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SEMIGROUPS GENERATED BY CERTAIN OPERATORS ON VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

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The operators C, K, L, T, T_l and T_r on the lattice $\mathscr{L}(\mathscr{CR})$ of varieties of completely regular semigroups have played an important role in recent studies of $\mathscr{L}(\mathscr{CR})$. Although each of these operators is idempotent, when applied in various combinations to the trivial variety they yield varieties for which the only upper bound is $\mathscr{CR}$. The semigroups generated by various subsets of $\{C, K, L, T, T_r, T_l\}$ are determined here in terms of generators and relations.

1. Introduction and summary. Completely regular semigroups (unions of groups) may be regarded as algebras with the operations of (binary) multiplication and (unary) inversion. As such they form a variety $\mathscr{CR}$ defined by the identities

$$(1) \qquad (ab)c = (ab)c, \quad a = aa^{-1}a, \quad aa^{-1} = a^{-1}a, \quad (a^{-1})^{-1} = a.$$

The lattice $\mathscr{L}(\mathscr{CR})$ of all subvarieties of $\mathscr{CR}$ turns out to be amenable to a thorough analysis both globally and locally. The former includes various (complete) congruences that emerge naturally in the study either of the varieties themselves or of the corresponding fully invariant congruences on a free completely regular semigroup $F\mathscr{CR}$ on a countably infinite set. Local studies of the lattice $\mathscr{L}(\mathscr{CR})$ usually amount to rather complete descriptions of relatively small intervals in $\mathscr{L}(\mathscr{CR})$ modulo $\mathscr{L}(\mathscr{G})$, the lattice of group varieties, starting from the bottom of the lattice.

In the local approach, a number of operators make their appearance in the description of certain varieties in terms of some of their proper subvarieties. But these operators may be defined on all of $\mathscr{L}(\mathscr{CR})$ thereby providing a certain amount of information for varieties scattered throughout $\mathscr{L}(\mathscr{CR})$ and hence may be used for a global study of this lattice. Another source of operators on $\mathscr{L}(\mathscr{CR})$ are the kernel and trace relations on the lattice of fully invariant congruences on $F\mathscr{CR}$ now translated into relations on $\mathscr{L}(\mathscr{CR})$.

Of the considerable literature on varieties of completely regular semigroups, we mention only the following ones because they are directly related to our object of study. We thus cite Jones **[6]**, [7], Kadourek **[8]**, Pastijn-Trotter **[10]** and Reilly **[13]** for various results concerning the operators under study.

In order to explain briefly what the subject of the paper is, we need some notation. For any $S \in \mathscr{CR}$, let $E(S)$ be the set of idempotents of S and $C(S)$ be the core of S, that is the subsemigroup of S generated by $E(S)$. We define operators C and L on $\mathscr{L}(\mathscr{CR})$ by

$$\mathscr{V}C = \{S \in \mathscr{CR} \mid C(S) \in \mathscr{V}\},$$

$$\mathscr{V}L = \{S \in \mathscr{CR} \mid eSe \in \mathscr{V} \text{ for all } e \in E(S)\}.$$

Note that $\mathscr{V}C$ (respectively $\mathscr{V}L$) consists of all S in $\mathscr{CR}$ all of whose idempotent generated subsemigroups (respectively submonoids) are contained in $\mathscr{V}$.

For a fully invariant congruence ρ on $F\mathscr{CR}$ (see above), let $[\rho]$ be the corresponding variety. Let ρ_K (respectively $\rho_T, \rho_{T_l}, \rho_{T_r}$) denote the least (automatically fully invariant) congruence on $F\mathscr{CR}$ with the same kernel (respectively trace, left trace, right trace) as ρ. We now define operators K, T, T_l, T_r on $\mathscr{L}(\mathscr{CR})$ by the requirement:

$$[\rho]P = [\rho_P] \qquad (P \in \{K, T, T_l, T_r\}).$$

These operators admit the following interpretations: for any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, and with $\mathscr{S}$ denoting the variety of semilattices,

$$\mathscr{V}K = \{S \in \mathscr{CR} \mid S/\tau \in \mathscr{V} \vee \mathscr{S}\},$$

$$\mathscr{V}T = \{S \in \mathscr{CR} \mid S/\mu \in \mathscr{V}\},$$

$$\mathscr{V}T_l = \{S \in \mathscr{CR} \mid S/\mathscr{L}^0 \in \mathscr{V}\},$$

in the usual notation with $\mathscr{L}^0$ the greatest congruence contained in $\mathscr{L}$ and a symmetric expression for $\mathscr{V}T_r$.

The principal results of the paper consist of complete descriptions of semigroups generated by certain subsets of $\{C, L, K, T, T_l, T_r\}$. In order to roughly state these descriptions, let $M_{P,Q}$ (respectively $N_{P,Q}$) be the monoid generated by C, L, P and Q (respectively P and Q) where $\{P, Q\}$ is any 2-element subset of $\{K, T, T_l, T_r\}$ and R be the monoid generated by C and L. We prove that

$$M_{P,Q} \cong R \times N_{P,Q},$$

that $N_{P,Q}$ is a free monoid on two idempotent generators (except when $\{P, Q\} = \{T, T_l\}$ or $\{T, T_r\}$), and give the multiplication table for the 5-element monoid R.

Section 2 contains the needed notation. The semigroup generated by C and L is described in §3. Properties of the operators K, T, T_l and T_r are discussed in §4. That C (respectively L) commutes with K, T, T_l and T_r is proved in §5 (respectively §6). The semigroups generated by pairs of these operators are determined in §7, whereas the semigroup generated by C, L, K and T is described in §8. A diagram of values of some of the elements of the semigroups of operators at the trivial variety is presented in §9.

2. Preliminaries. A semigroup which is a union of groups is said to be a *completely regular semigroup*. Basic information about such semigroups can be found in Howie **[5]**. The fundamental structure theorem for completely regular semigroups, due to Clifford, states that a semigroup S is completely regular if and only if S is a semilattice of completely simple semigroups. We will denote this by $S = \bigcup_{\alpha \in Y} S_\alpha$, where Y is a semilattice, and refer to the completely simple subsemigroups S_α as the *components* of S.

In a completely regular semigroup S, we use the following notation. If $x \in S$, then x^{-1} is the inverse of x in the maximal subgroup of S containing x. In addition, let $x^0 = xx^{-1}$. Also, $E(S)$ denotes the set of idempotents of S and Con S denotes the lattice of congruences on S.

Certain congruences on a completely regular semigroup are particularly important. Let $\rho \in \operatorname{Con} S$, $S \in \mathscr{CR}$. Then ρ is said to be *idempotent separating* if $e, f \in E(S)$ and $e\rho f$ imply $e = f$, while ρ is *idempotent pure* if $e \in E(S)$ and $e\rho a$ imply $a \in E(S)$. We will denote by $\mu = \mu_S$ (respectively $\tau = \tau_S$) the maximum idempotent separating (respectively idempotent pure congruence) on S. Also, for any equivalence relation λ on S, we denote by λ^0 the largest congruence on S contained in λ. It is sometimes useful to remember that $\mu = \mathscr{H}^0$.

The term *variety* means variety of completely regular semigroups as algebras with multiplication and inversion. We use the following notation for various varieties:

$\mathscr{T}$ — one element semigroups,
$\mathscr{LZ}$ — left zero semigroups,
$\mathscr{RZ}$ — right zero semigroups,
$\mathscr{RB}$ — rectangular bands,
$\mathscr{S}$ — semilattices,
$\mathscr{NB}$ — normal bands,
$\mathscr{B}$ — bands,
$\mathscr{G}$ — groups,
$\mathscr{LG}$ — left groups,
$\mathscr{RG}$ — right groups,
$\mathscr{CS}$ — completely simple semigroups,

$\mathcal{NBG}$ — normal bands of groups,
$\mathcal{R}e\mathcal{G}$ — rectangular groups,
$\mathcal{O}$ — orthodox completely regular semigroups,
$\mathcal{CR}$ — completely regular semigroups.

Moreover,

$\mathcal{L}(\mathcal{V})$ — the lattice of subvarieties of $\mathcal{V}$,
$F\mathcal{V}$ — the (relatively) free completely regular semigroup on a countably infinite set in a variety $\mathcal{V}$,
$F = F\mathcal{CR}$,
F_n — the free completely regular semigroup on a set of n elements,
$\mathcal{C}$ — the lattice of fully invariant congruences on F.

For a set A of operators on $\mathcal{L}(\mathcal{CR})$, we denote by $[A]$ the subsemigroup of the full transformation semigroup on $\mathcal{L}(\mathcal{CR})$ generated by A. The free semigroup on a nonempty set X is denoted by X^+. A semigroup given by generators G and relations R is denoted by $\langle G \mid R\rangle$. For a semigroup S, S^1 (respectively S^0) stands for S with an identity (respectively zero) adjoined. On any set X, ε denotes the equality relation. Proper inclusion of sets is denoted by $\subset$. The notation $|X|$ stands for the cardinality of a set X, $|w|$ stands for the length of the word w, and $|n|$ also denotes the usual absolute value of an integer n.

Undefined terminology and notation can be found in **[5]** and **[11]**.

3. The operators C **and** L. In this section we introduce two operators on $\mathcal{L}(\mathcal{CR})$ and describe the semigroup that they generate.

For any $S \in \mathcal{L}(\mathcal{CR})$, let $C(S)$ denote the subsemigroup of S generated by the idempotents of S. Then $C(S)$ is a completely regular subsemigroup of S called the *core* of S. The operator C is defined in $\mathcal{L}(\mathcal{CR})$ by:

$$\mathcal{V}C = \{S \in \mathcal{CR} \mid C(S) \in \mathcal{V}\}.$$

It is routine to verify that $\mathcal{V}C$ is closed under products, homomorphic images and (completely regular) subsemigroups and is, therefore, a variety. Clearly $(\mathcal{V}C)C = \mathcal{V}C$ so that $C^2 = C$. The operator C appeared in a special case in (**[13]**, Proposition 3.5).

The operator L is defined on $\mathcal{L}(\mathcal{CR})$ as follows:

$$\mathcal{V}L = \{S \in \mathcal{CR} \mid eSe \in \mathcal{V} \text{ for all } e \in E(S)\}.$$

This operator was introduced in [3]; see ([3], Proposition 4.1) where it is shown that $\mathscr{V}L \in \mathscr{L}(\mathscr{C}\mathscr{R})$ and that $(\mathscr{V}L)L = \mathscr{V}L$ or $L^2 = L$. Its restriction to $\mathscr{L}(\mathscr{C}\mathscr{S})$ was considered in ([12], §4).

