n_1 \exp H) = P_0 1(\exp(H) an^{-H}) \sim P_0 1(\exp H)$$

$$= \psi(H) e^{-\langle \rho, H \rangle}.$$

The Poisson kernel in $\overline{N}$ is defined by

$$P(n) = e^{-2\langle \rho, H(n) \rangle}.$$

For any function F in $\overline{N}$, we write $\tilde{F}(n) = F(n^{-1})$. Also, the normalized contraction of F is defined by $F_H(n) = e^{2\langle \rho, H \rangle} F(n^{-H})$ for $H \in \mathfrak{a}$. Then $\int F_H \, dn = \int F \, dn$ if $F \in L^1$. If $f \geq 0$ has compact support in $\overline{N}$, and n_1 stays in a compact set, it follows from (2.2) and (2.3) that

$$(2.4) \quad \mathcal{P}_0 f(n_1 \exp H) \leq C\psi(H)^{-1} e^{2\langle \rho, H \rangle} \int_{\overline{N}} f(n) e^{-\langle \rho, H((n_1^{-1}n)^{-H}) \rangle} \, dn$$

$$= C\psi(H)^{-1} f * \tilde{P}_H^{1/2}(n_1).$$

Here $\tilde{P}_H^{1/2}$ means $(\sqrt{\tilde{P}})_H$.

From Knapp and Williamson [9], Prop. 5.1, we know that $1/P$ is a finite sum of nonnegative homogeneous polynomials. This means that

$$(2.5) \quad 1/P(n^{-H}) = \sum_0^{\nu_0} e^{2\langle \beta_\nu, H \rangle} Q_\nu(n),$$

and each Q_ν is a nonnegative polynomial function of the canonical coordinates of n. Further, $Q_0 \equiv 1$. The β_ν are linear combinations of positive roots with nonnegative integer coefficients, and only β_0 is 0. Moreover, $\langle 2\rho - \beta_\nu, H \rangle \geq 0$ when $H \in \mathfrak{a}_+$, for all ν.

Given a fixed vector $H_0 \in \mathfrak{a}_+$, we can find a smooth homogeneous gauge $|\cdot|$ in $\overline{N}$. This is a function $\overline{N} \to \{t \geq 0\}$ which is smooth outside e and vanishes only at e. It also satisfies $|n^{-1}| = |n|$ and

$$\left| n^{tH_0} \right| = e^{-t} |n|, \qquad t \in \mathbf{R}.$$

This implies the quasinorm property

$$|nn'| \leq C(|n| + |n'|).$$

The ball $\{n: |n| \leq r\}$ is denoted by $B(r)$. Its Haar measure $|B(r)|$ is proportional to $r^{2\langle \rho, H_0 \rangle}$.

The gauge will be related to the canonical coordinates of n by

$$(2.6) \quad C^{-1} \max |x_j|^\alpha \leq |n| \leq C \max |x_j|^\beta$$

for some C and $\alpha, \beta > 0$.

Finally, we write $e_k = e^{-2^k}$, $k = 0, 1, \ldots$, and it will be convenient to let e_k for $k < 0$ denote large constants depending on the context.

3. Auxiliary results. We first give an $\overline{N}$ version of the method of k-pieces from Sjögren [14], cf. in particular Lemma 2. Let $L \subset \overline{N}$ be compact, and extend any function in L by 0 to all of $\overline{N}$.

LEMMA 3.1. (a) *Assume the sublinear operators* $(T_k)_1^\infty$ *are defined in* $L^1(L)$, *take values which are nonnegative measurable functions in* L, *and satisfy the following for some* $C_1 < \infty$.

(i) *Each* T_k *is of weak type* $(1,1)$ *with constant at most* C_1.

(ii) *The value* $T_k f(n)$, $n \in L$, *depends only on the restriction of* f *to the ball*
$nB(e_k)$.

(iii) *For some natural number* N,

$$\|T_k f\|_{L^\infty} \leq C_1 \sup_n |B(e_{k+N})|^{-1} \int_{nB(e_{k+N})} |f| \, dn.$$

Then the operator $f \to \sup_k T_k f$ *is of weak type* $(1,1)$ *with constant at most* $CC_1 N$, $C = C(L)$.

(b) *The hypothesis* (iii) *in* (a) *can be replaced by the following*:

(iii′) *Each* T_k *is given by*

$$T_k f(n) = \sup_{i \in I_k} |f| * K_i(n),$$

where I_k *is an index set and the kernels* K_i *satisfy* $\int K_i^*(n) \, dn \leq C_1$. *Here, for* $i \in I_k$,

$$K_i^*(n) = \sup_{n' \in B(e_{k+N})} K_i(nn').$$

Proof of (a). We shall apply Lemma 1 of [**14**], recalled below. Small values of k will cause minor complications, so we consider only $k \geq A$, where A is a suitable large integer. This is enough, because of (i). For each $k \geq A$, choose in $\overline{N}$ a lattice of cubes which are defined in terms of the canonical coordinates and whose sides are equal and approximately e_{k-A}. These cubes are called k-pieces. We also choose these lattices in such a way that each k-piece is the union of the $(k + 1)$-pieces contained in it. (Neglect the boundaries of the cubes.) Moreover, the ratio between the sides of a k-piece and a $(k + 1)$-piece should be odd.

Lemma 1 of [**14**] says that the operator $f \to \sup_k T_k' f$ is of weak type $(1,1)$ provided that

(α) the sublinear operators T_k' are uniformly of weak type $(1,1)$;

(β) the restriction of $T_k' f$ to a k-piece P depends only on $f \mid_P$;

(γ) $\|T_k' f\|_{L^\infty} \leq C \sup |P|^{-1} \int_P |f| \, dn$, the supremum taken over all $(k + N')$-pieces P, for some fixed $N' \in \{1, 2, \dots\}$.

From the proof in [14], it is immediately seen that the constant in the conclusion is $O(N')$.

Our T_k' will be slight modifications of the T_k. Because of (2.6), the variation of each coordinate in the ball $B(e_k)$ is at most e_{k-C}, which is much smaller than e_{k-A} if A is suitably chosen. The same holds for any ball $nB(e_k)$, $n \in L$. Thus (ii) implies that $T_k f(n)$ depends only on the restriction of f to the k-piece containing n, except when n is very close to the boundary of some k-piece. In this last case, we let $T_k' f(n) = 0$, otherwise $T_k' f(n) = T_k f(n)$. Then (β) holds, and also (α).

To obtain (γ) from (iii), it is enough to estimate the mean of $|f|$ in $nB(e_{k+N})$ by C times the largest of the means of $|f|$ in those $(k + N')$-pieces intersecting $nB(e_{k+N})$. For this we need only verify that these $(k + N')$-pieces have total measure at most $C|B(e_{k+N})|$, or that they are all contained in $nB(Ce_{k+N})$. But any point n' in such a piece is obtained by changing the coordinates of a point $n'' \in nB(e_{k+N})$ by at most $Ce_{k+N'-A}$. Since n' and n'' stay in a compact set, the coordinates of $(n'')^{-1}n'$ are $O(e_{k+N'-A})$. Then (2.6) implies $|(n'')^{-1}n'| \le e_{k+N'-A-C} \le e_{k+N}$, if $N' \ge N + A + C$. It follows that

$$n' = n''(n'')^{-1}n' \in nB(e_{k+N})B(e_{k+N}) \subset nB(Ce_{k+N}),$$

and (γ) is verified.

Thus $\sup_k T_k'$ is of weak type $(1,1)$, and the corresponding constant is at most $CN' \le CN$. To extend this to T_k, we must recover those values of $T_k f$ which we lost when defining T_k'. This is done as in the proof of Lemma 2 in [14]. Indeed, one can repeat the argument just given with the lattices translated half the side of an A-piece in one or more coordinate directions. Here translation is taken in the Euclidean sense. Since the ratios of the cube sides are odd, this will make us catch all of $T_k f$ in $2^{\dim \overline{N}}$ steps. Part (a) is proved.

Proof of (b). Assuming (iii'), we shall prove (iii). One can clearly assume $\operatorname{supp} K_i \subset L^{-1}L$. We consider the functions

$$K_i^{**}(n) = \sup_{|n'| \le e_{k+N+C}} K_i(n'n), \quad i \in I_k.$$

Take $n' \in B(e_{k+N+C})$ and write $n'n = nn^{-1}n'n$. The coordinates of $n^{-1}n'n$ are polynomials in the coordinates of n and n', and vanish for $n' = e$. They are thus bounded by const. times the largest coordinate of n' when n and n' stay bounded. Then $|n^{-1}n'n| \le e_{k+N}$ because of (2.6), if C is suitably chosen. Hence, $K_i^{**} \le CK_i^*$ and $\int K_i^{**} dn \le CC_1$.

With $i \in I_k$ and $f \geq 0$, we have

$$f * K_i(n_1) = \int f(n) K_i(n^{-1} n_1)\, dn \leq \int f(n) K_i^{**}(n' n^{-1} n_1)\, dn,$$

for any $n' \in B(e_{k+N+C})$. Thus,

$$f * K_i(n_1) \leq \int f(n)\, dn \big| B(e_{k+N+C}) \big|^{-1} \int_{|n'| \leq e_{k+N+C}} K_i^{**}(n' n^{-1} n_1)\, dn'.$$

Transforming $n \to n^{-1} n'$, we get

$$f * K_i(n_1) \leq \int K_i^{**}(n n_1)\, dn \big| B(e_{k+N+C}) \big|^{-1} \int_{|n'| \leq e_{k+N+C}} f(n^{-1} n')\, dn'.$$

Since $K_i^{**} \in L^1$, (iii) now follows, with $N + C$ instead of N. Lemma 3.1 is proved.

Next, we consider $P(n)^{1/2}$ and $\psi(H)$.

LEMMA 3.2. *For each $n \in \overline{N}$ the expression*

$$P_H^{1/2}(n) = e^{2\langle \rho, H \rangle} P(n^{-H})^{1/2}$$

is increasing as a function of $H \in \mathfrak{a}$.

