PACIFIC JOURNAL OF MATHEMATICS
Vol. 131, No. 2, 1988

# MENGER SPACES AND INVERSE LIMITS

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**M. Bestvina in 1984 characterized the Menger universal $n$-dimensional spaces. This characterization is used to identify certain inverse sequences having inverse limit homeomorphic to one of the Menger spaces. Specific models of the Menger spaces are then constructed in the Hilbert Cube as inverse limits of polyhedra. The union of these models is shown to be homeomorphic to the countably infinite dimensional space $\sigma$.**

**1. Introduction.** In 1984, M. Bestvina [**Be**] characterized the Menger universal $n$-dimensional compactum $\mu_n$ as follows.

THEOREM. *A space $X$ is homeomorphic to $\mu_n$ if and only if $X$ satisfies the following properties*:

1. *$X$ is compact and $n$-dimensional*,
2. *$X$ is $(n-1)$-connected $(C^{n-1})$*,
3. *$X$ is locally $(n-1)$-connected $(LC^{n-1})$, and*
4. *$X$ satisfies the disjoint $n$-cells property $(DD^nP)$.*

Using this characterization, Bestvina showed that the various constructions in the literature of compact universal $n$-dimensional spaces ([**Mg**], [**Lf**], [**Pa**]) all yield $\mu_n$. In addition, Bestvina showed that each $\mu_n$ is homogeneous. Prior to this result, there had been characterizations only of $\mu_0$ (the Cantor set) and $\mu_1$ (the universal curve) [**An**].

Using Bestvina's characterization, we identify certain inverse sequences that have $\mu_n$ as inverse limit. This leads to the construction of models of $\mu_n$ in the Hilbert Cube. These models can be described by putting restrictions on the coordinates of points in the Hilbert Cube. We also show that the union of certain of these models naturally yields the countably infinite dimensional space $\sigma$.

**2. Notation and terminology.** All spaces are assumed to be separable and metrizable. A reference for dimension theory is [**En**]. A space is $n-1$ *connected*, $C^{n-1}$, if each map of $S^k$, $k \leq n-1$, into the space extends to a map of the $k+1$ cell into the space. A space $X$ is *locally* $n-1$ *connected*, $LC^{n-1}$, if for each point $p \in X$, and for each neighborhood $U$ of $p$ there is a neighborhood $V$ of $p$, $V \subset U$, so that each map of $S^k$ into $V$, $k \leq n-1$, extends to a map of $B^{k+1}$ into $U$.

A space $X$ satisfies the *disjoint n-cells property*, $DD^nP$, if for each pair of maps $f, g\colon B^n \to X$ and for each $\varepsilon > 0$ there are maps $f_1, g_1\colon B^n \to X$ so that $f_1(B^n) \cap g_1(B^n) = \emptyset$ and so that $d(f_1, f) < \varepsilon$, and $d(g_1, g) < \varepsilon$.

We denote an inverse sequence of topological spaces

$$X_0 \overset{p_1}{\leftarrow} X_1 \overset{p_2}{\leftarrow} X_2 \cdots$$

by $\{X_i, p_i\}$. If $X$ is the inverse limit of such a sequence, we let $\pi_i\colon X \to X_i$ be projection onto the $i$th coordinate space and let $p_{ij}\colon X_j \to X_i$, $j \geq i$, be the map induced by the bonding maps. If $\{X_i, p_i\}$ is an inverse sequence, we assume $X_i$ is metrized by a metric $d_i$ so that diameter $(X_i) \leq 1/2^i$. We use the metric $d(x, y) = \Sigma_{i=0}^{\infty} d_i(x_i, y_i)$ on the product space $\prod_{i=0}^{\infty} X_i$.

We view the Hilbert Cube $Q$ as $\prod_{i=1}^{\infty} I_i$ where $I_i = [0, 1/2^i]$ and we view the $n$-cell $I^n$ as $\prod_{i=1}^{n} I_i$. $Q_j$ is $\prod_{i=j}^{\infty} I_i$ so that for each $n$, $Q = I^n \times Q_{n+1}$.