The calculation of the semigroup generated by C and L is quite simple as we shall now see.

LEMMA 3.1. *$LCL = CL$.*

Proof. For $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$ and $S \in \mathscr{V}LCL$, we successively obtain

$$\begin{aligned} eSe &\in \mathscr{V}LC && \text{for all } e \in E(S), \\ C(eSe) &\in \mathscr{V}L && \text{for all } e \in E(S), \\ fC(eSe)f &\in \mathscr{V} && \text{for all } e \in E(S) \text{ and } f \in E(eSe), \\ eC(eSe)e &\in \mathscr{V} && \text{for all } e \in E(S), \\ C(eSe) &\in \mathscr{V} && \text{for all } e \in E(S), \\ eSe &\in \mathscr{V}C && \text{for all } e \in E(S), \\ S &\in \mathscr{V}CL, \end{aligned}$$

so that $\mathscr{V}LCL \subseteq \mathscr{V}CL$; the opposite inclusion being trivial, we obtain the desired equality.

The next lemma is valid in any regular semigroup.

LEMMA 3.2. *If $e \in E(S)$ and S is a regular semigroup, then $C(eSe) = C(eC(S)e)$.*

Proof. Let $x \in C(eSe)$. Then $x = e_1 e_2 \cdots e_n$ for some $e_i \in E(eC(S)e)$ so that $x \in C(eC(S)e)$. This proves that $C(eSe) \subseteq C(eC(S)e)$; the opposite inclusion is obvious.

LEMMA 3.3. *$CLC = CL$.*

Proof. For $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$ and $S \in \mathscr{V}CLC$, we successively obtain

$$\begin{aligned} C(S) &\in \mathscr{V}CL && \\ eC(S)e &\in \mathscr{V}C && \text{for all } e \in E(S), \\ C(eC(S)e) &\in \mathscr{V} && \text{for all } e \in E(S), \\ C(eSe) &\in \mathscr{V} && \text{for all } e \in E(S) \text{ by Lemma 3.2}, \\ eSe &\in \mathscr{V}C && \text{for all } e \in E(S), \\ S &\in \mathscr{V}CL, \end{aligned}$$

so that $\mathscr{V}CLC \subseteq \mathscr{V}CL$; the opposite inclusion being trivial, the desired equality follows.

Applying C, L, LC and CL to the trivial variety $\mathcal{T}$, we see that they are all distinct. Then Lemmas 3.1 and 3.3 imply that $[C, L] = \{C, L, LC, CL\}$ with the multiplication table

	C	L	LC	CL
C	C	CL	CL	CL
L	LC	L	LC	CL
LC	LC	CL	CL	CL
CL	CL	CL	CL	CL

As a particular case of ([**3**], Theorem 2), we have the following useful observation.

LEMMA 3.4. *For $S \in \mathcal{CR}$, we have $C(S) = \bigcup_{x \in S} C(D_x)$.*

4. The kernel and trace operators. Of considerable importance in what follows is the anti-isomorphism from the lattice $\mathcal{L}(\mathcal{CR})$ of subvarieties of $\mathcal{CR}$ to the lattice $\mathcal{C}$ of fully invariant congruences on the free completely regular semigroup F on a countably infinite set of generators. We denote this correspondence by

$$\mathcal{V} \to \rho_{\mathcal{V}}, \quad \rho \to [\rho] \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}), \rho \in \mathcal{C}).$$

For any congruence ρ on S in $\mathcal{CR}$,

$$\ker \rho = \{x \in S \mid x \rho x^0\},$$

$$\operatorname{tr} \rho = \rho|_{E(S)},$$

$$\operatorname{ltr} \rho = (\rho \vee \mathcal{L})^0|_{E(S)},$$

$$\operatorname{rtr} \rho = (\rho \vee \mathcal{R})^0|_{E(S)},$$

and are called, respectively, the *kernel*, *trace*, *left trace* and *right trace* of ρ. Each of these objects determines an equivalence relation on the lattice $\operatorname{Con} S$:

$$\lambda K \rho \Leftrightarrow \ker \lambda = \ker \rho,$$

$$\lambda T \rho \Leftrightarrow \operatorname{tr} \lambda = \operatorname{tr} \rho,$$

$$\lambda T_l \rho \Leftrightarrow \operatorname{ltr} \lambda = \operatorname{ltr} \rho,$$

$$\lambda T_r \rho \Leftrightarrow \operatorname{rtr} \lambda = \operatorname{rtr} \rho.$$

The last two relations were introduced in ([**9**], §6).

Two fundamental observations on these relations are:

(i) ([**1**], Theorem 4.1). $K \cap T = \varepsilon$;

(ii) ([**9**], Theorem 6.12). $T_l \cap T_r = T$.

In what follows we will only be interested in the restrictions of these relations to $\mathscr{C}$. We will denote these restrictions by the same symbols K, T, T_l and T_r.

LEMMA 4.1. *The relations K, T, T_l and T_r are complete congruences on $\mathscr{C}$.*

Proof. See ([**8**], Theorem 11), ([**9**], Theorem 6.6).

If $\rho \in \mathscr{C}$, then it follows from Lemma 4.1 that each of the classes $\rho K, \rho T, \rho T_r$ and ρT_l has a minimum member which we shall denote by $\rho_K, \rho_T, \rho_{T_l}$ and ρ_{T_r}, respectively. We can repeat this process to obtain a network of congruences $\rho_{KT}, \rho_{KTK}, \rho_{KT_l}, \rho_{T_lT_r}$ etc.

In combination with the duality between varieties and fully invariant congruences, this enables us to introduce four operators on $\mathscr{L}(\mathscr{CR})$ defined as follows: for any $\rho \in \mathscr{C}$,

$$[\rho]K = [\rho_K], \quad [\rho]T = [\rho_T], \quad [\rho]T_l = [\rho_{T_l}], \quad [\rho]T_r = [\rho_{T_r}].$$

Clearly K, T, T_l and T_r are closure operators. In particular,

$$K^2 = K, \quad T^2 = T, \quad T_l^2 = T_l, \quad T_r^2 = T_r.$$

In order to work with these operators effectively, it is necessary to have alternative descriptions of $\mathscr{V}K$, $\mathscr{V}T$ etc. Toward this end, the following concept will prove useful.

Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$. The *Mal'cev product* $\mathscr{U} \circ \mathscr{V}$ of $\mathscr{U}$ and $\mathscr{V}$ is defined by

$$\mathscr{U} \circ \mathscr{V} = \{ S \in \mathscr{CR} \mid \text{there exists } \rho \in \operatorname{Con} S \text{ such that (i) if } (x\rho)^2 = x\rho, \text{ then } x\rho \in \mathscr{U} \text{ and (ii) } S/\rho \in \mathscr{V} \}.$$

In general $\mathscr{U} \circ \mathscr{V}$ need not be a variety ([7], Theorem 3.1).

LEMMA 4.2. ([7], Theorems 4.1 and 5.1).

(i) *If $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CS})$, then $\mathscr{U} \circ \mathscr{V} \in \mathscr{L}(\mathscr{CS})$.*

(ii) *If $\mathscr{U} \in \mathscr{L}(\mathscr{RB} \vee \mathscr{G})$ and $\mathscr{S} \subseteq \mathscr{V}$, then $\mathscr{U} \circ \mathscr{V} \in \mathscr{L}(\mathscr{CR})$.*

LEMMA 4.3. *Let $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$.*

(i) $\mathscr{V}K$

$$\begin{aligned} &= \mathscr{RB} \circ (\mathscr{V} \vee \mathscr{S}) = \mathscr{B} \circ (\mathscr{V} \vee \mathscr{S}) \\ &= \{ S \in \mathscr{CR} \mid S/\tau \in \mathscr{V} \vee \mathscr{S} \} \\ &= \{ S \in \mathscr{CR} \mid S/(\tau \cap \mathscr{D}) \in \mathscr{V} \vee \mathscr{S} \}. \end{aligned}$$

(ii) $\mathscr{V}T = \mathscr{G} \circ \mathscr{V} = \{S \in \mathscr{CR} \mid S/\mu \in \mathscr{V}\}$.
(iii) $\mathscr{V}T_l = \mathscr{L}\mathscr{G} \circ \mathscr{V} = \{S \in \mathscr{CR} \mid S/\mathscr{L}^0 \in \mathscr{V}\}$.
(iv) $\mathscr{V}T_r = \mathscr{R}\mathscr{G} \circ \mathscr{V} = \{S \in \mathscr{CR} \mid S/\mathscr{R}^0 \in \mathscr{V}\}$.

Proof. See ([7], Proposition 6.1 and Lemma 3.2) and ([**9**], Theorem 6.3).

THEOREM 4.4. *In F, we have*

$$\tau = \mu = \mathscr{L}^0 = \mathscr{R}^0 = \varepsilon.$$

Proof. It was established in ([**10**]), Lemma 5.11) that $\tau = \mu = \varepsilon$. Now suppose that $a, b \in F$ and $a\mathscr{L}^0 b$. Let $T = \{x_s \mid s \in F^1\}$ and define a multiplication on $S = F \cup T$ by

$$uv = \begin{cases} uv & \text{if } u, v \in F, \\ x_s & \text{if } v = x_s \in T, \\ x_{sv} & \text{if } u = x_s \in T, v \in F. \end{cases}$$

Then $S \in \mathscr{CR}$. Since S is countably infinite, it follows that there exists an epimorphism $\theta: F \to S$. Moreover, θ can be chosen to map the variables in a and b identically so that, in particular, $a\theta = a$, $b\theta = b$. Since S is a homomorphic image of F, we have $a\mathscr{L}^0 b$ in S. Hence

$$x_a = x_{1a} = x_1 a \mathscr{L} x_1 b = x_{1b} = x_b.$$

But T is a right zero semigroup. Therefore $x_a = x_b$ and $a = b$. Thus $\mathscr{L}^0 = \varepsilon$ on F and, by duality, $\mathscr{R}^0 = \varepsilon$.

