THE NORMAL INDEX OF A FINITE GROUP

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For a maximal subgroup M of a finite group G the normal index of M is the order of a chief factor H/K where H is minimal in the set of supplements of M in G. We obtain results about the normal index of M when M has composite index in G.

1. Introduction. The relationships between the properties of maximal subgroups of a finite group G and the structure of G have been studied by many people. In [3], [4] and [10] we investigated maximal subgroups of composite index, developing analogs of the Frattini subgroup and studying their role in the structure of groups. Here we obtain results which involve the normal index (introduced by Deskins in [5]) of a maximal subgroup M of a group G. The normal index of M, $\eta(G : M)$, is the order of a chief factor H/K of G when H is a minimal supplement of M in G. In §§2–4 we obtain extensions of results of Deskins [5], Beidleman and Spencer [2] and Mukherjee [9] based on $\eta(G : M)$ for the case when $[G : M]$ is composite.

All groups treated are finite, notation is standard (from [6] and [8]), and a maximal subgroup M of G is often denoted by $M <^{\cdot} G$. If $M <^{\cdot} G$ and $[G : M]$ is composite we call M c-*maximal* in G.

2. Normal index and solvability. If M is a maximal subgroup of a group G and H is a minimal normal supplement to M in G then for any chief factor H/K of G it follows that $K \subseteq M$ and $G = MH$. Therefore we have that $[G : M]$ divides $o(H/K) = \eta(G : M)$. For the sake of completeness we first describe some properties of the normal index which we shall use subsequently.

2.1 (Deskins [5, 2.1], Beidleman and Spencer [2, Lemma 1]). If M is a maximal subgroup of a group G then $\eta(G : M)$ is uniquely determined by M.

2.2 (Beidleman and Spencer [2, Lemma 2]). If N is a normal subgroup of a group G and M is a maximal subgroup of G such that $N \subseteq M$ then $\eta(G/N : M/N) = \eta(G : M)$.

It was announced by Deskins in [5, 2.5] that a group G is solvable if and only if $\eta(G : M) = [G : M]$ for each maximal subgroup M of G (a detailed proof appears in [2]). We extend this theorem by proving the following:

THEOREM 2.3. *A group G is solvable if and only if:*

$$\eta(G : M) = [G : M] \text{ for every } c\text{-maximal subgroup } M.$$

Proof. Let $\mathscr{J}$ be the family of all c-maximal subgroups of G. If $\mathscr{J}$ is empty then every maximal subgroup of G has prime index and so G is supersolvable (using a well known result of Huppert [7]) implying that G is solvable, proving the result. So we may assume that $\mathscr{J}$ is non-empty. We use induction on the order of G. If G is simple, then for any $M \in \mathscr{J}$ we have that $\eta(G : M) = o(G) = [G : M]$ implying that $o(M) = 1$, a contradiction. Therefore G is not simple. Now, suppose if possible, N_1, N_2 are two distinct minimal normal subgroups of G. Using 2.2 it is easy to see that the hypothesis holds for G/N_1 and G/N_2 and so by induction G/N_1 and G/N_2 are solvable implying that $G/(N_1 \cap N_2)$ is solvable. Thus G is solvable since $N_1 \cap N_2 = \langle 1 \rangle$, proving the result. Therefore we assume now that G has a unique minimal normal subgroup N. By the induction hypothesis G/N is solvable. Let p be the largest prime dividing the order of G. Let $\Phi_p(G)$ denote the intersection of all maximal subgroups M of G for which $[G : M]_p = 1$, (the subgroup $\Phi_p(G)$ was introduced by Deskins in [5]). If N is contained in every maximal subgroup L with $[G : L]_p = 1$, then $N \subseteq \Phi_p(G)$ and so N is solvable since $\Phi_p(G)$ is solvable ([10, Theorem 7(i)]). Since G/N is solvable this implies that G is solvable, proving the result. So we now suppose that there is some maximal subgroup L of G such that $[G : L]_p = 1$ and N is not contained in L. Since N is the unique minimal normal subgroup of G and N is not contained in L, it follows that the core of L is $\langle 1 \rangle$. Further, we have that $G = LN$ and $\eta(G : L) = o(N)$. Now, we claim that $[G : L]$ is composite. For, if not, suppose that $[G : L] = r$, a prime. Clearly, $r < p$. Then by considering the permutation representation of G on the r cosets of L and using the fact that the core of L is $\langle 1 \rangle$, we obtain that the order of G divides $r!$ which is absurd since p divides the order of G and $r < p$. Hence $[G : L]$ is composite. So by the hypothesis we get that $[G : L] = \eta(G : L)$ and consequently