A number $x$ in $[0, \frac{1}{2}]$ that can be written as $k/2^n$, $n \geq 1$, with $k$ and 2 relatively prime is called a dyadic rational of order $n$. So 0 and $\frac{1}{2}$ have order 1, $\frac{1}{4}$ has order 2, $\frac{1}{8}$ and $\frac{3}{8}$ have order 3 and so on.

**3. Inverse sequences.** There are a number of results in the literature giving conditions which imply that the inverse limit of $LC^n$ compacta is itself $LC^n$. Z. Cerin **[Ce]** shows that the inverse limit is $LC^n$ if and only if the inverse sequence is strongly $n$ $e$-movable. L. McAuley and E. Robinson **[M, R]** show that the inverse limit is $LC^n$ if each bonding map is $UV^n$.

For the examples we are interested in, we need conditions that yield both $C^{n-1}$ and $LC^{n-1}$. Conditions 2 and 3 in the next Theorem are sufficient for this purpose.

THEOREM 1. *Let $\{X_i, p_i\}$ be an inverse sequence of $LC^{n-1}$ n-dimensional compacta, satisfying the following conditions.*

1. *For each $i$ and map $f\colon B^n \to X_i$ there exists $j > i$ and maps $h_1, h_2\colon B^n \to X_j$ with $h_1(B^n) \cap h_2(B^n) = \emptyset$ and $p_{ij} \circ h_e = f$ for $e = 1, 2$.*
2. *$X_1$ is $C^{n-1}$.*
3. *There is a constant $c$ so that for each $i$, for each map $f\colon B^{k+1} \to X_i$, $k \leq n-1$, and for each map $g\colon S^k \to X_{i+1}$ with $p_{i+1} \circ g = f \mid S^k$, there is an extension $h\colon B^{k+1} \to X_{i+1}$ with $p_{m,i+1} \circ h$ within $c/2^{i+1}$ of $p_{m,i} \circ f$ for each $m \leq i$.*

*Then $X = \varprojlim \{X_i, p_i\}$ is homeomorphic to $\mu_n$.*

*Proof.*

compact $n$-dimensional.

Since each $X_i$ is compact and $n$-dimensional, $X$ is compact and less than or equal to $n$-dimensional. We show below that $X$ is $LC^{n-1}$ and satisfies $DD^nP$. A standard argument similar to that in **[Ca]** then shows that maps from $B^n$ into $X$ are approximable by embeddings. In particular, $X$ contains $n$-dimensional subspaces and is thus $n$-dimensional.

$DD^nP$.

Let $f_e$: $B_e^n \to X$, $e = 1,2$, be maps from $n$-cells $B_1^n$, $B_2^n$ into $X$ and let $\varepsilon > 0$ be given. Choose $N$ so that diameter $[\prod_{i=N}^{\infty} X_i] < \varepsilon$. By conditions 2 and 3, there is an arc $\alpha$ in $X_N$ connecting the image of $\pi_N \circ f_1$ to the image of $\pi_N \circ f_2$. We may view $\pi_N \circ f_1[B_1^n] \cup \alpha \cup \pi_N \circ f_2[B_2^n]$ as the image of the $n$-cell $B^n$ under a map $g$. Furthermore, we may assume that $B_e^n \subset B^n$ with $\pi_N \circ f_e = g \mid B_e^n$. For a similar argument see **[Ga]**.

