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# ON MATRICIALLY NORMED SPACES

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**Arveson and Wittstock have proved a "non-commutative Hahn-Banach Theorem" for completely bounded operator-valued maps on spaces of operators. In this paper it is shown that if $T$ is a linear map from the dual of an operator space into a $C^*$-algebra, then the usual operator norm of $T$ coincides with the completely bounded norm. This is used to prove that the Arveson-Wittstock theorem does not generalize to "matricially normed spaces". An elementary proof of the Arveson-Wittstock result is presented. Finally a simple bimodule interpretation is given for the "Haagerup" and "matricial" tensor products of matricially normed spaces.**

**1. Introduction.** A *function space* $V$ on a set $X$ is a linear subspace of the bounded complex functions on $X$. With the uniform norm, this is a normed vector space. Conversely, any (complex) normed vector space $V$ may be realized as a function space on the closed unit ball $X$ of the dual space $V^*$. Thus one may regard a normed vector space as simply an abstract function space.

An *operator space* $V$ on a Hilbert space $H$ is a linear subspace of the bounded operators on $H$. For each $n \in \mathbf{N}$, the operator norm associated with $H^n$ determines a distinguished norm on the $n \times n$ matrices over $V$. The second author recently gave an abstract characterization for the operator spaces by taking into consideration these systems of matrix norms. The operator spaces $V$ are characterized among the "matricially normed spaces" (see §2), by the "$L^\infty$-property": given matrices $v = [v_{ij}]$, $w = [w_{kl}]$ with $v_{ij}, w_{kl} \in V$,

$$\|v \oplus w\| = \max\{\|v\|, \|w\|\}.$$

On the other hand, the dual of an operator space is canonically an "$L^1$-matricially normed space", in the sense that its matrix norms satisfy

$$\|v \oplus w\| = \|v\| + \|w\|.$$

In this paper we shall begin a systematic study of the matricially normed spaces. Our main results are:

(a) We show in §2 that if $\varphi: V \to W$ is a linear map from an $L^1$-matricially normed space to an operator space, then the completely bounded norm of $\varphi$ coincides with the usual norm. As a corollary, we conclude that the Arveson-Wittstock Hahn-Banach Theorem for operator spaces cannot be extended to $L^1$-matricially normed spaces.

(b) In §3 we give a direct proof of the Arveson-Wittstock Hahn-Banach Theorem for operator spaces that uses only the $L^\infty$-matricial norm structure, rather than appealing to Wittstock's theory of sublinear set valued functions, or to Paulsen's reduction to the completely positive case (see **[1]**, **[14]**, **[9]**). Our approach was first considered by Haagerup (unpublished **[6]**). Using a result of Smith **[12]**, we have been able to substantially simplify his argument.

(c) In §4 we consider the Haagerup tensor product for matricially normed spaces. In particular, following a suggestion of B. E. Johnson, we consider the corresponding construction in the isomorphic category of *normed $\mathscr{F}$-bimodules*, $\mathscr{F}$ being the algebra of all finitely non-zero infinite matrices. We show that the Haagerup tensor product corresponds to the projective bimodule tensor product $\hat{\otimes}_{\mathscr{F}}$ (see (4.1)).

(d) Letting $\mathscr{M}(V, \mathscr{B}(H))$ be the completely bounded maps from a matricially normed space $V$ into $\mathscr{B}(H)$, we have that

$$\mathscr{M}(V, \mathscr{B}(H)) = [V \otimes_{\mathscr{M}} \mathscr{B}(H)_*]^*,$$

where $\otimes_{\mathscr{M}}$ is the "completely bounded tensor product" (see §3 and **[4]**). In §5 we show that for arbitrary matricially normed spaces, this tensor product is isometric to the projective $\mathscr{F} \otimes \mathscr{F}$-bimodule tensor product $\hat{\otimes}_{\mathscr{F} \otimes \mathscr{F}}$ in Johnson's category (see (5.2)).

We are indebted to Uffe Haagerup for providing us with a copy of **[6]**, and to Barry Johnson for his suggestions regarding $\mathscr{F}$-bimodules.

**2. Matricially normed spaces and a Hahn-Banach counter-example.** Given normed vector spaces $V$ and $W$, we let $\mathscr{B}(V, W)$ denote the bounded functions from $V$ to $W$ with the usual norm, and $\mathscr{B}(V) = \mathscr{B}(V, V)$. We say that a bounded linear surjection $\varphi: V \to W$ of normed vector spaces is a *quotient map*, if it induces an isometry $V/\ker\varphi \to W$. Equivalently, it maps the open unit ball of $V$ onto that of $W$. If $\varphi: V \to W$ is an isometric injection, then the adjoint map $\varphi^*: W^* \to V^*$ is a quotient map.