$$[G : L] = \eta(G : L) = o(N).$$

Since $[G : L]_p = 1$, it follows that p does not divide the order of N. Now, let M be any maximal subgroup of G with core $\langle 1 \rangle$. Then $G = MN$. Suppose, if possible, that $[G : M]$ is a prime t, say. Then clearly $t < p$. By representing G on the t cosets of M and using the fact that the core of M is $\langle 1 \rangle$, we obtain that the order of G divides $t!$, a contradiction as before. Therefore $[G : M]$ is composite. So, $[G : M] = \eta(G : M)$ which in turn is equal to $o(N)$. Thus G has a unique maximal normal subgroup and there is a common divisor of the indices of all the maximal subgroups with core $\langle 1 \rangle$. Therefore by using Baer [1, Lemma 3], we get that G has a solvable, normal subgroup K, $K \neq \langle 1 \rangle$. Clearly $N \subseteq K$ and so N is solvable. Since G/N is solvable, we now get that G is solvable.

The converse is a direct consequence of Deskins [5, 2.5].

3. Supersolvable groups. First we prove a lemma

LEMMA 3.1. *If M is a maximal subgroup of a group G such that $[G : M]$ is a square-free integer then $\eta(G : M) = [G : M]$.*

Proof. We use induction on the order of G. If G is simple then $\eta(G : M) = o(G)$ and so $o(G)$ is square-free, G is supersolvable and simple, hence $M = 1$ and $\eta(G : M) = [G : M] = p$. Assume that G is not simple. Let N be a minimal normal subgroup of G, $N \neq \langle 1 \rangle$. If $N \subseteq M$ then by 2.2 we have $\eta(G/N : M/N) = \eta(G : M)$ and since $\eta(G : M)$ is square-free, it follows that $\eta(G/N : M/N) = [G/N : M/N]$ implying that $\eta(G : M) = [G : M]$. Now, suppose that N is not contained in M. Then $G = MN$ and $\eta(G : M) = o(N)$, and $o(N)$ is a square-free integer. So N is supersolvable. So N is a cyclic group of prime order. $[G : M] = o(N)/o(M \cap N)$ is a divisor of $\eta(G : M) = o(N)$. So $M \cap N = \langle 1 \rangle$ and $[G : M] = o(N) = \eta(G : M)$, proving the result.

In Mukherjee [9, Corollary 2] it was established that a group G is supersolvable if and only if $\eta(G : M)$ is square-free for each maximal subgroup M of G. We extend this result by showing the following:

THEOREM 3.2. *If G is a group such that $\eta(G : M)$ is square-free for every c-maximal subgroup M then G is supersolvable.*

Proof. If every maximal subgroup of G is of prime index, then G is supersolvable. Use induction on the order of G. Let N be a

minimal normal subgroup of G. If M/N is a c-maximal subgroup of G/N then clearly M is a c-maximal subgroup of G and so $\eta(G : M)$ is square-free. Thus $\eta(G/N : M/N)$ is square-free. Therefore by the induction hypothesis, G/N is supersolvable. Now, if N_1 is a minimal normal subgroup of G, $N \neq N_1$, then the same arguments as above yield that G/N_1 is also supersolvable. Consequently, $G/N \cap N_1 \simeq G$ is supersolvable, proving the result. Hence we may now assume that N is the unique minimal normal subgroup of G. Now if N is contained in every maximal subgroup of G, then N is contained in the Frattini subgroup $\Phi(G)$. Since G/N is supersolvable and $N \subseteq \Phi(G)$, it follows that $G/\Phi(G)$ is supersolvable which implies that G is supersolvable. So, we assume that there exists some $M <\cdot G$ such that N is not contained in M. Then $G = MN$ and $\eta(G : M) = o(N)$. If $[G : M]$ is composite then by the hypothesis we have that $\eta(G : M)$ is square-free and so $o(N)$ is square-free which implies that N is supersolvable. So, N must be of prime order and G is supersolvable, proving the theorem. Thus we now assume that for any maximal subgroup M not containing N, $[G : M]$ is a prime and that there exists at least one such M.