COROLLARY 4.5. *If $\mathscr{V}$ is a proper subvariety of $\mathscr{CR}$, then so are $\mathscr{V}K, \mathscr{V}T, \mathscr{V}T_l$ and $\mathscr{V}T_r$.*

Proof. For example, consider $\mathscr{V}T_r$. If $\mathscr{V}T_r = \mathscr{CR}$, then $F \in \mathscr{V}T_r$ and so, by Lemma 4.3(iv), $F/\mathscr{R}^0 \in \mathscr{V}$. By Theorem 4.4, $\mathscr{R}^0$ is the identity congruence. Thus $F \in \mathscr{V}$ and $\mathscr{V} = \mathscr{CR}$.

The next result explains the local and global behaviour of these operators.

THEOREM 4.6. *Let $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, $\mathscr{V} \neq \mathscr{CR}$.*
(i) $\mathscr{V} \subset \mathscr{V}KT \subset \mathscr{V}(KT)^2 \subset \cdots \subset \mathscr{CR}$ *and* $\bigvee \mathscr{V}(KT)^n = \mathscr{CR}$.
(ii) $\mathscr{V} \subset \mathscr{V}KT_l \subset \mathscr{V}(KT_l)^2 \subset \cdots \subset \mathscr{CR}$ *and* $\bigvee \mathscr{V}(KT_l)^n = \mathscr{CR}$.
(iii) *If* $\mathscr{S} \subseteq \mathscr{V}$, *then* $\mathscr{V} \subset \mathscr{V}T_lT_r \subset \mathscr{V}(T_lT_r)^2 \subset \cdots \subset \mathscr{CR}$ *and* $\bigvee \mathscr{V}(T_lT_r)^n = \mathscr{CR}$.

Proof. It follows immediately from Corollary 4.5 that $\mathscr{V}(KT)^n$, $\mathscr{V}(KT_l)^n$ and $\mathscr{V}(T_lT_r)^n$ are proper subvarieties of $\mathscr{CR}$ for all positive integers n. The remainder of part (i) can be found in ([**10**]), Corollary 5.9). From Lemma 4.3(ii) and (iii), it is clear that $\mathscr{V}(KT)^n \subseteq \mathscr{V}(KT_l)^n$ for all positive integers n, and so it follows from (i) that $\mathscr{CR} = \bigvee \mathscr{V}(KT_l)^n$. Now, if $\mathscr{V}(KT_l)^m = \mathscr{V}(KT_l)^{m+1}$ for some m, then clearly $\mathscr{V}(KT_l)^m = \mathscr{V}(KT_l)^k$ for all $k \geq m$. But this would imply that $\mathscr{CR} = \bigvee \mathscr{V}(KT_l)^n = \mathscr{V}(KT_l)^m$, which would contradict the fact that $\mathscr{V}(KT_l)^n$ is a proper subvariety of $\mathscr{CR}$ for all n. Thus (ii) also holds. Now consider (iii).

Let $\mathscr{S} \subseteq \mathscr{V}$. In order to establish the last assertion, it suffices to show that $F_k \in \mathscr{S}(T_lT_r)^k$ for all positive integers k, where F_k is the free completely regular semigroup on k generators.

It is clear from Lemma 4.3(iii) that

$$\mathscr{G} \subseteq \mathscr{L}\mathscr{G} \circ \mathscr{S} = \mathscr{S}T_l \subseteq \mathscr{S}T_lT_r$$

so that $F_k \in \mathscr{S}(T_lT_r)^k$ holds for $k = 1$. Now suppose that $F_k \in \mathscr{S}(T_lT_r)^k$ and consider F_{k+1}.

Let K be the minimum component of F_{k+1} and define ρ on F_{k+1} by

$$a\rho b \Leftrightarrow \begin{cases} a, b \in K \text{ and } a\mathscr{R}b \\ a = b \quad \text{otherwise.} \end{cases}$$

Then ρ is a congruence on F_{k+1} and $\rho \subseteq \mathscr{R}$. Now let σ be the Rees congruence on F_{k+1}/ρ determined by the ideal K/ρ of F_{k+1}/ρ. Since K/ρ is a single $\mathscr{L}$-class of F_{k+1}/ρ, we have $\sigma \subseteq \mathscr{L}$. But $(F_{k+1}/\rho)/\sigma \cong F_{k+1}/K$, the Rees quotient for F_{k+1} determined by the ideal K.

For each component D of F_{k+1}/K covering the zero component, let $\overline{D}$ be the union of all components above it. On F_{k+1}/K define a relation ρ_D by

$$a\rho_D b \Leftrightarrow a = b \quad \text{or} \quad a, b \notin \overline{D}.$$

Then ρ_D is a congruence on F_{k+1}/K and $(F_{k+1}/K)/\rho_D \cong F_k^0$.

In addition, the intersection of all congruences ρ_D is the equality relation. We conclude that F_{k+1}/K is a subdirect product of copies of F_k^0. Since F_k, and therefore F_k^0, lies in $\mathscr{S}(T_lT_r)^k$, it follows that $F_{k+1}/K \in \mathscr{S}(T_lT_r)^k$. Thus $F_{k+1}/\rho \in \mathscr{S}(T_lT_r)^kT_l$ and $F_{k+1} \in \mathscr{S}(T_lT_r)^kT_lT_r = \mathscr{S}(T_lT_r)^{k+1}$, as required. Hence $\bigvee \mathscr{V}(T_lT_r)^n = \mathscr{CR}$. It now follows, as in part (ii), that $\mathscr{V}(T_lT_r)^n \subset \mathscr{V}(T_lT_r)^{n+1}$ for all positive integers n.

In Theorem 4.6, it is shown that $\mathscr{CR}$ is the join of an infinite family of proper subvarieties. In contrast, we have the following result.

THEOREM 4.7. *The variety $\mathscr{CR}$ is finitely join irreducible.*

Proof. Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR}), \mathscr{U} \neq \mathscr{CR}, \mathscr{V} \neq \mathscr{CR}$. Then there must exist a positive integer n such that $F_n \notin \mathscr{U} \cup \mathscr{V}$. Suppose that $F_n \in \mathscr{U} \vee \mathscr{V}$.

Then there exist $A \in \mathscr{U}, B \in \mathscr{V}$, a subdirect product R of A and B and an epimorphism $\theta : R \to F_n$. Since F_n is the free completely regular semigroup on n generators, there exists a homomorphism $\varphi : F_n \to R$ such that $\iota = \varphi\theta$, where ι is the identity mapping on F_n:

$$\begin{array}{ccc} F_n & \xrightarrow{\varphi} & R \leq A \times B \\ & \searrow^{\iota} & \downarrow \theta \\ & & F_n \end{array}$$

It follows that φ is a monomorphism and we may consider F_n as a subsemigroup of $A \times B$. Let π_A and π_B denote the projections of $A \times B$ onto A and B, respectively, and let $\rho = \pi_A \circ \pi_A^{-1}$, $\sigma = \pi_B \circ \pi_B^{-1}$. Then $\rho \cap \sigma = \varepsilon$. Since $F_n \notin \mathscr{U} \cup \mathscr{V}$, ρ and σ are both non-trivial congruences.

Let K be the minimum component of F_n and κ be the Rees congruence on F_n determined by the ideal K. By the remark prior to Theorem 5.12 in **[9]**, there are no idempotent pure congruences in F_n.

Hence, there exists $x \in F_n \setminus E(F_n)$ such that $x\rho x^0$. We prove next that there exists $k \in K$ such that $xk \neq x^0k$. If $x \in K$, take $k = x^0$. It remains to consider the case $x \notin K$. To this end, it suffices to construct a completely regular semigroup S with an epimorphism $\theta : F_n \to S$ such that $(xk)\theta \neq (x^0k)\theta$. For this, let $S = F_{n-1} \cup \{y_r \mid r \in F_n^1\}$ with the multiplication

$$ab = \begin{cases} ab & \text{if } a, b \in F_{n-1}, \\ y_r & \text{if } a = y_r, \\ y_{ar} & \text{if } b = y_r. \end{cases}$$

Then $S \in \mathscr{C}\mathscr{R}$ and

$$xy_1 = y_{x1} = y_x \neq y_{x^0} = y_{x^01} = x^0y_1. \tag{2}$$

Let $x_1, x_2, \ldots, x_n$ be the free generators of F_n. Define a mapping θ by

$$\theta : x_i \to x_i \quad \text{for } i = 1, 2, \ldots, n-1, \quad x_n \to y_1,$$

and extend it to a homomorphism of F_n into S. Letting

$$k = x_n(x_1 \cdots x_{n-1})^0,$$

we obtain $k \in K$ and $k\theta = y_1(x_1 \cdots x_{n-1})^0 = y_1$. Consequently, (2) gives $(xk)\theta \neq (x^0k)\theta$ and thus $xk \neq x^0k$. Since $xk\rho x^0k$, it follows that $\rho_1 = \rho \cap \kappa$ is not the equality relation. Again by **[9]**, ρ_1 is not idempotent pure. Therefore, there must be an element $a \in K \setminus E(K)$ with $a\rho_1 a^0$.

Let σ_1 be similarly derived from σ. Then there is an element $b \in K \setminus E(K)$ with $b\sigma_1 b^0$. Since K is completely simple, we may assume that $a\mathscr{H}b$. The restrictions of ρ_1 and σ_1 to H_a, say ρ_2 and σ_2, are then both non-trivial. But H_x is a free group ([**2**], Theorem 5.7) and so $\rho_2 \cap \sigma_2$ is also non-trivial. Hence $\rho_1 \cap \sigma_1$ is non-trivial and so also $\rho \cap \sigma$, which is a contradiction.

5. C commutes with K, T, T_l and T_r. In order to prove the statement in the title, we first consider how idempotent pure congruences restrict to the core.