This is almost Lemma 3.5 of Sjögren [13]. For the proof, we need only observe that all the terms in the sum in (2.5) become decreasing in H when multiplied by $e^{-4\langle \rho, H \rangle}$.

Via an integration, this lemma implies the known fact that the function ψ is increasing (Harish-Chandra [5], Lemma 36 p. 281). In [1], Theorem 6.6, van den Ban determined its order of magnitude at $+\infty$. We shall do this again, without insisting on the value of the corresponding constant. For the case $X = SL(3, \mathbf{R})/SO(3)$, see also Herz [8]. Let π be the polynomial

$$\pi(H) = \prod_{\alpha \in \Sigma_0^+} \langle \alpha, H \rangle, \qquad H \in \mathfrak{a}.$$

With any $H \in \mathfrak{a}$, we can associate the differential operator $\partial(H)$ defined in $\mathfrak{a}$ as differentiation along the vector H. Then ∂ can be extended to a homomorphism, also denoted ∂, from the ring of polynomials on $\mathfrak{a}$ into that of differential operators in $\mathfrak{a}$.

PROPOSITION 3.3. *For large $H \in \mathfrak{a}_+$, one has $\psi(H) \sim \pi(H)$.*

Proof. When $\lambda \in \mathfrak{a}_+$, one can expand $P_\lambda 1$:

$$(3.1) \qquad P_\lambda 1(\exp H) = \sum_{s \in W} c(s\lambda) e^{\langle s\lambda - \rho, H \rangle} \sum_{\mu \in \Lambda} \Gamma_\mu(s\lambda) e^{-\langle \mu, H \rangle},$$

see Helgason [7], Theorem IV.5.5 p. 430. Here c is Harish-Chandra's c-function, and Λ is the set of linear combinations of the simple positive roots with nonnegative integer coefficients. The quantities $\Gamma_\mu(\lambda)$ are given by $\Gamma_0(\lambda) = 1$ and by the recursion formula (12) of [7], IV.5 p. 427. To deal with the singularity of c at 0, we follow Harish-Chandra and write

$$P_0 1(\exp H) = c_1 \{ \partial(\pi)(\pi(\lambda) P_\lambda 1(\exp H)) \}_{\lambda = 0},$$

where the operator $\partial(\pi)$ is taken with respect to λ and $c_1 \neq 0$ is a constant. See also Exercise IV.B.1, p. 483 of [7]. Then (3.1) gives

$$(3.2) \quad \psi(H) = c_1 \left\{ \partial(\pi) \left(\sum_{s \in W} \pi(\lambda) c(s\lambda) e^{\langle s\lambda, H \rangle} \sum_{\mu \in \Lambda} \Gamma_\mu(s\lambda) e^{-\langle \mu, H \rangle} \right) \right\}_{\lambda = 0}.$$

Let $s \in W$. At $\lambda = 0$, the function $\pi(\lambda) c(s\lambda)$ is smooth and takes the value $c_2(-1)^s$, $c_2 \neq 0$, as seen from the explicit expression for $c(\lambda)$ (formula (43) of [7], IV.6 p. 447).

In the expansion of the derivative in (3.2), consider those terms obtained by letting $\partial(\pi)$ act only on $e^{\langle s\lambda, H \rangle}$ and taking only $\mu = 0$ in the last sum. These terms will sum up to

$$\sum_{s \in W} c_2(-1)^s \pi(sH) = c_2 \pi(H).$$

We shall see that the remaining part of (3.2) is much smaller, for large H.

Let

$$Q_\mu = \sup_{\pi'} \left| \{ \partial(\pi') \Gamma_\mu(\lambda) \}_{\lambda = 0} \right|,$$

the supremum taken over π' of type

$$\pi'(H) = \prod_{\alpha \in \Sigma'} \langle \alpha, H \rangle, \qquad \Sigma' \subset \Sigma_0^+.$$

We claim that there exists C such that for all $\mu \in \Lambda$

$$(3.3) \qquad\qquad Q_\mu \leq C e^{\langle \mu, \rho \rangle}.$$

This is a simple extension of the estimate for Γ_μ given in Lemma IV.5.3 of [7], where we have chosen $H = \rho$. The proof is similar: by differentiating the recursion formula for Γ_μ, one obtains a recursive inequality for Q_μ which is like (14) of [7], IV.5.3. After that, Helgason's arguments can be repeated and give (3.3).

Since now (3.3) allows us to sum in μ, it is not hard to see that the remaining part of (3.2) is $o(\pi(H))$ as $H \to +\infty$. The proposition follows.

4. The bidisk. Whenever convenient, we consider $\mathbf{T}$ as $\mathbf{R}/2\pi\mathbf{Z}$ rather than $\{|z| = 1\}$. Define for $(\theta_1, \theta_2) \in \mathbf{T}^2$ the region of restricted weakly tangential convergence by

$$\Omega_{\theta_1, \theta_2} = \left\{(z_1, z_2) = \left((1 - t_1)e^{i\varphi_1}, (1 - t_2)e^{i\varphi_2}\right) \in \mathbf{D}^2 : \right.$$

$$\left. 0 < t_1 \sim t_2 < 1/2, |\varphi_i - \theta_i| \le Ct_i \log t_i^{-1}, i = 1, 2\right\}.$$

THEOREM 4.1. *Let $f \in L^1(\mathbf{T}^2)$. Then for a.a. $(\theta_1, \theta_2) \in \mathbf{T}^2$*

$$\mathscr{P}_0 f(z_1, z_2) \to f(\theta_1, \theta_2)$$

as (z_1, z_2) tends to $(e^{i\theta_1}, e^{i\theta_2})$ staying in $\Omega_{\theta_1, \theta_2}$.

Proof. In the disk, the normalizing factor $P_0 1$ behaves like $(1 - |z|)^{1/2}\log(1 - |z|)^{-1}$, and in $\mathbf{D}^2$ like a tensor product of two such factors. Therefore, $\mathscr{P}_0 f(z_1, z_2)$ is given by a convolution in $\mathbf{T}^2$ of f and a kernel which is

$$\sim \frac{1}{\log t_1 \log t_2} \frac{1}{t_1 + |\theta_1|} \frac{1}{t_2 + |\theta_2|}, \qquad t_i = 1 - |z_i|.$$

Define in $\mathbf{T}^2$

$$K_t(\theta_1, \theta_2) = \frac{1}{(\log t)^2} \frac{1}{t + (|\theta_1| - t \log t^{-1})_+} \frac{1}{t + (|\theta_2| - t \log t^{-1})_+}.$$

If for simplicity we take $C = 1$ in the definition of $\Omega_{\theta_1, \theta_2}$, it follows that

$$(4.1) \qquad \sup_{\Omega_{\theta_1, \theta_2}} |\mathscr{P}_0 f| \le C \sup_{0 < t < 1/2} |f| * K_t(\theta_1, \theta_2).$$

Since convergence holds for continuous f, we need only deduce a weak type (1,1) estimate for the operator defined by the left-hand side here.

To do this, we divide $K_t(\theta_1, \theta_2)$ into four parts. That part where $|\theta_1|, |\theta_2| \le 2t \log t^{-1}$ leads to the mean value of $|f|$ in squares of side $4t \log t^{-1}$, which is easy to control. For $|\theta_1|, |\theta_2| > 2t \log t^{-1}$, we can estimate the kernel by

$$C\frac{1}{(\log t)^2} \frac{1}{t + |\theta_1|} \frac{1}{t + |\theta_2|}.$$

This corresponds to restricted nontangential convergence and can be controlled, see Theorem 4 of [14] or Theorem 5.1 below.

The third part of K_t is defined by $|\theta_1| \le 2t \log t^{-1} < |\theta_2|$. (The fourth part is analogous and not discussed.) We estimate the supremum in t of

the convolution of $|f|$ with the kernel

$$\frac{1}{t \log t^{-1}} \chi_{|\theta_1| \leq 2t \log t^{-1}} \frac{1}{\log t^{-1}} \frac{1}{t \log t^{-1} + |\theta_2|}.$$

Replacing $2t \log t^{-1}$ by t, we obtain the kernel

$$(4.2) \qquad L_t(\theta_1, \theta_2) = t^{-1} \chi_{|\theta_1| \leq t} \frac{1}{\log t^{-1}} \frac{1}{t + |\theta_2|}.$$

Assume $e_k \leq t \leq e_{k-1}$. If $e_{k-\nu} \leq |\theta_2| \leq e_{k-\nu-1}$ with $\nu \in \{0, 1, \ldots, k\}$, the last factor in (4.2) is of course dominated by $C(e_{k-\nu} + |\theta_2|)^{-1}$. This gives

$$(4.3) \qquad L_t(\theta_1, \theta_2) \leq Ct^{-1} \chi_{|\theta_1| \leq t} \sum_{\nu=0}^{k} \frac{2^{-k}}{e_{k-\nu} + |\theta_2|} \chi_{|\theta_2| \leq e_{k-\nu-1}}.$$

Here it is important that those factors which contain θ_2 do not depend on t. We must estimate

$$(4.4) \qquad \sum_{\nu=0}^{\infty} 2^{-\nu} \sup_{k \geq \nu} \sup_{e_k \leq t \leq e_{k-1}} |f| * \left(t^{-1} \chi_{|\theta_1| \leq t} \frac{2^{\nu-k}}{e_{k-\nu} + |\theta_2|} \chi_{|\theta_2| \leq e_{k-\nu-1}} \right).$$

For each ν, we apply Lemma 2 of [14] to the supremum in k here.

To verify the hypotheses of this lemma, we estimate the supremum in t in (4.4). By integrating first in θ_2 and then in θ_1, we see that this supremum is essentially the standard maximal function taken in θ_1 of a convolution in θ_2 of f and a function whose L^1 norm is bounded. Therefore, this supremum defines an operator of weak type (1,1), uniformly in k. Further, it depends only on the values of f in a rectangle of sides e_{k-1} and $e_{k-\nu-1}$, centered at the point considered. Finally, we have a stability condition like (iii') of our Lemma 3.1, where we can move at most e_k and $e_{k-\nu}$ in the first and second variables, respectively, when defining the starred kernels.