Choose $j_1 > N$ and maps $h_1, h_2$: $B^n \to X_{j_1}$ as in condition 1. Define $f_e^1$: $B_e^n \to X_{j_1}$, $e = 1,2$, by $f_e^1 = h_e \mid B_e^n$. Then

$$f_1^1[B_1^n] \cap f_2^1[B_2^n] = \varnothing \quad \text{and} \quad \pi_N \circ f_e = P_{n,j_1} \circ f_e^1.$$

Repeating this argument, we inductively define a sequence of integers $j_1 < j_2 < j_3 < \cdots$ and maps $f_e^m \to X_{j_m}$, $e = 1,2$, so that for each positive integer $m$,

$$P_{j_m, j_{m+1}} \circ f_e^{m+1} = f_e^m.$$

This procedure induces maps $g_1, g_2$: $B_e^n \to X$ with $\pi_i \circ g_e = \pi_i \circ f_e$ for $i \leq N$, and with $\pi_{j_1} \circ g_1[B_1^n] \cap \pi_{j_1} \circ g_1[B_2^n] = \varnothing$. It follows that $g_e$ is within $\varepsilon$ of $f_e$ and that $g_1[B_1^n] \cap g_2[B_2^n] = \varnothing$.

$C^{n-1}$.

Let $\varepsilon_i = c/2^i$. Let $f$: $S^k \to X$, $k \leq n-1$, be given. Since $X_1$ is $C^{n-1}$, $\pi_1 \circ f$ extends to a map $f_1$: $B^{k+1} \to X_1$. Use condition 3 to extend $\pi_2 \circ f$ to a map $f_2$: $B^{k+1} \to X_2$ so that

$$d(f_1, p_2 \circ f_2) < \varepsilon_2.$$

In this manner we inductively define a sequence of maps $f_m$: $B^{k+1} \to X_m$ with

$$d(p_{i,m} \circ f_m, p_{i,m-1} \circ f_{m-1}) < \varepsilon_m \quad \text{for } i \leq m-1,$$

and so that $f_m \mid S^k = \pi_m \circ f$.

Define a map $g\colon B^{k+1} \to \prod_{i=1}^{\infty} X_i$ by

$$g_i(x) = \lim_{m\to\infty} p_{i_m} \circ f_m(x).$$

Since the sequence $(p_{i_m} \circ f_m)_{m=1}^{\infty}$ is uniformly Cauchy, each $g_i$ is continuous. The definition of the functions $f_m$ shows that $g_i$ extends $\pi_i \circ f$. Finally, since $p_{i-1,i} \circ g_i = g_{i-1}$,

$$g(B^{k+1}) \subset X.$$

Then the required extension has been constructed.

$\underline{LC^{n-1}}$.

Given $q \in X$ and $\varepsilon > 0$, choose $N$ so that diameter $(\prod_{i=N}^{\infty} X_i) < \varepsilon/2$ and so that

$$\sum_{i=N}^{\infty} \varepsilon_i < \frac{\varepsilon}{8N}.$$

Use the fact that $X_N$ is $LC^{n-1}$ to choose $\delta_1 < \delta_2 < \varepsilon/8N$ so that any map of $S^k$, $k \leq n-1$, into the $\delta_1$ neighborhood of $q_N$ in $X_N$ extends to a map of $B^{k+1}$ into the $\delta_2$ neighborhood of $q_N$ in $X_N$. We may also require that any set of diameter $< 2\delta_2$ in $X_N$ has image in $X_i$ of diameter $< \varepsilon/8N$ under the map $p_{iN}$. This uses the uniform continuity of the bonding maps.

Let $f\colon S^k \to X$ be a map into the $\delta_1$ neighborhood of $q$ in $X$. Then $\pi_N \circ f(S^k)$ is contained in the $\delta_1$ neighborhood of $q_N$, and so there is an extension $g_N\colon B^{k+1} \to X_N$ with image of diameter $< \delta_2$.

For each $i \leq N$, we thus have maps $g_i \equiv p_{i,N} \circ g_N\colon B^{k+1} \to X_i$ with image of diameter $< \varepsilon/8N$, and with $g_i | S^k = \pi_i \circ f$.

Since $\sum_{i=N}^{\infty} \varepsilon_i < \varepsilon/8N$, we may now proceed exactly as in the proof of $C^{n-1}$ to construct an extension $h\colon B^{k+1} \to X$ of $f$ so that

$$d(\pi_i \circ h, g_i) < \frac{\varepsilon}{8N} \quad \text{for } i \leq N.$$

It follows that $h_i$ is a map into the $4 \cdot \varepsilon/8N = \varepsilon/2N$ neighborhood of $q_i$ in $X_i$ for $i \leq N$.