LEMMA 5.1. *Let $S \in \mathscr{CR}$ and $C = C(S)$. Then*

$$(\tau_S \cap \mathscr{D})|_C = \tau_C \cap \mathscr{D}.$$

Proof. Let $\tau = (\tau_S \cap \mathscr{D})|_C$ and $\sigma = \tau_C \cap \mathscr{D}$. Clearly $\tau \subseteq \sigma$. Since σ and τ are idempotent pure, it suffices to show that they have the same trace. So let $e, f \in E(S) = E$ and $e\sigma f$. Let $x, y \in S^1$ be such that $xey \in E$. Then

$$(eyx)^2 = ey(xey)x = ey(xey)^2x = (eyx)^3$$

so that, since eyx lies in a subgroup, $eyx \in E$. Thus $e(eyx) \in E$ and so, since $e\sigma f$ and $eyx \in C$, we must have $f(eyx) \in E$. Hence

$$(yxfe)^3 = yx(feyx)(feyx)fe = yx(feyx)fe = (yxfe)^2$$

and $yxfe \in E$. Thus $(yxfe)e \in E$ and, again since $e\sigma f$ and $yxfe \in C$, we obtain $yxfef \in E$. But $e\mathscr{D}f$ so that $f\mathscr{H}fef$. Since $e\tau_C f$ and τ_C restricted to any $\mathscr{H}$-class is trivial we get $fef = f$. Therefore $yxf \in E$ and

$$(xfy)^3 = xf(yxf)(yxf)y = xf(yxf)y = (xfy)^2.$$

Thus $xfy \in E$. By symmetry, we have that, for all $x, y \in S^1$,

$$xey \in E \Leftrightarrow xfy \in E$$

and therefore $e\tau f$. Hence $\sigma = \tau$, as required.

LEMMA 5.2. *Let $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$. Then $(\mathscr{V} \vee \mathscr{S})C = \mathscr{V}C \vee \mathscr{S}$.*

Proof. The claim is trivial if $\mathscr{S} \subseteq \mathscr{V}$. So suppose that $\mathscr{S} \nsubseteq \mathscr{V}$ so that $\mathscr{V} \subseteq \mathscr{CS}$.

We clearly have $\mathscr{V}C \vee \mathscr{S} \subseteq (\mathscr{V} \vee \mathscr{S})C$. So now let $S \in (\mathscr{V} \vee \mathscr{S})C$. Then $C(S) \in \mathscr{V} \vee \mathscr{S}$ and thus $C(S)$ is a normal band of groups. By ([**11**], IV.4.3), normal bands of groups can be characterized by the behaviour of their idempotents relative to their components. Since this pertains only to idempotents, it follows that S itself is a normal band of groups. If D is a component of S, then by Lemma 3.4 and ([**6**], Theorem 3.3), we obtain

$$C(D) = C(S) \cap D \in (\mathscr{V} \vee \mathscr{S}) \cap \mathscr{C}\mathscr{S} = (\mathscr{V} \cap \mathscr{C}\mathscr{S}) \vee (\mathscr{S} \cap \mathscr{C}\mathscr{S}) = \mathscr{V}.$$

Therefore $D \in \mathscr{V}C$ for each component D of S which together with S being a normal band of groups and the proof of ([**11**], IV.4.3), yields that S is a subdirect product of its components with a zero possibly adjoined. Since each of these, by the above, is contained in $\mathscr{V}C \vee \mathscr{S}$, we conclude that $S \in \mathscr{V}C \vee \mathscr{S}$.

We can now show that C commutes with K.

LEMMA 5.3. $KC = CK$.

Proof. First let $S \in \mathscr{C}\mathscr{R}$ and define a mapping χ by

$$\chi : c(\tau_S \cap \mathscr{D}) \to c(\tau_S \cap \mathscr{D}) \cap C(S) \qquad (c \in C(S)).$$

Lemma 5.1 asserts that $(\tau_S \cap \mathscr{D})|_{C(S)} = \tau_{C(S)} \cap \mathscr{D}$ which then implies that χ is a bijection of $C(S/(\tau_S \cap \mathscr{D}))$ onto $C(S)/(\tau_{C(S)} \cap \mathscr{D})$ since every homomorphism maps the core onto the core. It now follows by Lemma 5.1 that χ is also a homomorphism. Therefore

$$C(S/(\tau_S \cap \mathscr{D})) \cong C(S)/(\tau_{C(S)} \cap \mathscr{D}). \tag{3}$$

For $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$, we obtain

$$\begin{aligned}
S \in \mathscr{V}KC &\Leftrightarrow C(S) \in \mathscr{V}K \\
&\Leftrightarrow C(S)/(\tau_{C(S)} \cap \mathscr{D}) \in \mathscr{V} \vee \mathscr{S} && \text{by Lemma 4.3} \\
&\Leftrightarrow C(S/(\tau_S \cap \mathscr{D})) \in \mathscr{V} \vee \mathscr{S} && \text{by (3)} \\
&\Leftrightarrow S/(\tau_S \cap \mathscr{D}) \in (\mathscr{V} \vee \mathscr{S})C \\
&\Leftrightarrow S/(\tau_S \cap \mathscr{D}) \in \mathscr{V}C \vee \mathscr{S} && \text{by Lemma 5.2} \\
&\Leftrightarrow S \in \mathscr{V}CK && \text{by Lemma 4.3.}
\end{aligned}$$

Thus $\mathscr{V}KC = \mathscr{V}CK$, as required.

LEMMA 5.4. *Let* $S \in \mathscr{C}\mathscr{R}$ *and* $C = C(S)$. *Then*

$$\mathscr{L}^0|_C = (\mathscr{L}_C)^0, \quad \mathscr{R}^0|_C = (\mathscr{R}_C)^0, \quad \mu|_C = \mu_C.$$

Proof. Consider $\mathscr{L}^0$. Clearly $\mathscr{L}^0|_C \subseteq (\mathscr{L}_C)^0$. So let $a, b \in C$, $a\mathscr{L}_C^0 b$ and $x, y \in S^1$. Then $x^0ay^0\mathscr{L}_C x^0by^0$ since $a\mathscr{L}_C^0 b$, and $x^0ay\mathscr{L}x^0by$ since $\mathscr{L}$ is a right congruence. But clearly

$$xay\mathscr{L}x^0ay, \quad xby\mathscr{L}x^0by$$

so that $xay\mathscr{L}xby$. Thus $a(\mathscr{L}^0|_C)b$ and $\mathscr{L}^0|_C = (\mathscr{L}_C)^0$. Similarly for $\mathscr{R}$ and $\mu = \mathscr{H}^0 = \mathscr{R}^0 \cap \mathscr{L}^0$.

LEMMA 5.5. $TC = CT$, $T_lC = CT_l$, $T_rC = CT_r$.

Proof. Consider T_l. From Lemma 5.4, it follows that

$$C(S/\mathscr{L}^0) \cong C(S)/\mathscr{L}^0_{C(S)}.$$

For any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, we then obtain

$$S \in \mathscr{V}T_lC \Leftrightarrow C(S) \in \mathscr{V}T_l \Leftrightarrow C(S)/\mathscr{L}^0_{C(S)} \in \mathscr{V}$$
$$\Leftrightarrow C(S/\mathscr{L}^0) \in \mathscr{V} \Leftrightarrow S/\mathscr{L}^0 \in \mathscr{V}C \Leftrightarrow S \in \mathscr{V}CT_l.$$

Similarly for T_r and T.

6. L commutes with K, T, T_l and T_r. In order to prove the statement in the title, we require information on the restriction of congruences to certain submonoids.

LEMMA 6.1. *For any $S \in \mathscr{CR}$, $\tau_S|_{eSe} = \tau_{eSe}$ for all $e \in E(S)$.*

Proof. Let $e \in E(S)$, $\sigma = \tau_S|_{eSe}$ and $\tau = \tau_{eSe}$. Clearly $\sigma \subseteq \tau$. For the opposite inclusion, it suffices to show that $\operatorname{tr}\tau \subseteq \operatorname{tr}\sigma$. Hence let $f, g \in E = E(eSe)$ with $f\tau g$. Let $x, y \in S^1$ and assume that $xfy \in E(S)$. Then

$$[f(eyxe)]^3 = fey(xefey)^2xe = fey(xfy)^2xe$$
$$= fey(xfy)xe = (feyxe)^2$$

which implies that $f(eyxe) \in E$, since $f(eyxe)$ is contained in a subgroup of eSe. The hypothesis implies that $g(eyxe) \in E(S)$ whence

$$(xgy)^3 = x(geyxe)^2gy = x(geyxe)gy = (xgy)^2$$

so that $xgy \in E(S)$. By symmetry, we deduce that $f\sigma g$. Consequently $\sigma = \tau$, as required.

LEMMA 6.2. *Let $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$. Then $(\mathscr{V} \vee \mathscr{S})L = \mathscr{V}L \vee \mathscr{S}$.*

Proof. The claim is trivial if $\mathscr{S} \subseteq \mathscr{V}$. So suppose that $\mathscr{V} \subseteq \mathscr{CS}$. It is clear that $\mathscr{V}L \vee \mathscr{S} \subseteq (\mathscr{V} \vee \mathscr{S})L$.

Let $S \in (\mathscr{V} \vee \mathscr{S})L$. Then, for all $e \in E(S)$, $eSe \in \mathscr{V} \vee \mathscr{S}$. It follows that eSe is a normal band of groups. By ([**11**], IV.4.3), normal bands of groups can be characterized by the behaviour of their idempotents relative to the subsets eSe as e runs over all idempotents of S. It follows from this reference that since eSe is a normal band of groups for all $e \in E(S)$, we have that S itself is a normal band of groups. Moreover, for any $e \in E(S)$, using ([**6**], Theorem 3.3), we obtain

$$eD_ee \in (\mathscr{V} \vee \mathscr{S}) \cap \mathscr{CS} = (\mathscr{V} \cap \mathscr{CS}) \vee (\mathscr{S} \cap \mathscr{CS}) = \mathscr{V},$$

and thus $D_e \in \mathscr{V}L$. The proof of ([**11**], IV.4.3) implies that S is a subdirect product of the semigroups D_e with a zero possibly adjoined. Since each of these, by the above, is contained in $\mathscr{V}L \vee \mathscr{S}$, we conclude that $S \in \mathscr{V}L \vee \mathscr{S}$.

LEMMA 6.3. $KL = LK$.