Hence, Lemma 2 of [14] applies, with $N = 1$. This means that each term of (4.4) gives a weak type (1,1) operator with a constant which is $O(2^{-\nu})$. We can then sum in weak L^1 (cf. Stein [18], Sec. 6) and obtain an operator of weak type (1,1). This takes care of the third part of our kernel and completes the proof of Theorem 4.1.

Let d denote the hyperbolic distance in $\mathbf{D}$, normalized by $ds = (1 - |z|^2)^{-1}|dz|$, and D the corresponding bihyperbolic distance in $\mathbf{D}^2$. As shown by J. Taylor (unpublished), the convergence condition for $\mathscr{P}_0 f$ in $\mathbf{D}$,

$$(4.5) \qquad |\arg z - \theta| \leq C(1 - |z|)\log(1 - |z|)^{-1}$$

is equivalent to

$$(4.6) \qquad d(z, r_\theta) \leq \tfrac{1}{2} \log d(z, 0) + C,$$

up to the values of C. Here r_θ is the geodesic ray $\{re^{i\theta}: 0 \leq r < 1\}$, and we similarly let $R_{\theta_1, \theta_2} = \{(re^{i\theta_1}, re^{i\theta_2}): 0 \leq r < 1\} \subset \mathbf{D}^2$. Inequality (4.6) should of course be compared to the condition $d(z, r_\theta) \leq C$, which defines nontangential convergence in $\mathbf{D}$. In $\mathbf{D}^2$, restricted nontangential convergence is similarly described by $D((z_1, z_2), R_{\theta_1, \theta_2}) \leq C$.

A natural question is now whether the convergence condition of Theorem 4.1 can be similarly described in terms of D. We shall prove below that any domain W_{θ_1, θ_2} containing $\Omega_{\theta_1, \theta_2}$ which one can define by means of an inequality $D(z, R_{\theta_1, \theta_2}) \leq \psi(D(z, 0))$, $z \in \mathbf{D}^2$, for some function ψ, will be too large to allow a.e. convergence of $\mathscr{P}_0 f$, $f \in L^1$. Here it is assumed that W_{θ_1, θ_2} is obtained by rotating $W = W_{0,0}$, which we consider first.

If s_0 is large and $\theta = s_0 e^{-2s_0}$, the point $z = (e^{i\theta} \tanh s_0, e^{i\theta} \tanh s_0)$ belongs to $\Omega_{0,0}$. Clearly $D(z, 0) = s_0 \sqrt{2}$. We shall estimate the distance D_1 from z to $R_{0,0}$ and start by observing that $D_1 = d_1 \sqrt{2}$, where $d_1 = d(e^{i\theta} \tanh s_0, r_0)$.

To estimate d_1, we let $\gamma: [0, d_1] \to \mathbf{D}$ be the shortest geodesic arc from $e^{i\theta} \tanh s_0$ to r_0, parametrized by arc length. Then γ is a Euclidean circle arc, orthogonal to r_0 at $\gamma(d_1) = \tanh s$, and one has $s < s_0$. Let $\gamma(t_0)$ be the Euclidean midpoint of γ. Then $\arg \gamma(t_0) > \theta/2$, and $1 - |\gamma(t_0)| > (1 - \gamma(d_1))/2 \sim e^{-2s}$. Considering the variation of $|\gamma(t)|$ in $[0, t_0]$ and that of $\arg \gamma(t)$ in $[t_0, d_1]$, we find that the length d_1 is at least $s_0 - s - C + \theta/2 \cosh^2 s$. Now insert the value of θ and minimize with respect to s. We find $d_1 \geq \tfrac{1}{2} \log s_0 - C$ and thus

$$D_1 \geq \frac{1}{\sqrt{2}} \log s_0 - C = \frac{1}{\sqrt{2}} \log D(z, 0) - C.$$

Incidentally, this estimate for d_1 makes it easy to verify the equivalence of (4.5) and (4.6). This will be done in greater generality in the proof of Theorem 5.4.

This means that any candidate for W must contain the region

$$W = \left\{ z \in \mathbf{D}^2: D(z, 0) \text{ large and } D(z, R_{0,0}) \leq (1/\sqrt{2}) \log D(z, 0) - C \right\}.$$

We need an observation about this W. If $s_1 = s + \sqrt{2}^{-1} \log s - C_1$ and C_1 is large enough, the point $z = (\tanh s, e^{i\theta} \tanh s_1)$ belongs to W for $|\theta| \leq e^{-2s}$ and large s. This is seen by measuring a path from z to $R_{0,0}$ defined as follows. Keep the first coordinate constant at $\tanh s$ and move the second coordinate first radially from $e^{i\theta} \tanh s_1$ to $e^{i\theta} \tanh s$ and then

along the circle $|z_2| = \tanh s$ to $\tanh s$. One finds $D(z, R_{0,0}) \leq s_1 - s + C$. Since $D(z, 0) = s\sqrt{2} + O(\log s)$, it follows that $z \in W$.

To disprove a.e. convergence of $\mathscr{P}_0 f$, $f \in L^1(T^2)$, as z tends to $(e^{i\theta_1}, e^{i\theta_2})$ staying in W_{θ_1,θ_2}, we replace f by a point mass δ at $(0,0) \in \mathbf{T}^2$. Let θ_1 and θ_2 be small and positive, choose s so that $e^{-2s} = \theta_2$, and let s_1 be as before. Then $z = (e^{i\theta_1} \tanh s, \tanh s_1) \in W_{\theta_1,\theta_2}$ by the above, and

$$\mathscr{P}_0 \delta(z) \geq \frac{1}{Cs^2} \frac{1}{e^{-2s} + |\theta_1|} \frac{1}{e^{-2s_1}}.$$

For $\theta_2 < \theta_1$ we get

$$\sup_{W_{\theta_1,\theta_2}} \mathscr{P}_0 \delta \geq \frac{1}{C|\log \theta_2|^{2-\sqrt{2}}} \frac{1}{\theta_1 \theta_2}.$$

This is easily seen to disprove a weak type $(1,1)$ estimate for the maximal function corresponding to W. From general theorems, it now follows that a.e. convergence does not hold for all L^1 functions, see Chapter 2 of de Guzmán [4].

5. Results for general symmetric spaces. Take X as in §2, and let the gauge in $\overline{N}$ be defined with $H_0 = \rho$. We call $V \subset \mathfrak{a}_+$ a strict subcone of $\mathfrak{a}_+$ if V is a cone, i.e., $\mathbf{R}_+ V = V$, and the closure of V is contained in $\mathfrak{a}_+ \cup \{0\}$.

THEOREM 5.1. *Let $f \in L^1(K/M)$ and assume that $D \subset X$ is compact and V is a strict subcone of $\mathfrak{a}_+$. Then*

$$\mathscr{P}_0 f(k \exp(H) \cdot x) \to f(kM) \quad \text{as } H \to +\infty, H \in V,$$

uniformly for $x \in D$, for a.a. $kM \in K/M$. If f is continuous, one has admissible convergence, i.e., the condition $H \in V$ can be deleted, uniformly in K/M.

The result for continuous f here is also in van den Ban and Schlichtkrull [2, Cor. 16.6]. Before proving the theorem, we give a lemma about the behavior of $P(n)$ for large n.

LEMMA 5.2. *Given $\delta > 0$, there exists a constant $B < \infty$ such that*

$$P(n) \leq \delta P(n^H)$$

for $|n| > B$ and large $H \in \mathfrak{a}_+$.

Proof. Because of (2.5), we must prove

$$\sum_{\nu=0}^{\nu_0} e^{-2\langle \beta_\nu, H\rangle} Q_\nu(n) \leq \delta \sum_{\nu=0}^{\nu_0} Q_\nu(n).$$

For $\nu > 0$, we have termwise inequalities when H is large, since $\beta_\nu \neq 0$. Term number 0 in the left-hand side is 1, and it remains to see that $1 \leq \delta/P(n)$. But this holds for large n, see [9], Prop. 5.5. The lemma follows.

Proof of Theorem 5.1. Consider first continuous f in K/M. We must show that the operators which map f to the functions

$$kM \rightarrow \mathscr{P}_0 f(k \exp(H) \cdot x)$$

form an approximate identity in K/M as $H \rightarrow +\infty$, $x \in D$. Since the corresponding kernels are positive and constant functions are preserved, it is enough to prove convergence at $eM \in K/M$ for $f \geq 0$ vanishing near eM. Harnack's inequality (2.1) allows us to take $D = \{o\}$. Thus we must verify that $\mathscr{P}_0 f(\exp H) \rightarrow 0$ as $H \rightarrow +\infty$. Because of (2.2) and the definition of ψ, this holds if

$$(5.1) \quad \psi(H)^{-1} e^{2\langle \rho, H\rangle} \int_{|n| > \varepsilon} e^{-\langle \rho, H(n)\rangle} e^{-\langle \rho, H(n^{-H})\rangle} \, dn \rightarrow 0 \quad \text{as } H \rightarrow +\infty$$

for any $\varepsilon > 0$.

Let $\delta > 0$ and take B as in Lemma 5.2. Write the integral in (5.1) as $\int_{|n| > B} + \int_{\varepsilon < |n| < B}$. In the first term here, we apply Lemma 5.2 and then transform $n \rightarrow n^{-H}$. This gives

$$(5.2) \quad \psi(H)^{-1} e^{2\langle \rho, H\rangle} \int_{|n| > B} e^{-\langle \rho, H(n)\rangle} e^{-\langle \rho, H(n^{-H})\rangle} \, dn$$

$$\leq \sqrt{\delta}\, \psi(H)^{-1} e^{2\langle \rho, H\rangle} \int_{|n| > B} e^{-\langle \rho, H(n^H)\rangle} e^{-\langle \rho, H(n^{-H})\rangle} \, dn$$

$$\leq \sqrt{\delta}\, \psi(H)^{-1} e^{4\langle \rho, H\rangle} \int e^{-\langle \rho, H(n)\rangle} e^{-\langle \rho, H(n^{-2H})\rangle} \, dn$$

$$\leq \sqrt{\delta}\, \psi(2H)/\psi(H) \leq C\sqrt{\delta},$$

where the last inequality holds for large H because of Proposition 3.3.