We let $V \otimes W$ denote the tensor product of arbitrary vector spaces $V$ and $W$. If $V$ and $W$ are normed, we write $\| \ \|_\wedge$ and $\| \ \|_\lambda$ for the projective and injective cross norms on $V \otimes W$ (see **[13]**, §IV.2). These are defined by

$$\|u\|_\wedge = \inf\left\{\sum \|v_i\|\|w_i\|: u = \sum v_i \otimes w_i\right\},$$
$$\|u\|_\lambda = \sup\{|p \otimes q(u)|: p \in V^*, q \in W^*, \|p\| = \|q\| = 1\}.$$

We let $V \hat{\otimes} W$ and $V \otimes_\lambda W$ denote the corresponding normed spaces (we do not use the completions in this paper). It is easily verified that if $V \hookrightarrow V_1$ is an isometry, the corresponding map $V \otimes_\lambda W \hookrightarrow V_1 \otimes_\lambda W$ is again isometric, whereas if $V \to V_1$ is a quotient map, the same is true for the map $V \hat{\otimes} W \to V_1 \hat{\otimes} W$.

An *operator space* $V$ on a Hilbert space $H$ is a linear subspace of $\mathscr{B}(H)$. The latter is a von Neumann algebra, and we denote its predual by $\mathscr{B}(H)_*$. Given $\xi, \eta \in H$, we define $\omega_{\xi,\eta} \in \mathscr{B}(H)_*$ by

$$\omega_{\xi,\eta}(b) = b\xi \cdot \eta.$$

We let $\mathbf{M}_{m,n}$ (resp., $\mathbf{M}_n$ if $m = n$ ) denote the complex $m \times n$ matrices with the usual vector space operations and the operator norm. We let $\varepsilon_{ij} \in \mathbf{M}_{m,n}$ be the matrix units

$$\varepsilon_{ij} = \begin{array}{c} j \\ \downarrow \\ \begin{bmatrix} 0\cdots 0 \\ 1 \\ 0\cdots 0 \end{bmatrix} \end{array} \leftarrow i$$

Given a (complex) vector space $V$, we identify the vector space $\mathbf{M}_{m,n}(V)$ of $m \times n$ matrices $[v_{ij}]$ ($v_{ij} \in V$) with the algebraic tensor product $V \otimes \mathbf{M}_{m,n}$, letting $[v_{ij}] \mapsto \sum v_{ij} \otimes \varepsilon_{ij}$. We regard $V \otimes \mathbf{M}_n$ as an $\mathbf{M}_n$ bimodule by using the operations

$$\alpha(v \otimes \mu) = v \otimes \alpha\mu, \qquad (v \otimes \mu)\beta = v \otimes \mu\beta.$$

The corresponding operations in $\mathbf{M}_n(V)$ are determined by matrix multiplication. Given $n = n_1 + \cdots + n_r$, we identify $v \in \mathbf{M}_n(V)$ with the $r \times r$ matrix of rectangular matrices $[\tilde{v}_{pq}]$, where

$$\tilde{v}_{11} = [v_{ij}]_{i,j=1,\ldots,n_1} \in \mathbf{M}_{n_1}(V)$$
$$\tilde{v}_{12} = [v_{ij}]_{i=1,\ldots,n_1; j=n_1+1,\ldots,n_2} \in \mathbf{M}_{n_1,n_2}(V)$$

and so on. We also use the notation

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}.$$

A *matricially normed space* is a normed vector space $V$ together with an assignment of a norm to each of the matrix spaces $\mathbf{M}_n(V)$, such that

$\mathrm{M}_1$. $\|v \oplus 0_r\| = \|v\|$ $(v \in \mathbf{M}_n(V), r \in \mathbf{N})$.

$\mathrm{M}_2$. $\|\alpha v\| \leq \|\alpha\|\|v\|$ and $\|v\alpha\| \leq \|v\|\|\alpha\|$ for $v \in \mathbf{M}_n(V)$, $\alpha \in \mathbf{M}_n$.

Given $1 \leq p < \infty$, we say that $V$ is an *$L^p$-matricially normed space* if in addition we have that

$\mathrm{L}_p$. $\|v \oplus w\| = (\|v\|^p + \|w\|^p)^{1/p}$,

and that it is an *$L^\infty$-matricially normed space* if

$L_\infty$. $\|v \oplus w\| = \max\{\|v\|, \|w\|\}$.

Given a matricially normed space $V$ and a unitary $\alpha \in \mathbf{M}_n$, we have from $\mathrm{M}_2$ that $\|\alpha v\| = \|v\| = \|v\alpha\|$. In particular, row and column operations on matrices in $\mathbf{M}_n(V)$ are isometric. It follows that if $p, q \leq n$, then any of the embeddings of $\mathbf{M}_{p,q}(V)$ in $\mathbf{M}_n(V)$ obtained by letting $v \mapsto v'$, where $v'$ vanishes on $n - p$ and $n - q$ specified rows and columns, determines the same norm on $\mathbf{M}_{p,q}(V)$. In this sense we have norms provided on rectangular matrices over $V$. The relations $\mathrm{M}_1$ and $\mathrm{M}_2$ continue to hold for suitable rectangular matrices of elements of $V$ and of $\mathbf{C}$.