Proof. First let $S \in \mathscr{C}\mathscr{R}$ and $e \in E(S)$. Define a mapping χ by

$$\chi : a\tau \to a\tau \cap eSe \qquad (a \in eSe).$$

Lemma 6.1 asserts that $\tau|_{eSe} = \tau_{eSe}$ which implies that χ is a bijection of $(e\tau)(S/\tau)(e\tau)$ onto $(eSe)/\tau_{eSe}$. It now follows, by Lemma 6.1, that χ is also a homomorphism. Therefore

$$(e\tau)(S/\tau)(e\tau) \cong (eSe)/\tau_{eSe}. \tag{4}$$

For $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$, we obtain

$$\begin{aligned} S \in \mathscr{V}KL &\Leftrightarrow eSe \in \mathscr{V}K \text{ for all } e \in E(S) \\ &\Leftrightarrow (eSe)/\tau_{eSe} \in \mathscr{V} \vee \mathscr{S} \text{ for all } e \in E(S) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 4.3(i)} \\ &\Leftrightarrow (e\tau)(S/\tau)(e\tau) \in \mathscr{V} \vee \mathscr{S} \text{ for all } e \in E(S) \quad \text{by (4)} \\ &\Leftrightarrow S/\tau \in (\mathscr{V} \vee \mathscr{S})L \\ &\Leftrightarrow S/\tau \in \mathscr{V}L \vee \mathscr{S} \qquad\qquad \text{by Lemma 6.2} \\ &\Leftrightarrow S \in \mathscr{V}LK \qquad\qquad \text{by Lemma 4.3(i).} \end{aligned}$$

Thus $\mathscr{V}KL = \mathscr{V}LK$ and the proof is complete.

We now consider the interaction of L and T, T_l and T_r.

LEMMA 6.4. *Let* $S \in \mathscr{C}\mathscr{R}$ *and* $e \in E(S)$. *Then*

$$\mu|_{eSe} = \mu_{eSe}, \quad \mathscr{L}^0|_{eSe} = (\mathscr{L}_{eSe})^0, \quad \mathscr{R}^0|_{eSe} = (\mathscr{R}_{eSe})^0.$$

Proof. Consider $\mathscr{L}^0$. Clearly $\mathscr{L}^0|_{eSe} \subseteq (\mathscr{L}_{eSe})^0$. So let $a, b \in eSe$, $a(\mathscr{L}_{eSe})^0 b$ and $x, y \in S$. First note that, for any $w \in S$,

$$\begin{aligned} (exe)^0(xe)^0(exe)w &= (exe)^{-1}(exe)(xe)^0 exew \\ &= (exe)^{-1}(exe)(exe)w = (exe)w \end{aligned}$$

so that $(exe)w\mathscr{L}(xe)^0(exe)w$. Also

$$a(\mathscr{L}_{eSe})^0 b \Rightarrow (exe)a(eye)\mathscr{L}_{eSe}(exe)b(eye)$$

$$\Rightarrow (xe)^0(exe)a(eye)\mathscr{L}(xe)^0(exe)b(eye) \quad \text{by the above remarks}$$

$$\Rightarrow (xe)^0(exe)a(eye)(ey)^0\mathscr{L}(xe)^0(exe)b(eye)(ey)^0$$

$$\text{since } \mathscr{L} \text{ is a right congruence}$$

$$\Rightarrow (xe)^0 xeaey(ey)^0\mathscr{L}(xe)^0 xebey(ey)^0$$

$$\Rightarrow xeaey\mathscr{L}xebey$$

$$\Rightarrow xay\mathscr{L}xby.$$

Thus $a\mathscr{L}^0 b$. The case of $\mathscr{R}$ is symmetric and the case of μ follows from these two cases.

LEMMA 6.5. $TL = LT,\ T_l L = LT_l,\ T_r L = LT_r$.

Proof. Consider T_l. From Lemma 6.4 it follows that

$$(e\mathscr{L}^0)(S/\mathscr{L}^0)(e\mathscr{L}^0) \cong (eSe)/\mathscr{L}^0_{eSe}.$$

Hence for any $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$, we obtain

$$S \in \mathscr{V}T_l L \Leftrightarrow eSe \in \mathscr{V}T_l \quad \text{for all } e \in E(S)$$

$$\Leftrightarrow eSe/\mathscr{L}^0_{eSe} \in \mathscr{V} \quad \text{for all } e \in E(S)$$

$$\Leftrightarrow (e\mathscr{L}^0)(S/\mathscr{L}^0)(e\mathscr{L}^0) \in \mathscr{V} \quad \text{for all } e \in E(S)$$

$$\Leftrightarrow S/\mathscr{L}^0 \in \mathscr{V}L \Leftrightarrow S \in \mathscr{V}LT_l.$$

Thus $T_l L = LT_l$. That $T_r L = LT_r$ and $TL = LT$ follows similarly.

7. Semigroups generated by any two of K, T, T_l, T_r. In this section we deal with some technical preparations for the determination of various semigroups of operators in terms of generators and relations. We shall require some additional notation.

For reasons that will soon be apparent, we will be interested mostly in the free semigroup on $\{c, l, k, t, t_l, t_r\}$. An element $w = a_1 a_2 \cdots a_n \in A^+$ will be called *distinguished* if $a_i \neq a_{i+1}$ for $i = 1, \dots, n-1$. The set of distinguished elements in $\{a, b, c, \dots\}^+$ will be denoted by $\Delta_{abc\cdots}$.

If $v \in A^+$ and we wish to emphasize the variables $x_1, \dots, x_n \in A$ that appear in w, that we shall write $v(x_1, \dots, x_n)$. If S is a semigroup and

$s_1, \ldots, s_n \in S$, then we shall denote by $v(s_1, \ldots, s_n)$ the element obtained from v by substituting s_i for x_i, $i = 1, \ldots, n$.

In keeping with the notation ρ_K and ρ_T already introduced for the minimum congruence in the kernel and trace class of any congruence ρ on a completely regular semigroup S, we will now denote by ρ^K the maximum congruence in the kernel class of ρ; that is, ρ^K denotes the largest congruence on S with the same kernel as ρ. That ρ^K exists follows from Lemma 4.1 and basic information about ρ^K can be found in [**9**]. (Of course there is the dual concept of the maximum congruence ρ^T in the trace class of ρ, but we shall not require that here.)

The variety $\mathcal{R}g\mathcal{B}$ of *regular bands*, that is, the variety of bands defined by the identity $axya = axaya$, has a role to play in the next two results.

PROPOSITION 7.1 (*Kadourek* [**8**], *Proposition* 8.2). *For any* $\rho \in \mathcal{C}$, with $\rho \subseteq \rho_{\mathcal{R}g\mathcal{B}}$, $(\rho_T)^K = \rho_T$.

LEMMA 7.2. *Let* $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{C}\mathcal{R})$, $\mathcal{R}g\mathcal{B} \subseteq \mathcal{V}$ *and* $\mathcal{V} \not\subseteq \mathcal{U}$. *Then* $\mathcal{V}T \not\subseteq \mathcal{U}K$, $\mathcal{V}T_l \not\subseteq \mathcal{U}K$ *and* $\mathcal{V}T_r \not\subseteq \mathcal{U}K$.

Proof. Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{C}\mathcal{R})$ be such that $\mathcal{R}g\mathcal{B} \subseteq \mathcal{V}$, $\mathcal{V} \not\subseteq \mathcal{U}$ and suppose that $\mathcal{V}T \subseteq \mathcal{U}K$. Let $\rho = \rho_{\mathcal{U}}$ and $\sigma = \rho_{\mathcal{V}}$. Then $\sigma \subseteq \rho_{\mathcal{R}g\mathcal{B}}$ and $\rho_K \subseteq \sigma_T$. By ([**10**], Theorem 4.5) the mapping $\rho \to \rho^K$ ($\rho \in \mathcal{C}$) is order preserving. Hence

$$\begin{aligned} \rho_K \subseteq \sigma_T &\Rightarrow \rho^K = (\rho_K)^K \subseteq (\sigma_T)^K \\ &\Rightarrow \rho^K \subseteq (\sigma_T)^K = \sigma_T \quad \text{by Proposition 7.1} \\ &\Rightarrow \rho \subseteq \rho^K \subseteq \sigma_T \subseteq \sigma \\ &\Rightarrow \mathcal{V} \subseteq \mathcal{U} \end{aligned}$$

a contradiction. This establishes the first claim. The claims that $\mathcal{V}T_l \not\subseteq \mathcal{U}K$ and $\mathcal{V}T_r \not\subseteq \mathcal{U}K$ now follow from the fact that $\mathcal{V}T = \mathcal{V}T_l \cap \mathcal{V}T_r$.

LEMMA 7.3. *Let* $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{C}\mathcal{R})$ *and* $\mathcal{S} \subseteq \mathcal{V}, \mathcal{V} \not\subseteq \mathcal{U}$. *Then* $\mathcal{V}K \not\subseteq \mathcal{U}T$, $\mathcal{V}K \not\subseteq \mathcal{U}T_l$ *and* $\mathcal{V}K \not\subseteq \mathcal{U}T_r$.

Proof. Let $S \in \mathcal{V} \setminus \mathcal{U}$. Then $S^0 \in \mathcal{V} \setminus \mathcal{U}$ since $\mathcal{S} \subseteq \mathcal{V}$. Let $X = \{x_s \mid s \in S^1\}$ and define a multiplication $*$ on $R = S \cup X$ by

$$a * b = \begin{cases} ab & \text{if } a, b \in S, \\ x_s & \text{if } b = x_s \in X, \\ x_{sb} & \text{if } a = x_s \in X. \end{cases}$$

Then R is an ideal extension of the right zero semigroup X by S^0. Therefore $R \in \mathscr{V}K \setminus \mathscr{U}$.

However, suppose that $a, b \in S$ and $a\mathscr{L}^0 b$. Then $x_a = x_1 a \mathscr{L} x_1 b = x_b$. But X is a right zero subsemigroup of R. Therefore $x_a = x_b$ and $a = b$. Similarly $\mathscr{L}^0$ must be the equality on X and therefore on R. Thus $R/\mathscr{L}^0 \cong R \notin \mathscr{U}$ so that $R \notin \mathscr{U}T_l$. Hence $\mathscr{V}K \notin \mathscr{U}T_l$. Since $\mathscr{U}T \subseteq \mathscr{U}T_l$, it follows that $\mathscr{V}K \not\subseteq \mathscr{U}T$ and, by duality, $\mathscr{V}K \not\subseteq \mathscr{U}T_r$.