For the integral over $R = \{n : \varepsilon < |n| < B\}$, we first introduce the regions $R_m = \{n^{m\delta H} : n \in R\}$, $m = 1, 2, \ldots$. If H is large, the R_m will be pairwise disjoint and contained in $B(1)$. Estimating $e^{-\langle \rho, H(n)\rangle}$ by 1 and

transforming $n \to n^{-m\delta H}$, we get

$$\psi(H)^{-1} e^{2\langle \rho, H \rangle} \int_R e^{-\langle \rho, H(n) \rangle} e^{-\langle \rho, H(n^{-H}) \rangle} \, dn$$

$$\leq \psi(H)^{-1} e^{2(1+m\delta)\langle \rho, H \rangle} \int_{R_m} e^{-\langle \rho, H(n^{-(1+m\delta)H}) \rangle} \, dn.$$

For $1 \leq m \leq [\delta^{-1}]$, Lemma 3.2 implies that this is at most

$$\psi(H)^{-1} e^{4\langle \rho, H \rangle} \int_{R_m} e^{-\langle \rho, H(n^{-2H}) \rangle} \, dn.$$

In this integral, we insert a factor $e^{-\langle \rho, H(n) \rangle}$, which is bounded below in $B(1)$. Then we sum in m, getting

$$[\delta^{-1}] \psi(H)^{-1} e^{2\langle \rho, H \rangle} \int_{\varepsilon < |n| < B} e^{-\langle \rho, H(n) \rangle} e^{-\langle \rho, H(n^{-H}) \rangle} \, dn$$

$$\leq C\psi(H)^{-1} e^{4\langle \rho, H \rangle} \int_{B(1)} e^{-\langle \rho, H(n) \rangle} e^{-\langle \rho, H(n^{-2H}) \rangle} \, dn$$

$$\leq C\psi(2H)/\psi(H) \leq C.$$

Dividing by $[\delta^{-1}]$, we obtain an inequality which together with (5.2) implies (5.1). This proves the last statement in Theorem 5.1.

Now consider $f \in L^1(K/M)$. The method of [**15**], §3 p. 49, shows that it is enough to take $f \in L^1(\overline{N})$ with support contained in a compact set $L \subset \overline{N}$, and prove a.e. convergence in L only. Modifying the gauge in $\overline{N}$, we can assume $|n| \leq 1$ for $n \in L^{-1}L$.

As usual, the conclusion follows if we show that the operator $\tilde{M}$ defined for $n_1 \in L$ by

$$\tilde{M}f(n_1) = \sup_{\substack{H \in V \\ H \text{ large}}} \mathscr{P}_0 |f| (n_1 \exp H)$$

is of weak type $(1,1)$ in L. We now use (2.4). Lemma 3.3 implies that $\psi(H) \sim |H|^\sigma$ for $H \in V$, where $|H|^2 = \langle H, H \rangle$ and $\sigma = \text{card}\,\Sigma_0^+$. As a result, the operator

$$Mf = \sup_{\substack{H \in V \\ H \text{ large}}} |H|^{-\sigma} |f| * \tilde{P}_H^{1/2}$$

satisfies $\tilde{M}f \leq CMf$ in L.

The following lemma can be seen as a stronger form of (5.1).

LEMMA 5.3. *If $H \in \mathfrak{a}_+$ with $H \geq 2^k \rho$, $k \in \mathbf{N}$, then for $\nu = 0, 1, \ldots, k$*

$$e^{2\langle \rho, H \rangle} \int_{e_{k-\nu} < |n| < 1} P(n^{-H})^{1/2} \, dn \leq C 2^{-\nu} \psi(H).$$

Proof. For $m = 1, \ldots, 2^{\nu}$, we transform $n \to n^{-m2^{k-\nu}\rho}$ and obtain

$$e^{2\langle\rho,H\rangle}\int_{e_{k-\nu}<|n|<1} P(n^{-H})^{1/2}\, dn$$

$$\leq e^{2\langle\rho,H+m2^{k-\nu}\rho\rangle}\int_{e_{k-\nu}^{m+1}<|n|<e_{k-\nu}^{m}} P(n^{-H-m2^{k-\nu}\rho})^{1/2}\, dn$$

$$\leq e^{4\langle\rho,H\rangle}\int_{e_{k-\nu}^{m+1}<|n|<e_{k-\nu}^{m}} P(n^{-2H})^{l1/2}\, dn,$$

because of Lemma 3.2. Now we can sum in m and argue as when proving (5.1). The lemma follows.

We estimate the operator M. Any large $H \in V$ is smaller than some $2^{k}\rho$, $k \in \mathbf{N}$, with $|H| \sim |2^{k}\rho|$. By Lemma 3.2, it is therefore enough to take $H = 2^{k}\rho$, $k = 1, 2, \ldots$, in the definition of M. For $0 \leq \nu \leq k$, we set

$$K_{k}^{\nu}(n) = 2^{-k\sigma}e^{2^{k+1}\langle\rho,\rho\rangle}\tilde{P}(n^{-2^{k}\rho})^{1/2}\chi_{e_{k-\nu}<|n|<e_{k-\nu-1}},$$

except that e_{k} is replaced by 0. Let

$$M^{\nu}f = \sup_{k\geq\nu}|f|*K_{k}^{\nu}, \qquad \nu = 0, 1, \ldots,$$

so that

(5.3) $$Mf \leq C\sum_{\nu=0}^{\infty} M^{\nu}f.$$

We shall apply Lemma 3.1 to each M^{ν}. Fix ν and set $T_{k}f = |f|*K_{k+\nu+1}^{\nu}$ for $k = -1, 0, \ldots$. Lemma 5.3 gives assumption (i) with $C_{1} = C2^{-\nu}$. Further, (ii) is clear. To verify (iii′) with $N = \nu + 1$, we must estimate $(K_{k+\nu+1}^{\nu})^{*}$ and thus $\tilde{P}((nn')^{-2^{k+\nu+1}\rho})^{1/2}$ with $|n'| \leq e_{k+\nu+1}$. But then $|(n')^{-2^{k+\nu+1}\rho}| \leq 1$, and we can apply Harnack's inequality (2.1) to $\tilde{P}$, since $\tilde{P}(n) = P_{\rho}\delta(n)$, where δ is a unit point mass at $eM \in K/M$. This gives

$$\tilde{P}((nn')^{-2^{k+\nu+1}\rho}) \leq C\tilde{P}(n^{-2^{k+\nu+1}\rho}).$$

Since also $\mathrm{supp}(K_{k+\nu+1}^{\nu})^{*} \subset B(Ce_{k})\setminus B(e_{k+1}/C)$, Lemma 5.3 implies

$$\int (K_{k+\nu+1}^{\nu})^{*}\, dn \leq C\int K_{k+\nu+1}^{\nu}\, dn \leq C2^{-\nu}.$$

Lemma 3.1 now shows that M^{ν} is of weak type (1,1), with constant at most $C(\nu + 1)2^{-\nu}$. This means that the series in (5.3) can be summed in weak L^{1}, so that M and $\tilde{M}$ are of weak type (1,1). Theorem 5.1 is proved.

Next, let X be of rank 1. We let α and 2α be the positive roots and $m_\alpha > 0$ and $m_{2\alpha} \geq 0$ their multiplicities. The root space decomposition of $\bar{\mathfrak{n}}$ is $\bar{\mathfrak{n}} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$. In X we choose the invariant metric associated with the Killing form in $\mathfrak{p}$ and denote by d the corresponding distance. As usual, $\mathfrak{p}$ is the eigenspace of the Cartan involution in the Lie algebra of G with eigenvalue -1. Fix $H_0 \in \mathfrak{a}_+$ and choose an associated gauge in $\bar{N}$. Let r_{kM} for $k \in K$ be the ray $\{ k\exp(tH_0) \cdot o \colon t \geq 0 \}$.

THEOREM 5.4. *Let* rank $X = 1$. *Take* $f \in L^1(K/M)$, *and set* $q = 1/\langle 2\rho, H_0 \rangle$ *and* $\kappa = \sqrt{2m_\alpha + 8m_{2\alpha}}/(m_\alpha + 2m_{2\alpha})$. *Then for almost all* $n_1 \in \bar{N}$

$$(5.4) \qquad \mathscr{P}_0 f\big(n_1 \exp(tH_0)n \cdot x\big) \to f\big(k(n_1)M\big)$$

as $t \to +\infty$ *and* n *stays in the ball* $B(Ct^q)$ *and* x *in a compact subset of* X, *for any fixed* C. *Equivalently,* $\mathscr{P}_0 f(x) \to f(kM)$ *as* x *approaches* K/M *while satisfying* $d(x, r_{kM}) \leq \kappa \log d(x, o) + C$, *for almost all* $kM \in K/M$. *This holds for no larger values of* q *and* κ.

As shown below, the two descriptions of the approach to the boundary in this theorem are equivalent in the following sense. Fix $n_1 \in \bar{N}$. Given a constant $C_1 < \infty$ and a compact set $D \subset X$, there exists $C_2 < \infty$ such that

$$(5.5) \qquad d\big(x, r_{k(n_1)M}\big) \leq \kappa \log d(x, o) + C_2$$

for all

$$(5.6) \quad x = n_1 \exp(tH_0)n \cdot x' \quad \text{with } t \text{ large, } |n| \leq C_1 t^q, \text{ and } x' \in D.$$

Conversely, any point x far from o satisfying (5.5) can be written as (5.6), where C_1 is determined by C_2 and $D = \{o\}$.

When X is the disk **D**, Theorem 5.4 is equivalent to the result from [16]. For this space $\kappa = \sqrt{2}$, which will agree with (4.6) if the hyperbolic metric is properly normalized.