Because $\operatorname{diam}(\prod_{i=N}^{\infty} X_i) < \varepsilon/2$, it then follows that $h$ is a map into the $\varepsilon$ neighborhood of $q$ in $X$. □

## 4. Specific models in the Hilbert cube.

THEOREM 2. *Fix $n \geq 0$. Let $P_i \subset I^i$, $i \geq n$, be a sequence of compact $n$-dimensional $LC^{n-1}$ spaces so that*

(a) *$P_i \times \{0, 1/2^{i+1}\} \subset P_{i+1}$ and $P_{i+1} \subset P_i \times I_{i+1}$,*

(b) *$P_n$ is $C^{n-1}$, and*

(c) *There is a constant $c$ so that for each map $f\colon B^{k+1} \to P_i \times I_{i+1}$*

*$(k \leq n-1)$ with $f(S^k) \subset P_{i+1}$, there is a map $g$: $B^{k+1} \to P_{i+1}$ with $d(f, g) < c/2^{i+1}$ and with $f \mid S^k = g \mid S^k$.*
*Let $X = \bigcap_{i=n}^{\infty}(P_i \times Q_{i+1})$. Then $X \cong \mu_n$.*

*Proof.* $X$ is homeomorphic to $\varprojlim\{P_i, p_i\}$ where $p_i$ is the restriction of projection from $I^i$ onto $I^{i-1}$. So it suffices to check that conditions 1, 2, and 3 from Theorem 1 are satisfied.

Conditions (a) and (b) above directly imply that conditions 1 and 2 are satisfied.

For condition 3, let $f$: $B^{k+1} \to P_i$, $k \leq n-1$, and $g$: $S^k \to P_{i+1}$ be maps with

$$p_{i+1} \circ g = f \mid S^k.$$

Extend $g$ to a map $h$: $B^{k+1} \to P_i \times I_{i+1}$ so that $p_{i+1} \circ h = f$. Use condition (c) to approximate $h$ by a map

$$h_1 \colon B^{k+1} \to P_{i+1}, \quad \text{with}$$

$$d(h, h_1) < \frac{c}{2^{i+1}}, \quad \text{and with } h_1 \mid S^k = h \mid S^k.$$

Then $p_{j,i+1} \circ h_1$ is within $c/2^{i+1}$ of $p_{ji} \circ f$ for each $j \leq i$ and condition 1 is satisfied. □

We now construct a specific model satisfying the conditions in Theorem 2. Again, fix $n \geq 0$.

For $X = I^i$ or $Q$, let

$$X_* \equiv \{x \in X \mid \text{for each choice of } n+1 \text{ coordinates } x_{m_1}, \ldots, x_{m_{n+1}} \text{ of } X \text{ with } m_1 < \cdots < m_{n+1}, \text{ at least one of the coordinates is dyadic of order} \leq m_{n+1}\}.$$

Let $P_i = I^i_*$.

For $n = 1$, the one-dimensional polyhedra $P_1$, $P_2$, and $P_3$ are illustrated in Figure 1.

When $n = 0$, $P_i = \prod_{j=1}^{i}\{0, 1/2^j\}$, the corner points of the $i$-cell $I^i$.

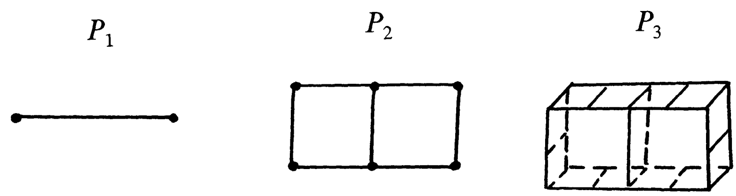


FIGURE 1

Let $X_n = \bigcap_{i=n}^{\infty}(P_i \times Q_{i+1})$. Then $X_n$ is $Q_*$. Note that $X_0$ is the Cantor Set consisting of the corner points of the Hilbert Cube. In Theorem 3 below, we show that $X_n \cong \mu_n$. Before proving this theorem, we provide an alternate description of $P_i$ that is easier to work with.