LEMMA 7.4. *Let* $\mathscr{U} \in \mathscr{L}(\mathscr{CR}), \mathscr{U} \neq \mathscr{CR}, v, w \in \Delta_{kt}, V = v(K,T), W = w(K,T)$ *and* $\mathscr{U}V = \mathscr{U}W$. *Then either* (i) $v = w$ *or* (ii) v *and* w *end in the same letter and differ in length by at most one.*

Proof. Let $\mathscr{W} = \mathscr{U}V = \mathscr{U}W$. By Corollary 4.5, $\mathscr{W} \neq \mathscr{CR}$. Since $v, w \in \Delta_{kt}$, v and w are both of length at least one. First suppose that v and w end in different letters. Without loss of generality, we can assume that $v = v_1 t$ and $w = w_1 k$ for some $v_1, w_1 \in \Delta_{kt} \cup \{\varnothing\}$. Letting $V_1 = v_1(K,T)$ and $W_1 = w_1(K,T)$, we have

$$\mathscr{W}T = \mathscr{U}VT = \mathscr{U}V_1TT = \mathscr{U}V_1T = \mathscr{U}V = \mathscr{W},$$
$$\mathscr{W}K = \mathscr{U}WK = \mathscr{U}W_1KK = \mathscr{U}W_1K = \mathscr{U}W = \mathscr{W},$$

which, since $\mathscr{U} \neq \mathscr{CR}$, contradicts Theorem 4.6(i). Therefore v and w must end in the same letter. If $|v| = |w|$, then since v and w are both distinguished elements (so that occurrences of t and k alternate) we must have $v = w$.

So suppose that $|v| \neq |w|$ and that v and w both end in k (a similar argument will handle the case where both end in t). Without loss of generality, we may assume that v is the shorter word. Suppose that $|v| + 2 \leq |w|$. Since v and w both end in k and are both distinguished elements of $\{k, t\}^+$, there must exist an element $u \in \Delta_{kt} \cup \{\varnothing\}$ such that $w = uvtk$. Thus

$$\mathscr{W} \subseteq \mathscr{W}TK = \mathscr{U}VTK \subseteq \mathscr{U}u(K,T)VTK = \mathscr{U}W = \mathscr{W}$$

so that $\mathscr{W} = \mathscr{W}TK$. Since $\mathscr{W} \neq \mathscr{CR}$, this contradicts Theorem 4.6(i). Thus v and w can differ in length by at most one.

LEMMA 7.5. *Let* $\mathscr{U} \in \mathscr{L}(\mathscr{CR}), \mathscr{NB} \subseteq \mathscr{U}, \mathscr{U} \neq \mathscr{CR}, \mathscr{U} \neq \mathscr{U}K, \mathscr{U} \neq \mathscr{U}T, v, w \in \Delta_{kt} \cup \{\varnothing\}$ *and* $\mathscr{U}v(K,T) = \mathscr{U}w(K,T)$. *Then* $v = w$.

Proof. By the hypothesis that $\mathscr{U} \neq \mathscr{U}T$, $\mathscr{U} \neq \mathscr{U}K$, we cannot have one of v, w equal to $\varnothing$, but not the other. Therefore we may assume that $v, w \in \Delta_{kt}$.

Suppose that $v \neq w$. By Lemma 7.4, we may assume that $|w| = |v| + 1$ and that v and w end in the same letter. Without loss of generality, let v and w end in k. We may also assume that v has been chosen to have the smallest length of any element in Δ_{kt} for which there exists $w \in \Delta_{kt}$ with $|w| = |v| + 1$ and $\mathcal{U}v(K,T) = \mathcal{U}w(K,T)$.

Let $v = v_1 k$ and $w = w_1 k$, where $v_1, w_1 \in \Delta_{kt}$. Let $V = v(K,T), V_1 = v_1(K,T), W = w(K,T)$ and $W_1 = w_1(K,T)$. By the minimality of $|v|, \mathcal{U}V_1 \neq \mathcal{U}W_1$. Since v, v_1, w, w_1 are distinguished, v_1 and w_1 must both end in t. Also, occurrences of k and t alternate in distinguished words and $|w_1| = |v_1| + 1$. Hence, we must have $w_1 = tv_1$ or $w_1 = kv_1$. Thus either $\mathcal{U}V_1 \subseteq \mathcal{U}TV_1 = \mathcal{U}W_1$ or $\mathcal{U}V_1 \subseteq \mathcal{U}KV_1 = \mathcal{U}W_1$. In both cases we conclude that $\mathcal{U}V_1 \subset \mathcal{U}W_1$ so that $\mathcal{U}W_1 \not\subseteq \mathcal{U}V_1$.

Since w_1 ends in $t, w_1(K,T)T = w_1(K,T)$ and so $\mathcal{U}W_1T = \mathcal{U}W_1$. But $\mathcal{U}W_1 \not\subseteq \mathcal{U}V_1$ implies $\mathcal{U}W_1T \not\subseteq \mathcal{U}V_1K$ by Lemma 7.2, so that

$$\mathcal{U}W_1 = \mathcal{U}W_1T \not\subseteq \mathcal{U}V_1K = \mathcal{U}V = \mathcal{U}W = \mathcal{U}W_1K$$

which is a clear contradiction.

LEMMA 7.6. *Let* $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$.
(i) $\mathcal{U}T_l \cap \mathcal{B} = \mathcal{LZ} \circ (\mathcal{U} \cap \mathcal{B})$.
(ii) $\mathcal{U}T_r \cap \mathcal{B} = \mathcal{RZ} \circ (\mathcal{U} \cap \mathcal{B})$.

Proof. By duality it suffices to prove (i). By Lemma 4.3(iii), we have $\mathcal{LZ} \circ (\mathcal{U} \cap \mathcal{B}) \subseteq \mathcal{LG} \circ \mathcal{U} = \mathcal{U}T_l$, while clearly $\mathcal{LZ} \circ (\mathcal{U} \cap \mathcal{B}) \subseteq \mathcal{B}$. Thus $\mathcal{LZ} \circ (\mathcal{U} \cap \mathcal{B}) \subseteq \mathcal{U}T_l \cap \mathcal{B}$. So let $S \in \mathcal{U}T_l \cap \mathcal{B}$. Then $S/\mathcal{L}^0 \in \mathcal{U}, S \in \mathcal{B}$ and $e\mathcal{L}^0 \in \mathcal{LG}$ for all $e \in E(S)$. Hence $e\mathcal{L}^0 \in \mathcal{LG} \cap \mathcal{B} = \mathcal{LZ}$ for all $e \in E(S)$, and $S/\mathcal{L}^0 \in \mathcal{U} \cap \mathcal{B}$. Thus $S \in \mathcal{LZ} \circ (\mathcal{U} \cap \mathcal{B})$, as required and the equality in (i) holds.

LEMMA 7.7. $TT_l = T_lT = T_l, TT_r = T_rT = T_r$.

Proof. It suffices to consider T_l. Let $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$. Clearly $\mathcal{U}T_l \subseteq \mathcal{U}TT_l$. So let $S \in \mathcal{U}TT_l$. Then $R = S/\mathcal{L}^0 \in \mathcal{U}T$ and $R/\mu_R \in \mathcal{U}$. But $\mathcal{L}_R^0$ is the equality congruence on R and, therefore, so also is μ_R. Hence

$$S/\mathcal{L}_S^0 = R \cong R/\mu_R \in \mathcal{U}$$

and so $S \in \mathcal{U}T_l$. Thus $\mathcal{U}TT_l = \mathcal{U}T_l$ and therefore $TT_l = T$.

It is also clear that $\mathcal{U}T_l \subseteq \mathcal{U}T_lT$. So let $S \in \mathcal{U}T_lT$. Then $R = S/\mu_S \in \mathcal{U}T_l$ and $R/\mathcal{L}_R^0 \in \mathcal{U}$. Since $\mu_S \subseteq \mathcal{L}_S^0$, it is easily seen that $\mathcal{L}_R^0 = \mathcal{L}_S^0/\mu_S$. Thus $S/\mathcal{L}_S^0 \cong R/\mathcal{L}_R^0 \in \mathcal{U}$ and $S \in \mathcal{U}T_l$. Hence $\mathcal{U}T_lT = \mathcal{U}T_l$ so that $T_lT = T_l$, as required.

We are finally ready for the main result of this section.

THEOREM 7.8. (i) $[K,T] \cong \langle k,t \mid k^2 = k, t^2 = t\rangle$.
(ii) $[K,T_l] \cong \langle k,t_l \mid k^2 = k,\ t_l^2 = t_l\rangle$.
(iii) $[K,T_r] \cong \langle k,t_r \mid k^2 = k,\ t_r^2 = t_r\rangle$.
(iv) $[T_l,T_r] \cong \langle t_l,t_r \mid t_l^2 = t_l,\ t_r^2 = t_r\rangle$.
(v) $[T,T_l] \cong \langle t,t_l \mid t^2 = t,\ t_l^2 = t_l,\ tt_l = t_lt = t_l\rangle \cong \langle t_l\rangle^1$
(vi) $[T,T_r] \cong \langle t,t_r \mid t^2 = t,\ t_r^2 = t_r,\ tt_r = t_rt = t_r\rangle \cong \langle t_r\rangle^1$.

Proof. Let $\varphi:\{k,t\}^+ \to [K,T]$ be the epimorphism defined by $k\varphi = K$, $t\varphi = T$. Let $\rho = \varphi \circ \varphi^{-1}$ and σ be the congruence on $\{k,t\}^+$ generated by the relation $\{(k^2,k),(t^2,t)\}$. Since $K^2 = K$ and $T^2 = T$, it follows that $\sigma \subseteq \rho$. To establish the reverse inclusion, let $p,q \in \{k,t\}^+$ and $p\rho q$. Using the relation defining σ, there exist $v,w \in \Delta_{kt}$ such that $p\sigma v$ and $q\sigma w$. Since $\sigma \subseteq \rho$, we have $v\rho w$. Thus $v(K,T) = v\varphi = w\varphi = w(K,T)$, and in particular $\mathcal{N}\mathcal{B}v(K,T) = \mathcal{N}\mathcal{B}w(K,T)$. But $\mathcal{N}\mathcal{B}K = \mathcal{B} \neq \mathcal{N}\mathcal{B}$ and $\mathcal{N}\mathcal{B}T = \mathcal{N}\mathcal{B}\mathcal{G} \neq \mathcal{N}\mathcal{B}$. Hence, by Lemma 7.5, $v = w$. Thus $p\sigma v = w\sigma q$, so that $p\sigma q$ and $\rho \subseteq \sigma$. Therefore $\rho = \sigma$ and (i) follows.

The proofs of parts (ii) and (iii) are similar to that of (i). Lemmas 7.4 and 7.5 were derived from Lemmas 7.2 and 7.3. These latter lemmas hold equally well for T_l and T_r as for T. Therefore, throughout Lemmas 7.4 and 7.5 we may replace T consistently by T_l or by T_r to obtain analogous results for K combined with T_l and for K combined with T_r. The final parts of the proofs of (ii) and (iii) are then similar to that of (i) above.