Let q be the largest prime dividing the order of G. Let $\Phi_q(G)$ denote the intersection of all maximal subgroups R of G such that $[G : R]_q = 1$. Now if N is contained in every maximal subgroup M_1 of G with $[G : M_1]_q = 1$ then $N \subseteq \Phi_q(G)$ and so N is solvable since $\Phi_q(G)$ is solvable [10, Theorem 7(i)]. So N, being a solvable, minimal normal subgroup, is elementary abelian. Now taking a subgroup M as in the end of the last paragraph, we have $G = MN$ and so $[G : M] = o(N)/o(M \cap N)$. Also $\eta(G : M) = o(N)$ and $[G : M]$ is a divisor of $\eta(G : M)$ (see the beginning of §2). Consequently $o(M \cap N) = 1$, $[G : M] = o(N)$, and so $o(N)$ is a prime. Thus N is cyclic and G is supersolvable, proving the result. Therefore we may now assume that N is not contained in some maximal subgroup M_1 with $[G : M_1]_q = 1$. So $G = M_1 N$. If $[G : M_1]$ is composite then as in the last paragraph G is supersolvable. Thus we assume that $[G : M_1]$ is a prime, say r. Clearly $r < q$. Now consider the permutation representation of G on the r cosets of M_1. If the core of M_1 is $\langle 1 \rangle$ then it follows that $o(G)$ divides $r!$, which is absurd. Therefore the core of M_1 is non-trivial and consequently $N \subseteq M_1$, a contradiction. Thus this possibility cannot arise. Hence in all cases G is supersolvable.

4. p-solvable and p-supersolvable groups. It is proved in Beidleman and Spencer [2, Theorem 1] that a group G is p-solvable if and only

if $\eta(G : M)_p = [G : M]_p$ for every maximal subgroup M of G. This result and a theorem of Mukherjee [9, Theorem 7] for p-supersolvable groups are both extended in the following.

THEOREM 4.1. *Let p be the largest prime factor dividing the order of a group G. Then*
   (i) *G is p-solvable if $\eta(G : M)_p = [G : M]_p$ for each c-maximal M.*
   (ii) *G is p-supersolvable if $\eta(G : M)_p = [G : M]_p = 1$ or p for each c-maximal M.*
(*Note: The converses of* (i) *and* (ii) *are easily seen to be true.*)