Fix $n$. Let $P_n' = I^n$. Let $P_n'$ be viewed as a cell complex consisting of rectilinear $n$-cells with sides of length $1/2^{n+1}$ by subdividing each factor $I_i$ of $I^n$ into subintervals of lengths $1/2^{n+1}$. Let $A_n$ be the $(n-1)$ skeleton of this cell complex.

Define $P_{n+1}' \subset I^{n+1}$ as

$$A_n \times I_{n+1} \cup P_n' \times \left\{0, \frac{1}{2^{n+1}}\right\}.$$

Note that $P_{n+1}'$ can be viewed as a cell complex consisting of rectilinear $n$-cells with sides of length $1/2^{n+2}$ by subdividing each factor $I_i$ of $I^{n+1}$ into subintervals of length $1/2^{n+2}$.

Inductively assume $P_j' \subset I^j$ has been defined so that $P_j'$ can be viewed as a cell complex consisting of rectilinear $n$-cells with sides of length $1/2^{j+1}$ by subdividing each factor $I_i$ of $I^j$ into subintervals of length $1/2^{j+1}$.

Let $A_j$ be the $n-1$ skeleton of this cell complex. Define $P_{j+1}' \subset I^{j+1}$ as

$$A_j \times I_{j+1} \cup P_j' \times \{0, 1/2^{j+1}\}.$$

LEMMA 1. *For each $i$, $P_i = P_i'$.*

*Proof*. The proof proceeds by induction. When $i = n$, $P_i = P_i' = I^n$.

Assume inductively that $P_i = P_i'$. Let $x$ be a point in $P_{i+1}$. We may view $x$ as $(a, b)$ where $a = (x_1, \dots, x_i)$ and $b = x_{i+1}$. Then $a \in P_i = P_i'$. If $i - n + 1$ coordinates of $a$ have order $\leq i + 1$, then $a \in A_i$ and so $(a,b) \in P_{i+1}'$. If fewer than $i - n + 1$ coordinates of $a$ have order $\leq i + 1$, then $b$ must have order $i + 1$. So $b \in \{0, 1/2^{i+1}\}$ and $(a, b) \in P_{i+1}'$.

Conversely, let $X = (a, b) \in P_{i+1}'$. Then $a \in P_i' = P_i$. If $b \in \{0, 1/2^{i+1}\}$, then $(a, b) \in P_{i+1}$. If $b \notin \{0, 1/2^{i+1}\}$, then $a$ must have $i - n + 1$ coordinates of order $\leq i + 1$ since $a$ is then in $A_i$. Again $(a, b) \in P_{i+1}$. □

THEOREM 3. $X_n \cong \mu_n$.

*Proof*. The alternate description of the $P_i$ given by Lemma 1 shows that each $P_i$ is a compact $n$-dimensional polyhedron. Thus each $P_i$ is $LC^{n-1}$. So it suffices to show that the $P_i \subset I^i$ satisfy conditions (a), (b), and (c) in Theorem 2. The alternate description of the $P_i$ shows directly that condition $a$ is satisfied.

$P_n$ is an $n$-cube and hence is $(n-1)$-connected. Therefore condition (b) is satisfied. For condition (c), let $f$ be a map from $B^{k+1}$ to $P_i \times I_{i+1}$ $(k \leq n-1)$ with $f(S^k) \subset P_{i+1}$. Using the alternate description, $P_{i+1}$ may be viewed as the $n$-skeleton of a cell complex $L$ with underlying space $P_i \times I_{i+1}$. We may assume that $L$ consists entirely of rectilinear $(n+1)$ cells with sides of length $1/2^{i+1}$.