Proof of Theorem 5.4. We first prove (5.4) in a rather sketchy way, since this is easy. One has

$$(5.7) \qquad P(n) \sim \big(1 + |n|\big)^{-4\langle \rho, H_0 \rangle},$$

see Helgason [6], Theorem IX.3.8 p. 414, and $\psi(H) \sim |H|$ for large H. As in the preceding proof, we see that we need only take $f \in L^1(\bar{N})$ of compact support. Then we must estimate the convolution of $|f|$ with the

kernel

(5.8) $n \to t^{-1} e^{2\langle \rho, tH_0 \rangle} \sup_{|n'| \le Ct^q} \tilde{P}(n^{-tH_0} n')^{1/2}.$

If $|n^{-tH_0}| > C_0 t^q$ and C_0 is large, one has $|n^{-tH_0}| \sim |n^{-tH_0} n'|$. Because of (5.7), the factor n' can then be deleted from (5.8). The preceding proof, or Michelson [12], Theorem 2.2 (ii), now allows us to control the convolution with this part of the kernel.

For $|n^{-tH_0}| \le C_0 t^q$, we estimate $\tilde{P}$ by 1 in (5.8). These n belong to the ball $B(C_0 t^q e^{-t})$. We simply obtain means of $|f|$ in translates of this ball, and thus a standard maximal function. This gives (5.4). To see that the given value of q is sharp, one uses approximated point masses.

In order to prove the equivalence, we let $n_1 \in \overline{N}$. Write

$$n_1 \exp(tH_0) = k(n_1) \exp(tH_0) \exp(H(n_1)) n(n_1)^{-tH_0},$$

where the last two factors stay in a compact set as t takes large values. Thus the curve $r_{n_1}\colon\ t \to n_1 \exp(tH_0) \cdot o,\ t \ge 0$, stays at a bounded distance from $r_{k(n_1)M}$ and conversely, so that $|d(x, r_{n_1}) - d(x, r_{k(n_1)M})| \le C$ for any x. This means that we can use r_{n_1} instead of $r_{k(n_1)M}$ in (5.5). Translating everything by n_1^{-1}, we may assume $n_1 = e$ since $d(x, o) - d(n_1^{-1}x, o)$ stays bounded. Further, we can take $D = \{o\}$, because the point x in (5.6) will move a bounded distance if x' is replaced by o. We can also normalize H_0 so that $\alpha(H_0) = 1$. Notice that $\kappa = q|H_0|$.

The geodesic $A \cdot o$ contains r_e. We first estimate the distance to $A \cdot o$ from $x = \exp(tH_0) n \cdot o$ or, equivalently, from $n \cdot o$. Let $\beta = \log_+ |n|$.

LEMMA 5.5. $|d(n \cdot o, A \cdot o) - |H_0|\beta| \le C.$

Proof. Consider first the path $s \to n \exp(-sH_0) \cdot o$, $0 \le s \le \beta$, starting at $n \cdot o$ and of length $|H_0|\beta$. The distance from its endpoint $\exp(-\beta H_0) n^{\beta H_0} \cdot o$ to the point $\exp(-\beta H_0) \cdot o \in A \cdot o$ is at most C, because $|n^{\beta H_0}| = 1$. It follows that $d(n \cdot o, A \cdot o) \le |H_0|\beta + C$.

It remains to prove that any curve γ from $n \cdot o$ to $A \cdot o$ has length at least $|H_0|\beta - C$. We can write γ as

$$\gamma(s) = n(s)\exp(-t(s)H_0) \cdot o, \qquad 0 \le s \le L,$$

where $n(s)$ and $t(s)$ take values in $\overline{N}$ and $\mathbf{R}$, respectively. Clearly $n(0) = n$, $t(0) = 0$, and $n(L) = e$. Since we need only consider geodesics, we may assume that γ is C^∞ and that $n'(s) \in \overline{\mathfrak{n}}$ never vanishes. This is because a geodesic parallel to the geodesic $n_1 A \cdot o$ at some point would coincide with $n_1 A \cdot o$.

The lifted curve $n(s)\exp(-t(s)H_0)$ in G has a tangent vector at s given by the left-invariant field, i.e., the $\mathfrak{g}$ vector,

$$-t'(s)H_0 + \mathrm{ad}(t(s)H_0)n'(s).$$

If $n'(s) = Y(s) + Z(s) \in \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, this vector equals

$$-t'(s)H_0 + e^{-t(s)}Y(s) + e^{-2t(s)}Z(s).$$

The properties of the Killing form then imply that

$$(5.9) \qquad |\gamma'(s)|^2 = t'(s)^2|H_0|^2 + e^{-2t(s)}|Y(s)|^2 + e^{-4t(s)}|Z(s)|^2.$$

Here the bars $|\cdot|$ denote Euclidean norms in $\mathfrak{a}$, $\mathfrak{g}_{-\alpha}$, and $\mathfrak{g}_{-2\alpha}$, respectively.

If $t(s) > \beta$ for some s, the desired lower estimate for the length of γ is clear from (5.9). Assume thus $t \le \beta$ in $[0, L]$. Then

$$|\gamma'(s)|^2 \ge t'(s)^2|H_0|^2 + e^{-2t(s)}\left(|Y(s)|^2 + e^{-2\beta}|Z(s)|^2\right).$$

Now reparametrize the curve so that $|Y(s)|^2 + e^{-2\beta}|Z(s)|^2 \equiv |H_0|^2$ and thus

$$|\gamma'(s)|^2 = |H_0|^2\left(t'(s)^2 + e^{-2t(s)}\right).$$

This parenthesis leads us to the curve $\tau(s) = s + ie^{-t(s)}$, $0 \le s \le L$, in the upper half-plane $\{y > 0\}$ with the hyperbolic metric $y^{-2}(dx^2 + dy^2)$. Indeed, $|\tau'(s)|^2 = t'(s)^2 + e^{-2t(s)}$. Now τ connects $\tau(0) = i$ to the line $l = \{x = L\}$. The shortest curve connecting i and l is an arc of a Euclidean circle hitting l and the real axis at right angles. Its hyperbolic length is greater than $\log L$. The length of γ is thus greater than $|H_0|\log L$.

To estimate L, consider the curve $s \to n(s)^{\beta H_0} \cdot o$, $0 \le s \le L$, in X. It goes from the point $n^{\beta H_0} \cdot o$ of the compact set $\{\tilde{n} \cdot o: |\tilde{n}| = 1\}$ to o. Therefore, its length is bounded below. But this length is by the above

$$\int_0^L e^{-\beta}|H_0| \, ds = |H_0|e^{-\beta}L.$$

It follows that $L \ge ce^{\beta}/|H_0|$, $c > 0$, so that the length of γ is at least $|H_0|\beta - C$. Lemma 5.5 is proved.

Roughly speaking, this lemma means the following. To reach $A \cdot o$ from $n \cdot o$, move first "upwards" (in the negative A direction) until the "horizontal" distance (in $\overline{N}$) to $A \cdot o$ is at most 1. Then move horizontally. Further, $\exp(-\beta H_0) \cdot o$ is a closest point in $A \cdot o$, up to an additive constant in the distance. A consequence is that

$$(5.10) \qquad \left|d(\exp(tH_0)n \cdot o, o) - |H_0||t - \beta|\right| \le |H_0|\beta + C,$$

as one sees by going via $\exp((t - \beta)H_0) \cdot o$.

To prove that (5.6) implies (5.5), let $x = \exp(tH_0)n \cdot o$ with $|n| \le Ct^q$ and $t > 0$ large. Then $\beta \le q \log t + C$, and (5.10) implies that $\log d(x, o) > \log t - C$. These inequalities and Lemma 5.5 yield (5.5).

Conversely, assume $d(x, r_e) \le \kappa \log d(x, o) + C$ with $d(x, o)$ large, and write $x = \exp(tH_0)n \cdot o$. Because of (5.10),

$$d(x, o) \le C(|t| + \beta) + C.$$

Combining this with the lower estimate of Lemma 5.5, we get

$$|H_0|\beta \le \kappa \log_+(|t| + \beta) + C.$$

If $|t|$ is not large, this implies $\beta \le C$ and $d(x, o) \le C$ which is excluded. If $|t| > C$, observe that $\beta < |t|$. When $t > C$, we obtain $\beta \le q \log t + C$ so that $|n| \le Ct^q$, and (5.6) follows. Finally if $t < -C$, then $d(x, r_e)$ is at least

$$d\big(\exp((t - \beta)H_0) \cdot o, r_e\big) - d\big(x, \exp((t - \beta)H_0) \cdot o\big)$$

$$\ge |H_0|(|t| + \beta) - |H_0|\beta - C = |H_0||t| - C.$$

This last quantity is not much smaller than $d(x, o)$, a contradiction which ends the proof of the equivalence.

Since the optimality of q implies that of κ, the proof of Theorem 5.4 is complete.

6. Admissible convergence in $SL(3, \mathbf{R})/SO(3)$. In this section and the next, X will be $G/K = SL(3, \mathbf{R})/SO(3)$. As $\mathfrak{a}$ we take the set of diagonal matrices in $\mathfrak{g} = \{\text{traceless } 3 \times 3 \text{ matrices}\}$. Denoting

$$H = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} \in \mathfrak{a},$$

we define $\mathfrak{a}_+$ by the condition $h_1 > h_2 > h_3$. Any $H \in \mathfrak{a}$ is then determined by the values of the two simple positive roots, $\alpha(H) = h_1 - h_2$ and $\beta(H) = h_2 - h_3$. The positive roots are α, β, and $\alpha + \beta$. Thus H is also determined by $\delta = e^{-\alpha(H)}$ and $\varepsilon = e^{-\beta(H)}$, and $H \to +\infty$ corresponds to $\delta, \varepsilon \to 0$. The group $\overline{N}$ consists of the matrices

$$n = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{bmatrix}, \qquad x, y, z \in \mathbf{R},$$

and we write simply $n = (x, y, z)$. Then $n^H = (\delta x, \varepsilon y, \delta\varepsilon z)$. The Poisson kernel in $\overline{N}$ is given by

$$P(n) = \frac{1}{(1 + x^2 + z^2)(1 + y^2 + (z - xy)^2)}.$$

This can be seen via a straightforward calculation and generalizes to $SL(n, \mathbf{R})/SO(n)$, see Bhanu Murthy [3]. By Lemma 3.3, or Herz [8],

$$(6.1) \qquad \psi(H) \sim \log 1/\delta \log 1/\varepsilon \log 1/\delta\varepsilon$$

for small δ, ε. Observe finally that the map

$$(6.2) \qquad (x, y, z) \to (-y, -x, xy - z)$$

is an automorphism of $\overline{N}$ and preserves $P(n)$.