Proof. (i) Let G be a non p-solvable group of minimal order among those satisfying the condition $\eta(G : M)_p = [G : M]_p$ for each c-maximal M, with p the maximal prime factor of $o(G)$. Clearly, G is non-abelian and has c-maximal subgroups. Furthermore G is not simple. For, if G were simple, consider $L <^{\cdot} G$ with $[G : L]_p = 1$. If $[G : L]$ is composite then $\eta(G : L)_p = o(G)_p = 1$, contrary to the choice of p. So $[G : L] = r$, a prime $< p$, which means that $o(G)$ divides $r!$, an impossibility. So G is not simple. Let N be a minimal normal subgroup of G. If N is not unique then choose another minimal normal subgroup $N_1 \neq N$. If p divides the orders of both G/N and G/N_1 then by the minimality of G and using 2.2, both G/N and G/N_1 are p-solvable and so $G/N \cap N_1 \simeq G$ is p-solvable and we are done. Now suppose p does not divide the order of, say, G/N, without loss of generality. If N_p is a Sylow p-subgroup of N, by the Frattini argument $G = N_G(N_p)N$. If $N_p \not\trianglelefteq G$, choose a maximal subgroup M of G which contains $N_G(N_p)$. Then $G = MN$ and clearly $[G : M]_p = 1$. If $[G : M]$ is composite, then by hypothesis $\eta(G : M)_p = [G : M]_p = 1$ and so $o(N)_p = 1$, an impossibility. Therefore $[G : M] = s$, a prime $< p$. Now if $\text{Core}_G(M) = \langle 1 \rangle$, then $o(G)$ divides $s!$, an impossibility. So $\text{Core}_G(M) \neq \langle 1 \rangle$. Since $o(G/\text{Core}_G(M))$ divides $s!$ a Sylow p-subgroup P of $\text{Core}_G(M)$ is a Sylow p-subgroup of G. Now $\text{Core}_G(M) \cap N = \langle 1 \rangle$ since otherwise $N \subseteq M$ implying $G = MN = M$, a contradiction. Thus $\text{Core}(M)$ and N centralize each other and so N_pP is a p-subgroup of G whose order is greater than $o(P)$ which is absurd since P is a Sylow p-subgroup of G. Hence $N_p \trianglelefteq G$ and so $N_p = N$. By the Schur-Zassenhaus theorem $G = NT$ where T is a p-complement. Now G/N and N are both p-solvable and so G is p-solvable, proving the result.

So we now assume that N is the unique minimal normal subgroup of G. If $N \subseteq M$ for every $M <^{\cdot} G$ with $[G : M]_p = 1$, then N is

contained in their intersection $\Phi_p(G)$ so N is solvable and we then obtain easily that G is p-solvable, proving the result. So suppose that N is not contained in some M where $M <^\cdot G$ and $[G : M]_p = 1$. Then $G = MN$. If $[G : M]$ is composite then proceeding as in the last paragraph we obtain a contradiction if $\mathrm{Core}_G(M) = \langle 1 \rangle$; and if $\mathrm{Core}_G(M) \neq \langle 1 \rangle$ then by the uniqueness of N, $N \subseteq M$ implying $G = MN = M$, a contradiction. Thus $[G : M]$ is composite. By using the hypothesis, $\eta(G : M)_p = [G : M]_p = 1$ and so $o(N)_p = 1$. Thus N is a p'-group and G/N is p-solvable giving that G is p-solvable.

(ii) Let G be a non p-supersolvable group of minimal order among those satisfying the condition $\eta(G : M)_p = [G : M]_p = 1$ or p for each c-maximal M. By (i) N is p-solvable. Let N be a minimal normal subgroup of G. If N is not unique then choose another minimal normal subgroup $N_1 \neq N$. If p divides the orders of both G/N and G/N_1, then as in (i) the result follows easily. Now suppose that p does not divide the order of, say, G/N_1. Since N_1 is either a p-group or a p'-group, it follows that N_1 is a Sylow p-subgroup of G. By Schur-Zassenhaus theorem $G = N_1 T$ where $N_1 \cap T = \langle 1 \rangle$. It is not hard to see that $T <^\cdot G$. Let $o(N_1) = p^m$. If $m > 1$, then since $[G : T] = o(N_1)$, $[G : T]$ is composite and by using the hypothesis we obtain that

$$\eta(G : T)_p = [G : T]_p = o(N_1)_p = 1 \text{ or } p$$

which contradicts the assumption $m > 1$. Thus $m = 1$, N_1 is cyclic of order p and since G/N_1 is trivially p-supersolvable, the result follows.

The following result is a generalization of Mukherjee [9, Theorem 8]. We omit the proof which is quite similar to the proof of Theorem 4.1.

THEOREM 4.2. *Let p be the largest prime dividing the order of a group G. Then G is p-nilpotent if and only if the following conditions hold:*

   (i) *$\eta(G : M)_p = [G : M]_p = 1$ or p for all c-maximal subgroups M of G.*

   (ii) *If $\eta(G : M) = p$ for some maximal subgroup M of G then $M <^\cdot G$.*

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