Let $\sigma$ be such an $n+1$ cell of $L$. Since $k+1 \leq n$, $f \mid f^{-1}(\sigma)$: $f^{-1}(\sigma) \to \sigma$ may be replaced by a map $g$: $f^{-1}(\sigma) \to \partial\sigma$ so that $g \mid f^{-1}(\partial\sigma) = f \mid f^{-1}(\partial\sigma)$ [**H, W**]. It follows that $d(f \mid f^{-1}(\sigma), g \mid f^{-1}(\sigma)) \leq \text{diameter}(\sigma) = (n+1)/2^{i+1}$.

By following the above procedure on each $n+1$ cell of $L$, one obtains a map $g$: $B^{k+1} \to P_{i+1}$ so that $g \mid S^k = f \mid S^k$, and so that $d(f, g) < (n+1)/2^{i+1}$. □

**5. Menger spaces and $\sigma$.** In this section, we show that if $X = \bigcup_{n=1}^{\infty} X_n$, then $X$ is homeomorphic to $\sigma$. Recall that $\sigma$ may be viewed as the set of points in Hilbert space having at most finitely many nonzero coordinates. In order to obtain the desired goal, we will show that $X$ satisfies the following characterization [**He**]: $X$ is a $\sigma$-manifold if and only if:

1. $X$ is an ANR.
2. $X$ is the countable union of finite dimensional compacta.
3. Each compact subset of $X$ is a strong $Z$-set in $X$.
4. For each integer $k$, mapping $f$: $\mathbf{R}^k \to X$, and $\varepsilon$: $X \to (0,1)$, there is an injection $f'$: $\mathbf{R}^k \to X$ with $d(f(x), f'(x)) < \varepsilon(x)$.

The last property is referred to as the Euclidean injection property (EIP). Condition 3 means that if $A$ is a compact subset of $X$, for each open cover $\mathscr{U}$ of $X$ and sequence of mappings $\alpha_1, \alpha_2, \ldots$ of $Q$ into $X$, there are $\mathscr{U}$-approximations $\beta_1, \beta_2, \ldots$ such that $\bigcup\{\beta_i(Q)\colon 1 \leq i < \infty\}$ misses a neighborhood of $A$ [**B, B, M, W**]. Condition 2 is satisfied since each $X_n$ is a compact finite dimensional set. The space $X$ will be shown to satisfy the other conditions through a sequence of results.

The first lemma involves approximating mappings of $\mathbf{R}^k$ into $Q$ by mappings into $X$. Throughout the remainder, by a basic open set $V$ in $Q$ we will mean an open set in $Q$ of the form $V = (\prod_{i=1}^{n} V_i) \times Q_{n+1}$ where $V_i$ is a connected open set in $I_i$. We let $D_k = \{(x_i) \in Q\colon x_i = 0 \text{ for } i > k\}$ and set $D = \bigcup_{n=0}^{\infty} D_k$.

LEMMA 2. *Let $V$ be a basic open set of $Q$. For $f$: $\mathbf{R}^k \to V$ and $\varepsilon$: $\mathbf{R}^k \to (0,1)$, there is a mapping $f'$: $\mathbf{R}^k \to D$ with $d(f(x), f'(x)) < \varepsilon(x)$.*

*Proof*. Let $f_i$ be the $i$th coordinate function of $f$. Define the $i$th coordinate function $f_i'$ of a function $f'$: $\mathbf{R}^k \to V$ by

$$f_i'(x) = \begin{cases} f_i(x) & \text{if } i \leq n \\ \operatorname{Max}\left\{ f_i(x) - \dfrac{\varepsilon(x)}{2^i}, 0 \right\} & \text{if } i > n. \end{cases}$$

Clearly, $d(f(x), f'(x)) < \varepsilon(x)$. □

We now turn to the problem of showing that $X$ is an ANR. According to Dugundji [**Du**], it suffices to show that given any open cover $\mathscr{U}$ of $X$, there is an open cover $\mathscr{V}$ of $X$ such that given any simplicial complex $K$, any partial realization of $K$ in $\mathscr{V}$ extends to a full realization of $K$ in $\mathscr{U}$. A partial realization of $K$ in $\mathscr{U}$ is a mapping $h$: $L \to X$ in which $L$ is a subcomplex of $K$ containing every vertex of $K$ and such that the sets $h(|L \cap s|)$ refine $\mathscr{U}$ where $s$ is a simplex of $K$. A full realization of $K$ in $\mathscr{U}$ is a partial realization of $K$ in $\mathscr{U}$ where $K = L$.