THEOREM 6.1. *Let* $p > 1$. *If* $X = SL(3, \mathbf{R})/SO(3)$ *and* $f \in L^p(K/M)$, *then* $\mathscr{P}_0 f$ *converges admissibly to* f *a.e. in* K/M.

The method of proof below is based on products of one-dimensional maximal operators. In a simpler form, it also applies to $\mathscr{P}_\lambda f$, $\lambda \in \mathfrak{a}_+$, and gives a shorter proof of the main result in [17] for this X. However, it does not generalize to an arbitrary symmetric space.

Proof. Theorem 5.1 gives the convergence of $\mathscr{P}_0 f$ for continuous f. As in the proof of that theorem, we can therefore take $f \in L^1(\overline{N})$ with support in a compact set L, and estimate the corresponding maximal function

$$Mf = \sup_{H \text{ large}} \psi(H)^{-1} |f| * \tilde{P}_H^{1/2}$$

in L. Assume $f \geq 0$. Clearly, $f * \tilde{P}_H^{1/2}(n_1) = \int f(n_1 n^H) P(n)^{1/2} \, dn$, and for $n_1 \in L$ we need only integrate over those n satisfying $n^H \in L^{-1}L$. Hence with $n_1 = (x_1, y_1, z_1)$,

$$(6.3) \quad Mf(x_1, y_1, z_1)$$

$$\leq \sup_{\delta, \varepsilon \text{ small}} \frac{C}{\log 1/\delta \log 1/\varepsilon \log 1/\delta\varepsilon}$$

$$\times \int \frac{f(x_1 + \delta x, y_1 + \varepsilon y, z_1 + \delta\varepsilon z + \delta xy_1)}{(1 + x^2 + z^2)^{1/2}(1 + y^2 + (z - xy)^2)^{1/2}} \, dx \, dy \, dz,$$

the integral taken over $\{|x| \leq C/\delta, \ |y| \leq C/\varepsilon, \ |z| \leq C/\delta\varepsilon\}$. Write this integral as

$$\int_{|z| < |xy|/2} + \int_{|z - xy| < |xy|/2} + \int_{\text{remaining } x, y, z} = I + II + III.$$

In I, we can estimate $P(n)^{1/2}$ by $C(1 + |x| + |z|)^{-1}(1 + |y| + |xy|)^{-1}$. This expression is even in z and decreasing in $|z|$, and its integral with respect to z in $|z| < |xy|/2$ is at most

$$C(1 + \log_+ |y|)(1 + |y| + |xy|)^{-1} \leq C \log 1/\varepsilon (1 + (1 + |x|)|y|)^{-1}.$$

Thus we can integrate with respect to z in I and estimate the result by means of the standard one-dimensional maximal operator taken in the z direction. This operator is denoted by $M_{0,0,1}$. This means that

$$I \leq C \log 1/\varepsilon \int_{\substack{|x| \leq C/\delta \\ |y| \leq C/\varepsilon}} \frac{M_{0,0,1} f(x_1 + \delta x, y_1 + \varepsilon y, z_1 + \delta x y_1)}{1 + (1 + |x|)|y|} \, dx \, dy.$$

The y integral here can similarly be estimated by means of the maximal operator $M_{0,1,0}$, so that

$$I \leq C \log 1/\varepsilon \int_{|x| < C/\delta} \frac{\log 1/\varepsilon + \log(1 + |x|)}{1 + |x|}$$

$$\times M_{0,1,0} M_{0,0,1} f(x_1 + \delta x, y_1, z_1 + \delta x y_1) \, dx.$$

For the remaining integration, we use the maximal operator in the direction $(1, 0, y_1)$, getting

$$I \leq C \log 1/\varepsilon \left(\log 1/\varepsilon \log 1/\delta + (\log 1/\delta)^2 \right)$$

$$\times M_{1,0,y_1} M_{0,1,0} M_{0,0,1} f(x_1, y_1, z_1).$$

The logarithmic factors here are controlled by those in (6.3).

We now claim that the operator $M_{1,0,y_1}$ is bounded on $L^p(\mathbf{R}^3)$. Indeed, for each fixed value of y_1, we have a maximal operator along a fixed direction in the xz plane. Thus,

$$\iint \left(M_{1,0,y_1} g(x_1, y_1, z_1) \right)^p dx_1 \, dz_1 \leq C \iint |g(x, y_1, z)|^p \, dx \, dz$$

for $g \in L^p(\mathbf{R}^3)$, uniformly in y_1. Integration in y_1 now gives the claimed boundedness. As a result, that part of M which corresponds to I is bounded on L^p.

The term II can be treated like I after the transformation (6.2), which interchanges δ and ε.

In III, we have $|z - xy| > |xy|/2$ and $|z| > |xy|/2$. Therefore, the argument used for I, and that of II, will apply. But in both arguments, the z integration will now produce a factor $\log 1/\delta\varepsilon$ instead of $\log 1/\varepsilon$ or $\log 1/\delta$. Since $\log 1/\delta\varepsilon \sim \max(\log 1/\delta, \log 1/\varepsilon)$, one argument or the other will give the right estimate for III. We conclude that M is bounded on L^p, and Theorem 6.1 is proved.

More algebraically, one could also have used the fact that the one-dimensional maximal operators in this proof are associated with one-parameter subgroups.

7. Improved restricted convergence in $SL(3, \mathbf{R})/SO(3)$. We use the notation from §6. Fix $H_0 \in \mathfrak{a}_+$ and write α and β for $\alpha(H_0)$ and $\beta(H_0)$, which are positive numbers.

THEOREM 7.1. *Let* $X = SL(3, \mathbf{R})/SO(3)$ *and take* $f \in L^1(K/M)$. *Set* $q = 2/(\alpha + \beta + \max(\alpha, \beta))$. *Then for a.a.* $n_1 \in \overline{N}$

$$\mathscr{P}_0 f(n_1 \exp(tH_0)n \cdot x) \to f(k(n_1)M)$$

as $t \to +\infty$, $|n| \leq Ct^q$, *and* $x \in X$ *stays in a compact set. This holds for no larger value of* q.

Formally, this result is like the first part of Theorem 5.4. Notice, however, that the widening of the convergence region caused by the factor n is now only in certain directions. Indeed, as t and n vary, the points $n_1 \exp(tH_0)n \cdot o$ describe a set whose dimension is smaller than that of X.

For the proof of Theorem 7.1, we need a proposition of independent interest. Consider for $\mu > 0$ the kernel

$$P_{\mu,0}^t(n) = \frac{t^{-1}e^{2\alpha t + 2\beta t}}{\left(1 + e^{\alpha t}|x| + e^{\alpha t + \beta t}|z|\right)^{1+\mu}\left(1 + e^{\beta t}|y| + e^{\alpha t + \beta t}|z - xy|\right)}$$

where $t > 1$ and $n = (x, y, z) \in \overline{N}$. Similarly, $P_{0,\mu}^t$ will denote the kernel obtained by interchanging the exponents $1 + \mu$ and 1 of the two factors in the denominator.

The automorphism (6.2) interchanges $P_{\mu,0}^t$ and $P_{0,\mu}^t$. These kernels, or rather $\tilde{P}_{\mu,0}^t$ and $\tilde{P}_{0,\mu}^t$, arise from normalized λ-Poisson integrals $\mathscr{P}_\lambda(n_1 \exp(tH_0))$ with a nonzero λ on the boundary of $\mathfrak{a}_+$. This is why the $P_{\mu,0}^t$, $t > 0$, form a family of normalized dilations of one function if the factor t^{-1} is deleted. Also,

$$(7.1) \qquad\qquad P_{\mu,0}^t(n'n) \sim P_{\mu,0}^t(n)$$

for $n' \in B(Ce^{-t})$ and any n. This is seen from Harnack's inequality or directly from the definition of the kernel. Let

$$M_{\mu,0}f = \sup_{t \text{ large}} |f| * \tilde{P}_{\mu,0}^t, \qquad f \in L^1(\overline{N}),$$

and similarly for $M_{0,\mu}$. These are the maximal functions corresponding to restricted convergence of $\mathscr{P}_\lambda f$ for $\lambda \in \partial\mathfrak{a}_+\backslash\{0\}$. A recent result by van den Ban and Schlichtkrull [2, Cor. 16.6] implies that this convergence holds for continuous f.

PROPOSITION 7.2. *If $L \subset \overline{N}$ is compact, $M_{\mu,0}$ and $M_{0,\mu}$ are of weak type* (1,1) *in L.*

This has the following consequence.

THEOREM 7.3. *Let $X = \mathrm{SL}(3, \mathbf{R})/\mathrm{SO}(3)$ and take $\lambda \in \partial \mathfrak{a}_+ \backslash \{0\}$ and $f \in L^1(K/M)$. Then $\mathscr{P}_\lambda f$ converges restrictedly to f a.e. in K/M.*

Proof of Proposition 7.2. Because of the map (6.2), we need only work with $M_{\mu,0}$. The idea of the proof is to apply Lemma 3.1, after some elementary reductions. Write temporarily $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ for $e^{\alpha t}x$, $e^{\beta t}y$, and $e^{\alpha t + \beta t}z$, respectively.

First of all, it is enough to treat that part of $P_{\mu,0}^t$ defined by

$$(7.2) \qquad\qquad |\tilde{x}| + |\tilde{z}| \le e^{\eta t}$$

for any fixed $\eta > 0$. Indeed, if this inequality is false, $(1 + |\tilde{x}| + |\tilde{z}|)^\mu > t^2/C$. Then $P_{\mu,0}^t$ is dominated by the kernel defining $\mathscr{P}_0 f$, which is controlled by Theorem 5.1.