In any matricially normed space $V$, we have that if $v = [v_{ij}] \in \mathbf{M}_n(V)$,

$$\|v_{ij}\| \leq \|v\| \leq \sum \|v_{ij}\| \tag{2.1}$$

(see **[11]**). It readily follows that if $V$ is complete, then the same is true for each of the spaces $\mathbf{M}_n(V)$. Given a $2 \times 2$ matrix of matrices $v = [v_{ij}]$, it is also shown in **[11]** that

$$\|v_{11} \oplus v_{22}\| \leq \|[v_{ij}]\|. \tag{2.2}$$

Given a linear map $\varphi\colon V \to W$ of matricially normed spaces, we define a map $\varphi_{m,n} = \varphi \otimes \mathrm{id}\colon \mathbf{M}_{m,n}(V) \to \mathbf{M}_{m,n}(W)$ by $\varphi_{m,n}([v_{ij}]) = [\varphi(v_{ij})]$, and we let $\varphi_n = \varphi_{n,n}$. We define

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbf{N}\},$$

and we say that $\varphi$ is *completely bounded* if $\|\varphi\|_{cb} < \infty$, a *complete contraction* if $\|\varphi\|_{cb} \leq 1$, and a *complete isometry* or *complete quotient map* if each $\varphi_n$ is isometric, or a quotient map, respectively. It is immediate that if $p \geq \max\{m, n\}$, $\|\varphi_{m,n}\| \leq \|\varphi_p\|$. We let $\mathscr{M}(V, W)$ denote the *normed vector space* of all completely bounded linear maps $\varphi : V \to W$ together with the corresponding norm $\| \ \|_{cb}$. It should be noted that we do not attempt to place a matricial norm structure on $\mathscr{M}(V, W)$ in this paper.

As we remarked in §1, any operator space $V$ is provided with matrix norms, namely, we may let $\mathbf{M}_n(V)$ have the relative norm in $\mathbf{M}_n\mathscr{B}(H) \cong \mathscr{B}(H^n)$. It is trivial that with these norms $V$ is an $L^\infty$-matricially normed space.

If $W$ is a vector subspace of a matricially normed space $V$, then identifying $\mathbf{M}_n(W)$ with a subspace of $\mathbf{M}_n(V)$, the relative norms determine a matricial norm structure on $W$, and we say that *$W$ is a matricially normed subspace of $V$*.

If $V$ is matricially normed, then regarding $\mathbf{M}_n(V^*)$ as the dual of $\mathbf{M}_n(V)$ under the pairing

$$\langle [v_{ij}], [f_{ij}] \rangle = \sum f_{ij}(v_{ij}),$$

$V^*$ is matricially normed by the dual norms on $\mathbf{M}_n(V^*)$ (see **[11]**). We call $V^*$ the *dual matricially normed space* of $V$. If $V \subseteq \mathscr{B}(H)$ is an operator space, and we are given $\xi, \eta \in H^n$, then for any $v \in \mathbf{M}_n(V)$,

$$\omega_{\xi,\eta}(v) = v\xi \cdot \eta = \sum v_{ij}\xi_j \cdot \eta_i = \sum \omega_{\xi_j,\eta_i}(v_{ij}),$$

hence with the above convention,

$$\omega_{\xi,\eta} = [\omega_{\xi_j,\eta_i}]. \tag{2.3}$$

It will be convenient to regard the dual of $\mathbf{M}_n$ as also being the predual, and we denote it by $\mathbf{M}_{n^*}$.

Finally if $W$ is a closed subspace of a matricially normed space $V$, then it follows from (2.1) that $\mathbf{M}_n(W)$ is closed in $\mathbf{M}_n(V)$. Identifying $\mathbf{M}_n(V/W)$ with $\mathbf{M}_n(V)/\mathbf{M}_n(W)$, we may let $\mathbf{M}_n(V/W)$ have the quotient norm for each $n$. It is readily verified that $V/W$ is thereby matricially normed. This is called the *quotient matricially normed space.* If $V$ is an $L^p$-matricially normed space then subspaces and quotients again have that property (for the latter, see (2.2)). In general we have that a map $\varphi\colon V \to W$ is a complete quotient map if and only if it induces a complete isometry of $V/\ker\varphi$ onto $W$.