PROPOSITION 1. *X is an ANR.*

*Proof*. Let $\mathscr{U}$ be an open cover of $X$. Choose $\mathscr{V}$ to be a locally finite open cover of $X$ refining $\mathscr{U}$ such that each open set in $\mathscr{V}$ is of the form $V' = V \cap X$ where $V$ is a basic open set in $Q$. Given a partial realization $h$: $L \to X$ in $\mathscr{V}$, it suffices to show that a mapping $h$: $\operatorname{Bd} I^k \to V' \in \mathscr{V}$ can be extended to $h'$: $I^k \to V'$ to conclude that $h$ can be extended to a full realization of $K$ in $\mathscr{V}$, and hence in $\mathscr{U}$. Given a $k$-simplex $s$ in $K$, the image of all vertices of $s$ lie in at least one element $V'$ of $\mathscr{V}$. If $V_s' = \bigcap\{V' \in \mathscr{V}$: the image of all vertices of $s$ lie in $V'\}$, then $V_s' = V_s \cap X$ where $V_s$ is a basic open set. Thus extending the mapping of $|L \cap s|$ to $|s|$ in $V_s$ extends the mapping in each basic set $V' \in \mathscr{V}$ containing the image of the vertices of $s$. Since $V_s$ is contractible, $h$: $\operatorname{Bd} I^k \to V_s'$ can be extended to $h_1$: $I^k \to V_s$. It now follows from Lemma 2 that the mapping $h_1$ restricted to the interior of $I^k$ can be approximated by a mapping $g$: $\operatorname{int} I^k \to V_s'$ which extends by $h$ to all of $I^k$, yielding $h'$: $I^k \to V_s'$. □

PROPOSITION 2. *X satisfies the EIP.*

*Proof*. Recall that $D = \{(x_i) \in Q\colon\ x_i = 0$ for almost all $i\}$. By Lemma 2, it suffices to show that $D$ has the EIP. Let $f$: $R^k \to D$ and $\varepsilon$: $R^k \to (0, 1)$ be given. Let $\Pi$: $Q \to I^{2k+1}$ be projection onto the first $2k + 1$ coordinates. It is well-known [**H, W**] that a map of $R^k$ into $I^{2k+1}$

can be approximated by an injection. Let $g\colon R^k \to I^{2k+1}$ be such that $d(\Pi \circ f(x), g(x)) < \varepsilon(x)$ for each $x$ in $R^k$. Define $f'\colon R^k \to D$ to be a map whose first $2k + 1$ coordinates are given by $g$ and whose remaining coordinates are the same as $f$. Then $f'$ shows that $D$ has the EIP. □

The final necessary result is that each compact subset of $X$ be a strong $Z$-set. This will be accomplished by first showing that each compact subset of $X$ is a $Z$-set in $Q$, and then getting the stronger property in $X$. Recall that a closed subset $A$ of an ANR $X$ is a $Z$-set if the relative homology groups $H_*(U, U - A; Z) = 0$ for each open set $U$ in $X$ and $A$ is 1-LCC embedded in $X$.