Similarly, we can restrict $P_{\mu,0}^t$ by

$$(7.3) \qquad\qquad 10\big(1 + |\tilde{x}| + |\tilde{z}|\big) < 1 + |\tilde{y}| + |\tilde{z} - \tilde{x}\tilde{y}|,$$

This is because that part of $P_{\mu,0}^t$ corresponding to the opposite inequality can be estimated by

$$\frac{Ce^{2\alpha t + 2\beta t}}{\big(1 + |\tilde{x}| + |\tilde{z}|\big)^{1 + \mu/2}\big(1 + |\tilde{y}| + |\tilde{z} - \tilde{x}\tilde{y}|\big)^{1 + \mu/2}}.$$

This is essentially the kernel obtained for $\mathscr{P}_\lambda f$ with some $\lambda \in \mathfrak{a}_+$. But such $\mathscr{P}_\lambda f$ are controlled by the results of Stein [18] or Sjögren [17], §7.

We next verify that it is enough to prove the proposition with $P_{\mu,0}^t$ replaced by the kernel

$$(7.4) \qquad\qquad \frac{t^{-1}e^{2\alpha t + 2\beta t}}{\big(1 + |\tilde{x}|\big)^{1 + \gamma}\big(1 + |\tilde{z}|\big)^{1 + \gamma}\big(1 + |\tilde{y}|\big)},$$

for $\gamma > 0$.

In case $|\tilde{y}| \le |\tilde{z} - \tilde{x}\tilde{y}|$, we see from (7.3) that $|\tilde{z}| < |\tilde{z} - \tilde{x}\tilde{y}|/2$, which implies $|\tilde{z} - \tilde{x}\tilde{y}| \sim |\tilde{x}\tilde{y}|$. Hence, $|\tilde{x}| < |\tilde{x}\tilde{y}|$ because of (7.3), so that $|\tilde{y}| > 1$ and $1 + |\tilde{y}| + |\tilde{z} - \tilde{x}\tilde{y}| \sim (1 + |\tilde{x}|)|\tilde{y}|$. Then $P_{\mu,0}^t$ is dominated by (7.4), with $\gamma = \mu/2$.

In the opposite case $|\tilde{y}| > |\tilde{z} - \tilde{x}\tilde{y}|$, (7.3) implies $|\tilde{z}| < |\tilde{y}|$. Then $|\tilde{x}\tilde{y}| < 2|\tilde{y}|$, so that $|\tilde{x}| < 2$ and (7.4) again dominates $P_{\mu,0}^t$.

We are thus led to integrals of $f(n_1 n)$ against the kernel (7.4) restricted by (7.2). Given a large t, choose an integer k so that $2^{k-1} < t \le 2^k$. If η is small, (7.2) then implies $|x| \le e_{k-2}^\alpha$ and $|z| \le e_{k-2}^{\alpha+\beta}$. Now we make a decomposition of (7.4) like that of L_t in (4.3). We arrive at the kernel

$$\sum_{\nu=0}^{k} \frac{2^{-k} e^{2\alpha t + \beta t}}{\left(1 + e^{\alpha t}|x|\right)^{1+\gamma} \left(1 + e^{\alpha t + \beta t}|z|\right)^{1+\gamma} \left(e_{k-\nu}^\beta + |y|\right)} \chi_{k,\nu},$$

where $\chi_{k,\nu}$ is the characteristic function of the set $\{ |x| \le e_{k-2}^\alpha,\ |z| \le e_{k-2}^{\alpha+\beta},\ |y| \le e_{k-\nu-1}^\beta \}$. Let $K_t^\nu(n)$ denote term number ν here, $n = (x, y, z)$, and set

$$T_k^\nu f(n_1) = \sup_{2^{k-1} < t \le 2^k} \int |f(n_1 n)| K_t^\nu(n)\, dn, \qquad k \ge \nu.$$

By means of Lemma 3.1, we shall prove that the operator

$$f \to \sup_{k \ge \nu} T_k^\nu f$$

is for each ν of weak type (1,1), with constant at most $C(\nu + 1)2^{-\nu}$. This will allow summation in ν and complete the proof.

Thus we must verify (i), (ii), (iii') and start with (i). Clearly,

$$T_k^\nu f(n_1) \le \sup_t 2^{-\nu} \int \frac{e^{2\alpha t + \beta t}\, dx\, dz}{\left(1 + e^{\alpha t}|x|\right)^{1+\gamma}\left(1 + e^{\alpha t + \beta t}|z|\right)^{1+\gamma}}$$

$$\times \int_{|y| \le e_{k-\nu-1}^\beta} \frac{2^{\nu-k} f(n_1 n)}{e_{k-\nu}^\beta + |y|}\, dy.$$

The inner integral here is a convolution of f and a measure which is independent of t and of total mass at most C. This integral thus defines a bounded operator $L^1 \to L^1$. The integration in x and z amounts to applying to f a one-parameter two-dimensional maximal operator. As an operator in the plane, it is of weak type (1,1), with a uniform constant. The fact that it acts here in the plane spanned by $(1, 0, y_1)$ and $(0, 0, 1)$ in $\mathbf{R}^3$ causes no problem, cf. the operator $M_{1,0,y_1}$ in §6. Thus T_k^ν is of weak type (1,1) with constant $O(2^{-\nu})$, which is (i).

For (ii) we simply observe that $T_k^\nu f(n_1)$ is determined by the restriction of f to $n_1 B(e_{k-\nu-2})$, if the gauge in $\overline{N}$ is suitably normalized.

Finally we must estimate

$$\int (\tilde{K}_t^\nu)^* dn = \int \sup K_t^\nu(n' n)\, dn,$$

the supremum taken over $|n'| \leq e_{k+C}$. If $n \in L$ and C is large, the coordinates of $n'n$ and n here will differ by at most e_k^α, e_k^β, and $e_k^{\alpha+\beta}$, respectively. The denominator in the expression for K_t^ν is then essentially the same at $n'n$ and n. The support of $\sup K_t^\nu(n'n)$ is only slightly larger than that of $K_t^\nu(n)$. Thus, $\tilde{K}_t^\nu$ and $(\tilde{K}_t^\nu)^*$ have the same L^1 bound $C2^{-\nu}$, and (iii') is proved.

Now Lemma 3.1 applies with $C_1 = 2^{-\nu}$ and $N = \nu + C$. Proposition 7.2 is proved.

Proof of Theorem 7.1. As when proving Theorem 5.1, we take $0 \leq f \in L^1(\overline{N})$ supported in a compact set L. Then we must estimate $\mathscr{P}_0 f$ at the point $n_1 \exp(tH_0)n$, or equivalently at

$$g = n_1 \exp((t + q \log t)H_0)n,$$

for $n_1 \in L$, large t, and $|n| \leq Ct^q$. One has $g = n_1 n' \exp((t + q \log t)H_0)$, where $|n'| \leq Ce^{-t}$. Then (2.4) implies

$$(7.5) \quad \mathscr{P}_0 f(g) \leq C \int f(n_1 n'n) t^{-3+2\alpha q + 2\beta q} e^{2\alpha t + 2\beta t} P\left(n^{-(t+q\log t)H_0}\right)^{1/2} dn,$$

and here

$$(7.6) \quad P\left(n^{-(t+q\log t)H_0}\right)^{1/2}$$
$$\sim \frac{1}{\left(1 + t^{\alpha q}e^{\alpha t}|x| + t^{\alpha q + \beta q}e^{\alpha t + \beta t}|z|\right)\left(1 + t^{\beta q}e^{\beta t}|y| + t^{\alpha q + \beta q}e^{\alpha t + \beta t}|z - xy|\right)}$$

with $n = (x, y, z)$. By decomposing $\overline{N}$ into several subsets, we shall divide the integral in (7.5) into parts which are considered separately. The decomposition will depend on the relative sizes of the terms in the denominator of (7.6), with the powers of t deleted. Parts 1 and 2 will mostly be handled by a method due to T. Lyons (oral communication). The idea is to estimate $\mathscr{P}_0 f(g)$, or parts of it, by $\mathscr{P}_\lambda f(g')$, where λ may be different from 0 and $g' \cdot o$ is a point further from the boundary than $g \cdot o$. We shall have $g' = n_1 n' \exp(tH_0)$. For Part 3, we use more direct estimates.

Part 1: $\max(1, e^{\alpha t}|x|) \leq e^{\alpha t + \beta t}|z|$ and $\max(1, e^{\beta t}|y|) \leq e^{\alpha t + \beta t}|z - xy|$. The kernel in (7.5) is dominated by

$$t^{-3} \frac{1}{|z||z - xy|}$$

$$\leq \frac{Ct^{-3}e^{2\alpha t + 2\beta t}}{\left(1 + e^{\alpha t}|x| + e^{\alpha t + \beta t}|z|\right)\left(1 + e^{\beta t}|y| + e^{\alpha t + \beta t}|z - xy|\right)}.$$

The corresponding part of $\mathcal{P}_0 f(g)$ is thus dominated by

$$\mathcal{P}_0 f\big(n_1 n' \exp(tH_0)\big) = \mathcal{P}_0 f\big(n_1 \exp(tH_0) n''\big),$$

where n'' stays in a compact set. This is controlled by Theorem 5.1.

Part 2: $\max(e^{\alpha t}|x|, e^{\alpha t + \beta t}|z|) \leq 1$. This will also include the case $\max(e^{\beta t}|y|, e^{\alpha t + \beta t}|z - xy|) \leq 1$, as seen via (6.2).

2a: $e^{\beta t}|y| \leq 1$. Now $(x, y, z) \in B(Ce^{-t})$, and this part of $\mathcal{P}_0 f(g)$ is majorized by

$$(7.7) \qquad t^{-3 + 2\alpha q + 2\beta q} e^{2\alpha t + 2\beta t} \int_{n_1 n' B(Ce^{-t})} f(n)\, dn.$$

Changing C, we have $n_1 n' B(Ce^{-t}) \subset n_1 B(Ce^{-t})$. Since $|B(Ce^{-t})| \sim e^{-2\alpha t - 2\beta t}$ and the exponent of t in (7.7) is negative, the expression (7.7) is dominated by the mean value of f in $n_1 B(Ce^{-t})$. This leads to the standard maximal operator in $\overline{N}$, which is of weak type (1,1).