We now prove part (iv). Let $\varphi:\{t_l,t_r\}^+ \to [T_l,T_r]$ be the epimorphism defined by $t_l\varphi = T_l$, $t_r\varphi = T_r$. Let $\rho = \varphi \circ \varphi^{-1}$ and σ be the congruence on $\{t_l,t_r\}^+$ defined by the relation $\{(t_l^2,t_l),\ (t_r^2,t_r)\}$. Since $T_l^2 = T_l$ and $T_r^2 = T_r$, it follows that $\sigma \subseteq \rho$. To establish the reverse inclusion, let $p,q \in \{t_l,t_r\}^+$ and $p\rho q$. Using the relation defining σ, there exist $v,w \in \Delta_{t_lt_r}$ such that $p\sigma v$ and $q\sigma w$. Since $\sigma \subseteq \rho$, we have $v\rho w$. Thus $v(T_l,T_r) = v\varphi = w\varphi = w(T_l,T_r)$. Now let T_l^*, T_r^* be the operators defined on $\mathcal{L}(\mathcal{B})$ by

$$\mathcal{U}T_l^* = \mathcal{L}\mathcal{Z} \circ \mathcal{U}, \quad \mathcal{U}T_r^* = \mathcal{R}\mathcal{Z} \circ \mathcal{U}.$$

By repeated application of Lemma 7.6, we have that for all $\mathcal{U} \in \mathcal{L}(\mathcal{B})$,

$$\begin{aligned}\mathcal{U}v\left(T_l^*,T_r^*\right) &= \mathcal{U}v(T_l,T_r) \cap \mathcal{B}\\ &= \mathcal{U}w(T_l,T_r) \cap \mathcal{B} = \mathcal{U}w\left(T_l^*,T_r^*\right).\end{aligned}$$

By ([**8**], Example 10), this implies that $v = w$. Thus $p\sigma v = w\sigma q$ so that $p\sigma q$ and $\sigma \subseteq \rho$. Thus $\sigma = \rho$ and (iv) holds.

Parts (v) and (vi) follow directly from Lemma 7.7.

8. The semigroup generated by C, L, K **and** T. In §§3 and 4 we have considered the semigroups $[C, L]$ and $[K, T]$ and in §§5 and 6 we have shown that the operators C and L commute with the operators K, T, T_l and T_r. In this section we will show that the relations between C, L, K and T already introduced are sufficient to describe $[C, L, K, T]$.

We require three technical preliminary lemmas.

LEMMA 8.1. *Let* $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR}), \mathscr{S} \subseteq \mathscr{U}, \mathscr{NB} \subseteq \mathscr{V}, \mathscr{U} \subset \mathscr{V}$ *and* $w \in \Delta_{kt}$. *If either* (i) $\mathscr{V}T = \mathscr{V}$ *and* $w = kw_1$ *for some* $w_1 \in \Delta_{kt} \cup \{\varnothing\}$ *or* (ii) $\mathscr{V}K = \mathscr{V}$ *and* $w = tw_1$ *for some* $w_1 \in \Delta_{kt} \cup \{\varnothing\}$, *then* $\mathscr{U}w(K, T) \subset \mathscr{V}w(K, T)$.

Proof. The assumption that $\mathscr{U} \subseteq \mathscr{V}$ implies that $\mathscr{U}w(K, T) \subseteq \mathscr{V}w(K, T)$ for all $w \in \Delta_{kt}$. So the objective is to show that under certain circumstances, inequality is preserved. We will only consider case (i), since case (ii) will follow in a similar manner using Lemma 7.3. We will argue by induction on the length of w_1.

First suppose that $w_1 = \varnothing$. Since $\mathscr{V} \not\subseteq \mathscr{U}$, it follows from Lemma 7.2 that $\mathscr{V}T \not\subseteq \mathscr{U}K$ so that $\mathscr{V} = \mathscr{V}T \not\subseteq \mathscr{U}K$ and therefore

$$\mathscr{V}w(K, T) = \mathscr{V}K \not\subseteq \mathscr{U}K = \mathscr{U}w(K, T).$$

Now suppose that $|w_1| \geq 1$ and that the desired inequality holds for any words of the form $kw_1' \in \Delta_{kt}$, where $|w_1'| < |w_1|$. We have two cases.

Case 1. $w_1 = w_2k$ for some $w_2 \in \Delta_{kt} \cup \{\varnothing\}$. In fact, we must have $w_2 \neq \varnothing$ since otherwise we obtain $w = kw_1 = kw_2k = k\varnothing k = kk$ which is not distinguished. Thus $|w_2| \geq 1$ and, since $w_1 = w_2k, w_2$ must end in t. Hence $w_2(K, T)T = w_2(K, T)$. By the induction hypothesis, $\mathscr{U}Kw_2(K, T) \subset \mathscr{V}Kw_2(K, T)$. In particular, $\mathscr{V}Kw_2(K, T) \not\subseteq \mathscr{U}Kw_2(K, T)$. Therefore, by Lemma 7.2,

$$\mathscr{V}Kw_2(K, T)T \not\subseteq \mathscr{U}Kw_2(K, T)K = \mathscr{U}w(K, T)$$

so that

$$\mathscr{V}Kw_2(K, T) = \mathscr{V}Kw_2(K, T)T \not\subseteq \mathscr{U}w(K, T)$$

and

$$\mathscr{V}w(K, T) = \mathscr{V}Kw_2(K, T)K \not\subseteq \mathscr{U}w(K, T).$$

Case 2. $w_1 = w_2t$ for some $w_2 \in \Delta_{kt} \cup \{\varnothing\}$. The proof in this case is similar to that of Case 1.

LEMMA 8.2. *Let* $r, s \in \Delta_{cl}$ *and* $v, w \in \Delta_{kt} \cup \{\varnothing\}$ *be such that* $r(C, L)v(K, T) = s(C, L)w(K, T)$. *Then* $v = w$.

Proof. Let $R = r(C, L), S = s(C, L), V = v(K, T), W = w(K, T)$ and $\mathscr{U} = \mathscr{O}\mathscr{N}\mathscr{B}\mathscr{G}$. Then $\mathscr{U}C = \mathscr{U}L = \mathscr{U}$ so that $\mathscr{U}R = \mathscr{U}S = \mathscr{U}$. Hence $\mathscr{U}V = \mathscr{U}W$. By Lemma 7.4, either (i) $v = w$, as claimed, or (ii) $|v| = |w| + 1$ and v and w end in the same letter. So let (ii) hold. Since occurrences of k and t alternate in v and w and they end in the same letter, we must have $v = kw$ or $v = tw$.

Let $v = kw$. If $w = \varnothing$, then $\mathscr{U}V = \mathscr{U}K = \mathscr{O} \neq \mathscr{O}\mathscr{N}\mathscr{B}\mathscr{G} = \mathscr{U}W$, a contradiction. Hence $w \neq \varnothing$ and, since v is distinguished, w must begin with t. Let $\mathscr{V} = \mathscr{U}K = \mathscr{O}$. Then $\mathscr{V}K = \mathscr{V}, \mathscr{U} \subset \mathscr{V}$ and $w = tw_1$ for some $w_1 \in \Delta_{kt} \cup \{\varnothing\}$. By Lemma 8.1(ii),

$$\mathscr{U}w(K,T) \subset \mathscr{V}w(K,T) = \mathscr{U}Kw(K,T) = \mathscr{U}v(K,T)$$

which is a contradiction. Thus $v = w$.

The case when $v = tw$ is treated similarly.

Lemma 8.3. *For distinct $A, B \in \{\varnothing, C, L, CL, LC\}$, there exist $\mathscr{U} = \mathscr{U}_{A,B}, \mathscr{V} = \mathscr{V}_{A,B} \in \mathscr{L}(\mathscr{C}\mathscr{R})$ such that*

(i) $\mathscr{N}\mathscr{B} \subseteq \mathscr{U} \cap \mathscr{V}$,

(ii) *for $\mathscr{X} \in \{\mathscr{U}, \mathscr{V}\}$, either $\mathscr{X}A \subset \mathscr{X}B$ or $\mathscr{X}B \subset \mathscr{X}A$,*

(iii) $\mathscr{U}AK = \mathscr{U}A, \mathscr{U}BK = \mathscr{U}B$,

(iv) $\mathscr{V}AT = \mathscr{V}A, \mathscr{V}BT = \mathscr{V}B$.

Proof. For all choices of distinct elements $A, B, \in \{\varnothing, C, L, CL, LC\}$, we display in the table below suitable choices for $\mathscr{U}$. The square in the row labelled with A and the column labelled with B has the form

<table>
<tr><td colspan="2">$\mathscr{U}$</td></tr>
<tr><td>$\mathscr{P}$</td><td>$\mathscr{Q}$</td></tr>
</table>

where $\mathscr{P} = \mathscr{U}A$ and $\mathscr{Q} = \mathscr{U}B$.

In all cases $\mathscr{U}$ is either $\mathscr{B}$ or $\mathscr{O}$ so that $\mathscr{U}$ satisfies the requirement (i). In addition, in all cases either $\mathscr{P} \subset \mathscr{Q}$ or $\mathscr{Q} \subset \mathscr{P}$ so that $\mathscr{U}$ also satisfies the requirement (ii).