PROPOSITION 3. *$X_n$ is a $Z$-set in $Q$.*

*Proof*. Since $X_n$ is a finite dimensional subset of $Q$,

$$H_*(U, U - X_n; Z) = 0$$

[**D, W**]. Thus it suffices to show that $X_n$ is 1-LCC in $Q$. Let $p$ be a point in $X_n$ and $U$ be an open set containing $p$. Choose $V$ to be a basic open set containing $p$ with $V \subset U$. For $g\colon S^1 \to V - X_n$, there is a homotopy $h_t\colon S^1 \to V$ with $h(S^1 \times I) \subset V - X^n$, $h_0 = g$, and $h_1(S^1) \subset D_k \cap V$ for some $k > n + 2$. Since $X_n \cap D_k$ is a tame, $n$-dimensional subpolyhedron of $D_k$, $h_1(S^1)$ contracts in $D_k \cap V$ missing $X_n$, and $X_n$ is 1-LCC in $Q$. □

COROLLARY. *Each compact subset $C$ of $X$ is a $Z$-set in $Q$.*

*Proof*. Since $C = \bigcup_{n=1}^{\infty}(X_n \cap C)$, and $X_n \cap C$ is a $Z$-set in $Q$, it follows [**C, D, M**] that $C$ is a $Z$-set in $Q$. □

LEMMA 3. *For each open cover $\mathscr{V}$ of $Q$ and each map $f\colon Q \to Q$ there is a $\mathscr{V}$-approximation $f'$ so that $f'(Q) \subset D$.*

*Proof*. The proof is similar to the proof of Lemma 2 and is left to the reader.

PROPOSITION 4. *Every compact subset of $X$ is a strong $Z$-set in $X$.*

*Proof*. Let $C$ be a compact subset of $X$, $\mathscr{U}$ an open cover of $X$, and $\{\alpha_i\}$ a sequence of mappings of $Q$ into $X$. We may suppose that $\mathscr{U}$ is the restriction of an open cover $\mathscr{W}$ of $Q$. Let $\mathscr{V}$ be an open star refinement of $\mathscr{W}$. Since $C$ is a $Z$-set in $Q$ and hence a strong $Z$-set in $Q$ [**B, B, M, W**], there are $\mathscr{V}$-approximations $\beta_i'$ to $\alpha_i$ in $Q$ such that for some neighborhood $N$ of $C$ in $Q$, $\beta_i'(Q)$ misses $N$ for each $i$. Let $M$ be a neighborhood of $C$ in $Q$ whose closure is contained in $N$. By Lemma 3, each $\beta_i'$ has a $\mathscr{V}$-approximation $\beta_i$ that takes $Q$ into $D$. We may further assume that each approximation $\beta_i$ is so close to $\beta_i'(Q)$ misses $M$. Since $\beta_i'$ is a $\mathscr{V}$-approximation of $\alpha_i$ and $\beta_i$ is a $\mathscr{V}$-approximation of $\beta_i'$, $\beta_i$ is a $\mathscr{W}$-approximation of $\alpha_i$. However, $\beta_i(Q)$ is contained in $D$ which lies in $X$, so $\beta_i$ is a $\mathscr{U}$-approximation of $\alpha_i$, and our proof is complete. □

THEOREM 4. *$X$ is homeomorphic to $\sigma$.*

*Proof*. Since $X$ satisfies the characterization theorem, $X$ is a $\sigma$-manifold. We have not shown that $X$ is homeomorphic to $\sigma$. However, a $\sigma$-manifold may be factored as $|K| \times \sigma$ where $K$ is a countable, locally finite simplicial complex [**Ch**]. It follows from Lemma 2 that $\pi_n(X) = 0$ for all $n$, so $\pi_n(|K|) = 0$ for all $n$, and $|K|$ is contractible. Thus $X$ is contractible and homeomorphic to $\sigma$ since they have the same homotopy type [**Ch**]. □

It should be noted that a more general result follows from the above proofs. The following theorem is immediate.

THEOREM 5. *Let $X = \bigcup_{n=1}^{\infty} X_n$, where each $X_n$ is a compact, finite dimensional Z-set in $Q$, with $X$ containing the set of all points in $Q$ having at most finitely many nonzero coordinates. Then $X$ is homeomorphic to $\sigma$.*

Note that this immediately implies that $D$ is homeomorphic to $\sigma$.

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Received September 29, 1986.

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