2b: $e^{\beta t}|y| > 1$. Here $e^{\alpha t + \beta t}|z - xy| \leq 2e^{\beta t}|y|$, and we can estimate the kernel of (7.5) by

$$\frac{Ct^{-3 + 2\alpha q + 2\beta q} e^{2\alpha t + 2\beta t}}{\big(1 + e^{\alpha t}|x| + e^{\alpha t + \beta t}|z|\big)^2 t^{\beta q} e^{\beta t}|y|}$$

$$\leq \frac{Ct^{-3 + 2\alpha q + \beta q} e^{2\alpha t + 2\beta t}}{\big(1 + e^{\alpha t}|x| + e^{\alpha t + \beta t}|z|\big)^2 \big(1 + e^{\beta t}|y| + e^{\alpha t + \beta t}|z - xy|\big)}$$

$$= Ct^{-2 + 2\alpha q + \beta q} P_{1,0}^t(n).$$

Again, the exponent of t is nonpositive. Proposition 7.2 gives the necessary estimate, since (7.1) takes care of the factor n' of (7.5).

Part 3: $\max(1, e^{\alpha t + \beta t}|z|) \leq e^{\alpha t}|x|$. This also takes care of the case $\max(1, e^{\alpha t + \beta t}|z - xy|) \leq e^{\beta t}|y|$.

3a: $e^{\beta t}|y| \leq 2$. We can assume that $e^{\alpha t + \beta t}|z - xy| > 1$, since the contrary case is covered by Part 2. For such n, (7.6) is at most $Ct^{-2\alpha q - \beta q} e^{-2\alpha t - \beta t}|x|^{-1}|z - xy|^{-1}$. Let

$$K_t(n) = \frac{Ct^{-3 + \beta q} e^{\beta t}}{\big(e^{-\alpha t} + |x|\big)\big(e^{-\alpha t - \beta t} + |z - xy|\big)} \chi_{|y| \leq 2e^{-\beta t}}.$$

That part of (7.5) corresponding to 3a can be estimated by

$$\int f(n_1 n) K_t\big((n')^{-1}n\big)\, dn \le \int f(n_1 n) K_t^*(n)\, dn,$$

where

$$K_t^*(n) = \sup K_t\big((n')^{-1}n\big)$$

and the supremum is taken over $n' \in B(Ce^{-t})$. Comparing the coordinates of $(n')^{-1}n$ and n, we see that

$$K_t^*(n) \le \frac{Ct^{-3+\beta q}e^{\beta t}}{\big(e^{-\alpha t}+|x|\big)\big(e^{-\alpha t-\beta t}+|z-xy|\big)}\chi_{|y|\le Ce^{-\beta t}}.$$

The automorphism (6.2) transforms $K_t^*(n)$ to

$$\frac{t^{-3+\beta q}e^{\beta t}}{\big(e^{-\alpha t}+|y|\big)\big(e^{-\alpha t-\beta t}+|z|\big)}\chi_{|x|<Ce^{-\beta t}} = L_t(n),$$

say. We write

$$(7.8) \quad \sup_t \int f(n_1 n) L_t(n)\, dn \le \sum_k \sup_{2^{k-1}\le t\le 2^k} \int f(n_1 n) L_t(n)\, dn.$$

For $2^{k-1} \le t \le 2^k$,

$$L_t(n) \le C\frac{2^{-3k+\beta qk}e^{\beta t}}{\big(e_k^\alpha +|y|\big)\big(e_k^{\alpha+\beta}+|z|\big)}\chi_{|x|\le Ce^{-\beta t}}.$$

Here we have no t dependence in the factors containing y and z. Thus we can estimate $\int f(n_1 n)L_t(n)\, dn$ for $2^{k-1} \le t \le 2^k$ by integrating first in these variables. This means convolving f with a finite measure, which preserves L^1. The integral in x can then be estimated by means of the standard maximal function, taken in the direction $(1, 0, y_1)$. Thus, term number k in the sum in (7.8) defines an operator of weak type (1,1). The associated constant is seen to be $O(2^{-3k+\beta qk+k+k}) = O(2^{(\beta q-1)k})$. Since $\beta q < 1$, the sum in (7.8) will be in weak L^1, and we get an estimate which ends Part 3a.

3b: $e^{\beta t}|y| > 2$. Now $|z - xy| \sim |xy|$, and the expression in (7.6) is bounded by $Ct^{-2\alpha q-\beta q}e^{-2\alpha t-\beta t}|x|^{-1}|xy|^{-1}$. We proceed as in 3a and obtain kernels K_t and K_t^* with

$$K_t^*(n) \le C\frac{t^{-3+\beta q}e^{\beta t}}{\big(e^{-\alpha t}+|x|\big)^2\big(e^{\beta t}+|y|\big)}\chi_t(n),$$

χ_t being the characteristic function of $\{e^{\alpha t + \beta t}|z| \le C(1 + e^{\alpha t}|x|)\}$. Assume again that $2^{k-1} \le t \le 2^k$. Then

$$(7.9) \qquad K_t^*(n) \le C\sum_j 2^{-3k + \beta qk} e^{\beta t} (2^{-j} e^{\alpha t})^2 \frac{1}{e_k^\beta + |y|} \chi_{t,j}(n)$$

$$\equiv C\sum_j K_t^{(j)}(n),$$

where $\chi_{t,j}$ is the characteristic function of $\{|x| \le 2^j e^{-\alpha t}, |z| \le C2^j e^{-\alpha t - \beta t}\}$ and j ranges from 0 to an upper bound which is $O(2^k)$.

Let us estimate

$$(7.10) \qquad \sup_{2^{k-1} \le t \le 2^k} \int f(n_1 n) K_t^{(j)}(n)\, dn.$$

We integrate first in y and obtain a convolution of f with a finite measure. As before, the integral in x and z is then estimated by means of a one-parameter two-dimensional maximal operator. This gives a weak type $(1,1)$ estimate in $\mathbf{R}^3$ for the operator (7.10), and the associated constant is seen to be $O(2^{(\beta q - 2)k})$.

To sum in j, we use the addition theorem for the weak L^1 quasinorm $\|\cdot\|$ from Stein [18], §6. It says that

$$\left\| \sum_1^N f_i \right\| \le C \log N \sum_1^N \|f_i\|, \qquad N \ge 2.$$

Thus the operator

$$f \to \sup_{2^{k-1} \le t \le 2^k} \int f(n_1 n) K_t^*(n)\, dn$$

is of weak type $(1,1)$, with a constant which is $O(k2^{(\beta q - 1)k})$. This allows us to sum in k and finish Part 3b.

It only remains to verify that the value of q is sharp. Let m denote Lebesgue measure in $[-1, 1]$ and set

$$\mu = \delta \otimes m \otimes \delta.$$

Here δ is the Dirac measure at $0 \in \mathbf{R}$, and the tensor product refers to the coordinates (x, y, z) of $\overline{N}$. Take any $q > 0$. At a point $n_0 = (0, y_0, 0)$, $|y_0| \le 1$, one has for large t

$$(7.11) \quad \mathscr{P}_0 f(n_0 \exp(tH_0)) \sim t^{-3} e^{2\alpha t + 2\beta t} \int_{|y| \le 1} \frac{dy}{1 + e^{\beta t}|y_0 + y|}$$

$$\sim t^{-2} e^{2\alpha t + \beta t}.$$

Let $n_1 = (x_1, y_1, z_1)$ be a point with $|x_1| \leq t^{\alpha q} e^{-\alpha t}$, $|y_1| \leq 1$, and $|z_1 - x_1 y_1| \leq t^{\alpha q + \beta q} e^{-\alpha t - \beta t}$. Then the point $n' = (-x_1, 0, x_1 y_1 - z_1)$ satisfies $n_1 n' = (0, y_1, 0)$, and $n = (n')^{-tH_0}$ stays in a ball $B(Ct^q)$. Because of (7.11) we get

$$\sup_{|n| \leq Ct^q} \mathscr{P}_0 \mu(n_1 \exp(tH_0)n) \geq ct^{-2} e^{2\alpha t + \beta t},$$

for some $c > 0$. The set of such n_1 has measure $8t^{2\alpha q + \beta q} e^{-2\alpha t - \beta t}$. When $2\alpha q + \beta q > 2$, this is enough to disprove the weak type (1,1) of the maximal function associated to H_0 and q. The symmetry in α and β obtained from (6.2) now implies that the value of q in the theorem is sharp for weak type (1,1).

The measure μ leads to a counterexample to a.e. convergence by a rather standard method which we briefly describe. By approximating and contracting μ, one constructs a sequence (f_k) of nonnegative functions in $L^1(\overline{N})$ supported near e such that $\sum \int f_k \, dn < \infty$ and for some $t_k \to +\infty$ and $\omega_k \to +\infty$

$$\sup_{|n| \leq Ct_k^q} \mathscr{P}_0 f_k(n_1 \exp(t_k H_0)n) \geq \omega_k$$

on a set A_k. Moreover, $\sum |A_k| = +\infty$ and the A_k are contained in a small neighborhood of 0.

It is enough to take a small ball B centered at e and verify the following: Given $k' \in \mathbf{N}$, there exist $k'' > k'$ and points $n_k \in B$, $k' < k \leq k''$, such that the union of translates $\bigcup\{n_k A_k: k' < k \leq k''\}$ has measure at least $|B|/2$. Indeed, then we can iterate this construction to form segments of a sequence $(n_k)_1^\infty$ such that the set of points belonging to infinitely many sets $n_k A_k$ has positive measure. The required counterexample f will then be the sum of the corresponding translates of the f_k.

To make this construction, assume n_k has been chosen for $k' < k < j$ and that the set $M_{j-1} = \bigcup\{n_k A_k: k' < k < j\}$ has measure smaller than $|B|/2$. Considering the average of $|M_{j-1} \cap nA_j|$ as n runs over B, we find an n_j with $|M_{j-1} \cap n_j A_j| \leq |A_j|/2$. Thus $|(n_j A_j) \setminus M_{j-1}| \geq |A_j|/2$, and the rest is easy.

Theorem 7.1 is proved.

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