THEOREM 2.1. *Given a normed vector space $V$, the norms on the matrix spaces $V \otimes_\lambda \mathbf{M}_n$ determine an $L^\infty$-matricial structure on $V$, whereas the norms on the matrix spaces $V \hat{\otimes} \mathbf{M}_{n^*}$ determine an $L^1$-matricial structure on $V$.*

*Proof*. To prove the first assertion, we embed $V$ into $l^\infty(X)$ for some set $X$, and then find a faithful representation of $l^\infty(X)$

on a Hilbert space $H$. This provides $l^\infty(X)$ with an $L^\infty$-matricial structure, and we let $V$ have the relative structure. The matrix norms for $l^\infty(X)$ are determined by the isometries:

$$\mathbf{M}_n(l^\infty(X)) \cong l^\infty(X) \otimes_\lambda \mathbf{M}_n$$

(see **[13]**, Th. IV. 4.14). Since the maps $V \otimes_\lambda \mathbf{M}_n \hookrightarrow l^\infty(X) \otimes_\lambda \mathbf{M}_n$ are isometric, the relative $L^\infty$-matricial structure is given by the isometries:

$$\mathbf{M}_n(V) \cong V \otimes_\lambda \mathbf{M}_n.$$

For the second, we let $X$ be the open unit ball of $V$, and we define $l(X) \subseteq l^1(X)$ to be the functions on $X$ vanishing off finitely many points, together with the $l^1$ norm. Letting $\delta(x)$ denote the characteristic function of the singleton $\{x\}$, the map

$$\theta : l(X) \to V : \sum \alpha_k \delta(x_k) \mapsto \sum \alpha_k x_k$$

is a quotient map of normed vector spaces. Representing $l^\infty(X)$ as a von Neumann algebra on a Hilbert space $H$, it follows that $l^1(X)$ is then isometric to the predual of this von Neumann algebra, and thus $l(X)$ inherits a corresponding $L^1$-matricial norm structure. We let $V \cong l(X)/\ker\theta$ have the quotient $L^1$-matricial structure. The matrix norms on $l(X)$ are determined by the isometries

$$\mathbf{M}_n(l(X)) \cong l(X)\hat{\otimes}\mathbf{M}_{n^*}$$

(see **[13]** Th. IV.7.17, and p. 261). Since

$$l(X)\hat{\otimes}\mathbf{M}_{n^*} \to V\hat{\otimes}\mathbf{M}_{n^*}$$

is a quotient map, the matricial norm on $V$ are given by the isometries

$$\mathbf{M}_n(V) \cong V\hat{\otimes}\mathbf{M}_{n^*}. \qquad \square$$

The following results were proved in **[11]**:

THEOREM 2.2. *Suppose that $V$ is matricially normed. Then*

(1) *$V$ is completely isometric to an operator space if and only if it is an $L^\infty$-matricially normed space.*

(2) *$V$ is an $L^1$-matricially normed space if and only if $V^*$ is an $L^\infty$-matricially normed space.* □

In light of (1), we shall also refer to $L^\infty$-matricially normed spaces as *abstract operator spaces*, or more simply, *operator spaces.*

COROLLARY 2.3. *Suppose that $V$ is a dual $L^1$-matricially normed space. Then it is a quotient of $\mathscr{B}(H)_*$ for some Hilbert space $H$.*

*Proof*. We may assume that $V = (V_*)^*$ for some matricially normed space $V_*$. Since the canonical map $j\colon V_* \to V^*$ is completely isometric, the dual $j^*\colon V^{**} \to V$ is a complete quotient map. From Theorem 2.2, $V^*$ is an $L^\infty$-matricially normed space, and there is a complete isometry $\varphi\colon V^* \to \mathscr{B}(K)$ for some Hilbert space $K$. Thus $\varphi^*\colon \mathscr{B}(K)^* \to V^{**}$ is a quotient map. We have that $\mathscr{B}(K)^* = \mathscr{R}_*$, where $\mathscr{R} = \mathscr{B}(K)^{**}$ is a von Neumann algebra on some Hilbert space $H$. Since $\mathscr{R}$ is weak* closed in $\mathscr{B}(H)$, $\mathscr{R}_* \cong \mathscr{B}(H)_*/\mathscr{R}_\perp$, where $\mathscr{R}_\perp$ is the annihilator of $\mathscr{R}$. The composition of these quotient maps gives us a quotient map $\mathscr{B}(H)_* \to V$. □

It was shown in **[11]** that if $A$ and $B$ are $C^*$-algebras, then there are no completely bounded maps from $A$ into $B^*$. This is even the case if $A = B = \mathbf{C}$. In the reverse direction we have:

THEOREM 2.4. *Suppose $V$ is an $L^1$-matricially normed space and $W$ is an operator space. Then for any linear map $\varphi\colon V \to W$ we have that*

$$\|\varphi\|_{cb} = \|\varphi\|.$$

*Proof*. It suffices to show that if $\|\varphi\| < \infty$, then for each $n \in \mathbf{N}$, $\|\varphi_n\| \leq \|\varphi\|$. Let us first assume that $V = \mathscr{B}(H)_*$ for some Hilbert space $H$, and that $W \subseteq \mathscr{B}(K)$. Each function $f \in \mathscr{B}(H)_*$ with $\|f\| \leq 1$ is a norm limit of convex combinations of functions of the form $\omega_{\xi,\eta}$ (see **[3]** §I.3.3, proof of Lemma 3). We may apply this to $\mathbf{M}_n(\mathscr{B}(H)_*) = [\mathbf{M}_n(\mathscr{B}(H))]_*$. Given vectors $\theta, \zeta \in K^n$, the function $f \mapsto |\varphi_n(f)\theta \cdot \zeta|$ is convex on the unit ball of $\mathbf{M}_n(\mathscr{B}(H)_*)$, and thus assumes its maximum on functions of the form $\omega_{\xi,\eta}$, $\xi, \eta \in H^n$:

$$\|\varphi_n\| = \sup\{|\varphi_n(\omega_{\xi,\eta})\theta \cdot \zeta|\colon \|\xi\|, \|\eta\|, \|\theta\|, \|\zeta\| \leq 1, \quad \xi, \eta \in H^n, \theta, \zeta \in K^n\}.$$

Using (2.3), we have

$$\begin{aligned}|\varphi_n(\omega_{\xi,\eta})\theta\cdot\zeta| &= |[\varphi(\omega_{\xi_j,\eta_i})](\theta_j)\cdot(\zeta_i)| \\ &= \left|\sum_{i,j}\varphi(\omega_{\xi_j,\eta_i})\theta_j,\zeta_i\right| \\ &\leq q\|\varphi\|\sum_{i,j}\|\xi_j\|\|\eta_i\|\|\theta_j\|\|\zeta_i\| \\ &= \|\varphi\|\left[\sum\|\xi_j\|\|\theta_j\|\right]\left[\sum\|\eta_i\|\|\zeta_i\|\right] \\ &\leq \|\varphi\|\left[\sum\|\xi_i\|^2\right]^{1/2}\left[\sum\|\theta_i\|^2\right]^{1/2}\left[\sum\|\eta_i\|^2\right]^{1/2}\left[\sum\|\zeta_i\|^2\right]^{1/2} \\ &= \|\varphi\|\|\xi\|\|\theta\|\|\eta\|\|\zeta\|,\end{aligned}$$

and thus $\|\varphi_n\| \leq \|\varphi\|$.

Now let us suppose that $V$ is a dual $L^1$-matricially normed space. From Corollary 2.3 there is a quotient map $\pi\colon \mathscr{B}(H)_* \to V$ in the category of matricially normed spaces, i.e., each of the maps $\pi_n\colon \mathbf{M}_n(\mathscr{B}(H)_*) \to \mathbf{M}_n(V)$ is a quotient map. Since $\pi_n$ maps the open unit ball onto the open unit ball of the image, it follows that

$$\|\varphi_n\| = \|\varphi_n\circ\pi_n\| = \|(\varphi\circ\pi)_n\| = \|\varphi\circ\pi\| = \|\varphi\|.$$

Finally suppose that $V$ is a general $L^1$-matricially normed space. Since $V^{**}$ is a dual $L^1$-matricially normed space, the map $\varphi^{**}\colon V^{**}\to W^{**}$ satisfies $\|\varphi^{**}\|_{cb} = \|\varphi^{**}\|$. For any $n$ we have that $(\varphi^{**})_n = (\varphi_n)^{**}$, hence $\|(\varphi^{**})_n\| = \|\varphi_n\|$, and $\|\varphi\|_{cb} = \|\varphi^{**}\|_{cb} = \|\varphi^{**}\| = \|\varphi\|$. □

COROLLARY 2.5. *Given $L^1$-matricially normed spaces $V\subseteq W$, a completely bounded map $\varphi\colon V\to\mathbf{M}_2$ need not have an extension $\psi\colon W\to\mathbf{M}_2$ satisfying $\|\psi\|_{cb} = \|\varphi\|_{cb}$.*

*Proof*. $\mathbf{M}_2$ is not injective as a normed vector space since it is not isometric to $l_4^\infty$ (this is a consequence of [7], Th. 7), which is the only 4-dimensional injective normed vector space (see [8] §3.11, Th. 6). Thus we may find normed vector spaces $V\subseteq W$ and a map $\varphi\colon V\to\mathbf{M}_2$ which does not have an extension $\psi\colon W\to\mathbf{M}_2$ with $\|\psi\| = \|\varphi\|$. From above we may extend the norm structure on $W$ to an $L^1$-matricial norm structure. Letting $V$ have the relative matricial norm structure, $V$ and $W$ are $L^1$-matricial norm spaces. Since $\|\varphi\|_{cb} = \|\varphi\|$ and $\|\psi\|_{cb} = \|\psi\|$, we cannot find an extension $\psi\colon W\to\mathbf{M}_2$ with $\|\psi\|_{cb} = \|\varphi\|_{cb}$. □

It should be noted that in the Proof of Corollary 2.5, although $M_n(W) \cong W\hat{\otimes}M_{n*}$ we cannot conclude that $M_n(V) \cong V\hat{\otimes}M_{n*}$.