<table>
<tr><td>A \ B</td><td colspan="2">C</td><td colspan="2">L</td><td colspan="2">CL</td><td colspan="2">LC</td></tr>
<tr><td rowspan="2">$\varnothing$</td><td colspan="2">$\mathscr{B}$</td><td colspan="2">$\mathscr{O}$</td><td colspan="2">$\mathscr{B}$</td><td colspan="2">$\mathscr{B}$</td></tr>
<tr><td>$\mathscr{B}$</td><td>$\mathscr{O}$</td><td>$\mathscr{O}$</td><td>$\mathscr{O}L$</td><td>$\mathscr{B}$</td><td>$\mathscr{O}L$</td><td>$\mathscr{B}$</td><td>$\mathscr{O}$</td></tr>
<tr><td rowspan="2">C</td><td colspan="2" rowspan="2"></td><td colspan="2">$\mathscr{B}$</td><td colspan="2">$\mathscr{B}$</td><td colspan="2">$\mathscr{O}$</td></tr>
<tr><td>$\mathscr{O}$</td><td>$\mathscr{B}$</td><td>$\mathscr{O}$</td><td>$\mathscr{O}L$</td><td>$\mathscr{O}$</td><td>$\mathscr{O}L$</td></tr>
<tr><td rowspan="2">L</td><td colspan="2" rowspan="2"></td><td colspan="2" rowspan="2"></td><td colspan="2">$\mathscr{B}$</td><td colspan="2">$\mathscr{B}$</td></tr>
<tr><td>$\mathscr{B}$</td><td>$\mathscr{O}L$</td><td>$\mathscr{B}$</td><td>$\mathscr{O}$</td></tr>
<tr><td rowspan="2">CL</td><td colspan="2" rowspan="2"></td><td colspan="2" rowspan="2"></td><td colspan="2" rowspan="2"></td><td colspan="2">$\mathscr{B}$</td></tr>
<tr><td>$\mathscr{O}L$</td><td>$\mathscr{O}$</td></tr>
</table>

By Lemmas 5.3 and 6.3, K commutes with C and L and therefore also with A and B. It is easily seen that $\mathscr{B}K = \mathscr{B}$ and $\mathcal{O}K = \mathcal{O}$. Consequently, since in all cases $\mathscr{U}$ is either $\mathscr{B}$ or $\mathcal{O}$, we have that $\mathscr{U}K = \mathscr{U}$. Hence

$$\mathscr{U}AK = \mathscr{U}KA = \mathscr{U}A, \quad \mathscr{U}BK = \mathscr{U}KB = \mathscr{U}B$$

and (iii) holds.

For each choice of A and B, we now take $\mathscr{V} = \mathscr{U}T$. Then $\mathscr{N}\mathscr{B} \subseteq \mathscr{U} \subseteq \mathscr{U}T = \mathscr{V}$ so that $\mathscr{V}$ satisfies (i). By (ii), either $\mathscr{U}A \subset \mathscr{U}B$ or $\mathscr{U}B \subset \mathscr{U}A$. First suppose that $\mathscr{U}A \subset \mathscr{U}B$. By (iii) we have that $(\mathscr{U}A)K = \mathscr{U}A$ and $(\mathscr{U}B)K = \mathscr{U}B$ and hence, by Lemma 8.1(ii), $\mathscr{U}AT \subset \mathscr{U}BT$. By Lemmas 5.5 and 6.5, T commutes with C and L and therefore with A and B. Hence

$$\mathscr{V}A = \mathscr{U}TA = \mathscr{U}AT \subset \mathscr{U}BT = \mathscr{U}TB = \mathscr{V}B.$$

Similarly, if $\mathscr{U}B \subset \mathscr{U}A$ we obtain $\mathscr{V}B \subset \mathscr{V}A$ so that $\mathscr{V}$ satisfies (ii). Finally, again since T commutes with A and B and also since $T^2 = T$,

$$\mathscr{V}AT = \mathscr{V}TA = \mathscr{U}TTA = \mathscr{U}TA = \mathscr{V}A,$$
$$\mathscr{V}BT = \mathscr{V}TB = \mathscr{U}TTB = \mathscr{U}TB = \mathscr{V}B.$$

Thus $\mathscr{V}$ satisfies (iv).

LEMMA 8.4. *Let $w \in \Delta_{kt} \cup \{\varnothing\}$ and $W = w(K,T)$. Then the following are distinct*: W, CW, LW, CLW, LCW.

Proof. If $w = \varnothing$, then the result follows from the description of the semigroup $[C, L]$ in §3. Now let $A, B \in \{\varnothing, C, L, CL, LC\}$ and suppose first that w begins with the letter t. By Lemma 8.3, there exists a variety $\mathscr{U} = \mathscr{U}_{A,B}$, satisfying Lemma 8.3(i)–(iii). By Lemma 8.1(ii), we must have $\mathscr{U}AW \neq \mathscr{U}BW$ so that $AW \neq BW$. Suppose, on the other hand, that w begins with the letter k. Let $\mathscr{V} = \mathscr{V}_{A,B} \in \mathscr{L}(\mathscr{C}\mathscr{R})$ satisfy Lemma 8.3(i), (ii) and (iv). By Lemma 8.1(i), we must have $\mathscr{V}AW \neq \mathscr{V}BW$ so that again $AW \neq BW$.

We now apply these observations to the semigroup $[C, L, K, T]$.

THEOREM 8.5.

$$[C, L, K, T] \cong \langle c, l, k, t \mid c^2 = c,\ l^2 = l,\ k^2 = k,$$
$$t^2 = t,\ clc = lcl = cl,\ ck = kc,$$
$$ct = tc,\ lk = kl,\ lt = tl\rangle.$$

Proof. Let N be the semigroup on the right hand side of the statement of the theorem. Let $M = \{c, l, k, t\}^+$ and $\varphi : M \to [C, L, K, T]$ be the epimorphism defined by $c\varphi = C, l\varphi = L, k\varphi = K, t\varphi = T$. Let $\rho = \varphi \circ \varphi^{-1}$ and σ be the congruence on M generated by the defining relations for N.

It follows from the fact that the operators C, L, K and T are all idempotent and from Lemmas 3.1, 3.3, 5.3, 5.5, 6.3 and 6.5 that $\sigma \subseteq \rho$. Now suppose that $p\rho q$. Using the relations defining σ, there exist $r, s \in \Delta_{cl}$ and $v, w \in \Delta_{kt}$ such that $p\sigma rv$ and $q\sigma sw$. Since $\sigma \subseteq \rho$, this implies that $rv\rho sw$. Hence $(rv)\varphi = (sw)\varphi$ so that $r(C, L)v(K, T) = s(C, L)w(K, T)$. By Lemma 8.2, $v = w$ and, by Lemma 8.4, $r = s$. Thus $p\sigma rv = sw\sigma q$. Hence $\rho \subseteq \sigma$, as required.

COROLLARY 8.6. $[C, L, K, T]^1 \cong [C, L]^1 \times [K, T]^1$.

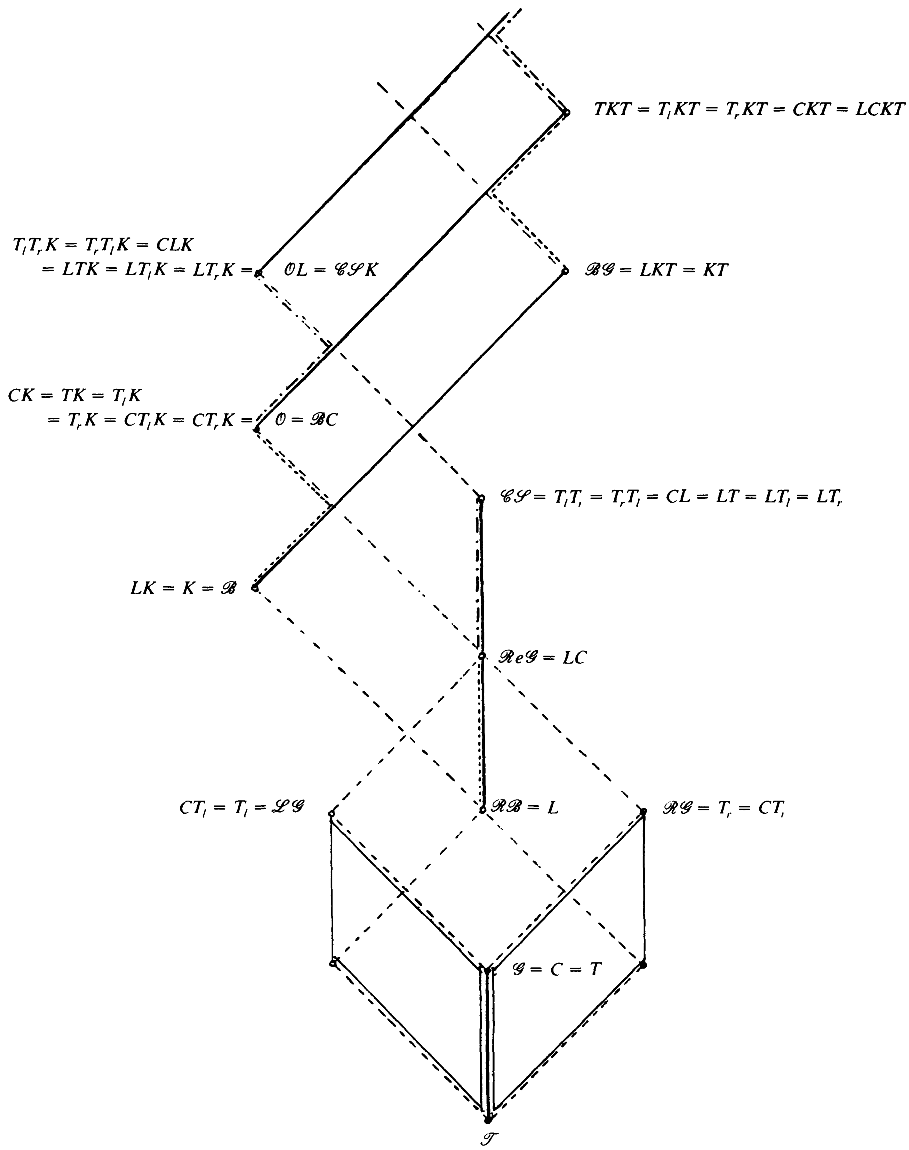
Proof. This is immediate from Theorem 8.5.

Similar descriptions to that of $[C, L, K, T]$ in Theorem 8.5 can be obtained for $[C, L, K, T_l], [C, L, K, T_r]$ and $[C, L, T_l, T_r]$ by substituting T_l for T etc. in all arguments. The nature of the semigroup generated by C, L, K, T, T_l and T_r is left open.

9. Evaluation at the trivial variety. The diagram below presents a few varieties obtained by repeated application of the operators C, L, K, T, T_l and T_r to the trivial variety $\mathscr{T}$. In order to save writing $\mathscr{T}$ in the expressions of the form $\mathscr{T}C, \mathscr{T}T, \mathscr{T}CT_r$, etc., we have written only C, T, CT_r, etc. For example, the vertex labelled $\mathscr{G} = C = T$ stands for the variety of groups; this variety can also be described as $\mathscr{T}C$ or $\mathscr{T}T$. The following legend is used to denote the classes induced by these operators:

....	C
-·-·-·-·-·-·-	L
-------------	K
━━━━━━━━━━ (thick)	T
────────── (thin)	T_l, T_r

We omit the proofs of the statements implicit in the diagram.



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