**3. The Hahn-Banach Theorem for operator spaces.** Consider the action of $\mathbf{M}_p(\mathbf{M}_n)$ on $(\mathbf{C}^n)^p$. Under the identification of $\mathbf{M}_n \otimes \mathbf{M}_p \cong \mathbf{M}_p(\mathbf{M}_n)$ we have that

$$\alpha \otimes I_p = \sum_i \alpha \otimes \varepsilon_{ii} = \begin{bmatrix} \alpha \cdots 0 \\ 0 \cdots \alpha \end{bmatrix},$$

$$I_n \otimes \beta = \sum_{ij} \beta_{ij} I_n \otimes \varepsilon_{ij} = \begin{bmatrix} \beta_{11} I_n \cdots \beta_{1p} I_n \\ \beta_{p1} I_n \cdots \beta_{pp} I_n \end{bmatrix}.$$

Thus given $\xi \in (\mathbf{C}^n)^p$,

$$(\alpha \otimes I_p(\xi))_i = \alpha \xi_i, \qquad (I_n \otimes \beta(\xi))_i = \sum_j \beta_{ij} \xi_j.$$

The following result is due to R. Smith [**12**]. We have included a proof that may be more familiar to operator algebraists, and that has an obvious extension to finite von Neumann algebras.

LEMMA 3.1. *Suppose that $p \geq n$, and the $\xi_0$ is a vector in $(\mathbf{C}^n)^p$. Then there exists a unitary matrix $U \in \mathbf{M}_p$ and a vector $\xi \in (\mathbf{C}^n)^n$ such that $(I_n \otimes U)\xi_0 = \xi \oplus o_{p-n}$.*

*Proof*. The representation of $\alpha \mapsto \alpha \otimes I_n$ of $\mathbf{M}_n$ on $(\mathbf{C}^n)^n$ has a separating vector, and thus any state on $\mathbf{M}_n$ is a vector state in this representation, i.e., it has the form $\omega_\xi$ for some vector $\xi \in (\mathbf{C}^n)^n$. it follows that given an arbitrary vector $\xi_0 \in (\mathbf{C}^n)^p = (\mathbf{C}^n)^n \oplus (\mathbf{C}^n)^{p-n}$, we choose $\xi \in (\mathbf{C}^n)^n$ with $\omega_{\xi_0,\xi_0} = \omega_{\xi\oplus 0,\xi\oplus 0}$ on $\mathbf{M}_n$. From elementary operator algebra theory, the map $(\alpha \otimes I_p)(\xi_0) \mapsto (\alpha \otimes I_p)(\xi \oplus 0)$ extends to a partial isometry $U_0'$ of $\mathbf{M}_n \otimes I_p(\xi_0)$ onto $\mathbf{M}_n \otimes I_p(\xi \oplus 0)$, which lies in $(\mathbf{M}_n \otimes I_p)'$. Letting $E'$ and $F'$ be the domain and range projections of $U_0'$, $E'$ and $F'$ are equivalent in $(\mathbf{M}_n \otimes I_p)'$, and thus the same is true for $I - E'$ and $I - F'$. Adding a corresponding partial isometry to $U_0'$, we obtain a unitary $U_0$ in $(\mathbf{M}_n \otimes I_p)'$ with $U_0 \xi_0 = \xi \oplus 0$. Since $(\mathbf{M}_n \otimes I_p)' = I_n \otimes \mathbf{M}_p$, $U_0$ must have the form $I_n \otimes U$, with $U$ a unitary in $\mathbf{M}_p$. □

Generalizing [**12**], we have:

LEMMA 3.2. *Suppose that $V$ is a matricially normed space. Then given a linear map $\varphi\colon V \to \mathbf{M}_n$, we have that $\|\varphi\|_{cb} = \|\varphi_n\|$.*

*Proof*. Given $p > n$ and unit vectors $\xi, \eta \in (\mathbf{C}^n)^p$, we may choose unitary matrices $U, V \in \mathbf{M}_p$ and unit vectors $\xi'$, $\eta' \in (\mathbf{C}^n)^n$ such that $\xi = (I_n \otimes U)(\xi' \oplus o_{p-n})$ and $\eta = (I_n \otimes V)(\eta' \oplus o_{p-n})$. Letting $E = [I_n\ O]$ and $U' = I_n \otimes UE^*$, $V' = I_n \otimes VE^*$ (these are $p \times n$ matrices over $\mathbf{M}_n$), it follows that for $v \in \mathbf{M}_p(V)$ with $\|v\| \leq 1$,

$$\begin{aligned}|\varphi \otimes I_p(v)\xi \cdot \eta| &= |\varphi \otimes I_n(V'^* v U')\xi' \cdot \eta'| \\ &\leq \|\varphi_n\| \|V'^* v U'\| \|\xi'\| \|\eta'\| \leq \|\varphi_n\|. \qquad \square\end{aligned}$$

COROLLARY 3.3. *If $V$ is matricially normed and $f \in V^*$, then $\|f\|_{cb} = \|f\|$.*

Given a matricially normed space $V$ an operator space $W$ and elements $f \in V^*$ and $w \in W$, we define a linear map $\theta_{f\otimes w}\colon V \to W$ by $\theta_{f\otimes w}(v) = f(v)w$.

COROLLARY 3.4. *Given a matricially normed space $V$, and operator space $W$ and elements $f \in V^*$, $w \in W$ we have that $\|\theta_{f\otimes w}\|_{cb} \leq \|f\|\|w\|$.*

*Proof*. We have that if $v \in \mathbf{M}_n(V)$, $\|v\| \leq 1$,

$$\begin{aligned}\|(\theta_{f\otimes w})_n(v)\| &= \|[f(v_{ij})w]\| = \|f_n(v)(w \oplus \cdots \oplus w)\| \\ &\leq \|f_n(v)\| \|w \oplus \cdots \oplus w\| \leq \|f\|\|w\|. \qquad \square\end{aligned}$$

Let us suppose that $V$ is a matricially normed space. We have a natural pairing between $\mathcal{M}(V, \mathbf{M}_n)$ and the algebraic tensor product $V \otimes \mathbf{M}_{n^*}$ defined by $\langle \varphi, v \otimes g\rangle = g(\varphi(g))$. In fact $\mathcal{M}(V, \mathbf{M}_n)$ is in this manner isometric to the dual of $V \otimes \mathbf{M}_{n^*}$ when the latter is provided with a suitable norm $\| \ \|_{\mathcal{M}}$ described below.

Given $v \in \mathbf{M}_n(V)$ and $g \in \mathbf{M}_n(\mathbf{M}_{n^*})$, we define $v \times g \in V \otimes \mathbf{M}_{n^*}$ by

$$v \times g = \sum v_{ij} \otimes g_{ij}.$$

A simple calculation shows that for any $\alpha \in \mathbf{M}_{m,n}$, $\beta \in \mathbf{M}_{n,m}$, $v \in \mathbf{M}_n(V)$ and $g \in \mathbf{M}_m(\mathbf{M}_{n^*})$, we have

$$(\alpha v \beta) \times g = v \times (\alpha^{\mathrm{tr}} g \beta^{\mathrm{tr}}), \tag{3.1}$$

where tr indicates the transposed matrix. For any $\varphi \in \mathcal{M}(V, \mathbf{M}_n)$,

$$\langle \varphi_n(v), g\rangle = \sum \langle \varphi(v_{ij}), g_{ij}\rangle = \langle \varphi, v \times g\rangle,$$

and thus

$$(3.2)\quad \begin{aligned} \|\varphi\|_{cb} &= \|\varphi_n\| \\ &= \sup\{|\langle\varphi_n(v), g\rangle| : \|v\|, \|g\| \le 1, v \in \mathbf{M}_n(V), g \in \mathbf{M}_n(\mathbf{M}_{n^*})\} \\ &= \sup\{|\langle\varphi, v \times g\rangle| : \|v\|, \|g\| \le 1, v \in \mathbf{M}_n(V), g \in \mathbf{M}_n(\mathbf{M}_{n^*})\}. \end{aligned}$$

We define the *completely bounded norm* $\| \ \|_{\mathscr{M}}$ on $V \otimes \mathbf{M}_{n^*}$ by

$$\|F\|_{\mathscr{M}} = \inf\left\{\sum \|v_i\|\|g_i\| : F = \sum v_i \times g_i, v_i \in \mathbf{M}_n(V), g_i \in \mathbf{M}_n(\mathbf{M}_{n^*})\right\}.$$

It is evident that this is a seminorm. To see that $\| \ \|_{\mathscr{M}}$ is a norm, let us suppose that $F = \sum w_i \otimes h_i \neq 0$, $w_i \in V$, $h_i \in \mathbf{M}_{n^*}$. We may assume that the $h_i$ form a basis for $\mathbf{M}_{n^*}$, and that $w_1 \otimes h_1 \neq 0$. Letting $\beta_i \in \mathbf{M}_n$ be a dual basis for the $h_i$, and choosing $f \in V^*$ with $f(w_1) \neq 0$, we have that $\langle\theta_{f\otimes\beta_1}, F\rangle = f(w_1) \neq 0$. Since we have

$$|\langle\theta_{f\otimes\beta_1}, F\rangle| \le \|\theta_{f\otimes\beta_1}\|_{cb}\|F\|_{\mathscr{M}},$$

it follows that $\|F\|_{\mathscr{M}} \neq 0$.

We let $V \otimes_{\mathscr{M}} \mathbf{M}_{n^*}$ denote $V \otimes \mathbf{M}_{n^*}$ with the completely bounded norm, and we call it the *completely bounded tensor product*. It follows from (3.2) that if $\varphi \in \mathscr{M}(V, \mathbf{M}_n)$ and $F \in V \otimes_{\mathscr{M}} \mathbf{M}_{n^*}$ we have

$$(3.3)\qquad \|\varphi\|_{cb} = \sup\{|\langle\varphi, F\rangle| : \|F\|_{\mathscr{M}} \le 1, F \in V \otimes_{\mathscr{M}} \mathbf{M}_{n^*}\},$$

and thus

$$\mathscr{M}(V, \mathbf{M}_n) \cong [V \otimes_{\mathscr{M}} \mathbf{M}_{n^*}]^*.$$

We say that a function $g = [g_{ij}] \in \mathbf{M}_n(\mathbf{M}_{n^*})$ is *non-singular* if the $g_{ij}$ are linearly independent functions in $\mathbf{M}_{n^*}$.

LEMMA 3.5. *Given $g \in \mathbf{M}_n(\mathbf{M}_{n^*})$ with $g = \omega_{\xi,\eta}$ where $\xi, \eta \in (\mathbf{C}^n)^n$, $g$ is non-singular if and only if $\xi$ and $\eta$ are cyclic vectors for $\mathbf{M}_n \otimes I_n$.*

*Proof.* $g$ will be non-singular if and only if the entries $g_{ij} = \omega_{\xi_j\eta_i}$ span $\mathbf{M}_{n^*}$, or equivalently, $g_{ij}(\alpha) = 0$ for all $i, j$ implies that $\alpha = 0$. Thus $g$ is non-singular if and only if

$$\alpha\xi_j \cdot \eta_i = 0 \qquad \text{for all } i, j \Rightarrow \alpha = 0.$$

It is clear that a necessary and sufficient condition for this is that the $\xi_j$ and the $\eta_i$ span $\mathbf{C}^n$. The $\xi_j$ are linearly independent if and only if given any $n$ vectors $\zeta_j$, there is a matrix $\beta \in \mathbf{M}_n$ with $\beta\xi_j = \zeta_j$, or letting

$\zeta = (\zeta_1, \ldots, \zeta_n)$ be the corresponding vector in $(\mathbf{C}^n)^n$, $\zeta = (\beta \otimes I_n)\xi$. Thus the $\xi_j$ span $\mathbf{C}^n$ if and only if $\xi$ is cyclic for $\mathbf{M}_n \otimes I_n$. Since the same applies to $\eta$, we are done. □

From the definition of the norm, if $F \in V \otimes_{\mathcal{M}} \mathbf{M}_{n^*}$ satisfies $\|F\| < 1$, then there is a finite sum representation

$$F = \sum t_i v_i \times g_i, \qquad (v_i \in \mathbf{M}_n(V), g_i \in \mathbf{M}_n(\mathbf{M}_{n^*})) \tag{3.4}$$

where $\|v_i\| < 1$, $\|g_i\| = 1$, $t_i \geq 0$, and $\sum t_i = 1$. The following may be regarded as the major distinguishing feature of the operator spaces. It was first proved by Haagerup [**6**] using a rather different argument.

LEMMA 3.6. *Suppose that $V$ is an operator space and that $F \in V \otimes \mathbf{M}_{n^*}$ satisfies $\|F\| < 1$. Then $F = v \times g$ where $v \in \mathbf{M}_n(V)$ and $g \in \mathbf{M}_n(\mathbf{M}_{n^*})$ satisfy $\|v\| < 1$, $\|g\| = 1$. Furthermore, $g$ may be chosen non-singular.*

*Proof*. We may assume that $F$ has the representation (3.4). Since any function $g_i \in \mathbf{M}_n(\mathbf{M}_{n^*})$ with $\|g_i\| = 1$ is a convex combination of functionals of the form $\omega_{\xi_j, \eta_j}$, where $\xi_j, \eta_j$ are unit vectors in $(\mathbf{C}^n)^n$, $F$ has the representation

$$F = \sum_{k=1}^{p} t_k (v_k \times \omega_{\xi_k, \eta_k}),$$

with $\xi_k, \eta_k$ unit vectors in $(\mathbf{C}^n)^n$, $v_k \in \mathbf{M}_n(V)$ satisfying $\|v_k\| < 1$, $t_k \geq 0$, and $\sum t_k = 1$. Since $V$ is an operator space, $\tilde{v} = v_1 \oplus \cdots \oplus v_p \in \mathbf{M}_{np}(V)$ satisfies $\|\tilde{v}\| < 1$. We have that

$$F = \tilde{v} \times \omega_{\xi_0 \eta_0},$$

where

$$\xi_0 = (t_1^{1/2}\xi_1) \oplus \cdots \oplus (t_p^{1/2}\xi_p), \qquad \eta_0 = (t_1^{1/2}\eta_1) \oplus \cdots \oplus (t_p^{1/2}\eta_p)$$

are unit vectors in $(\mathbf{C}^n)^{np}$. From Lemma 3.1, we may choose unitary matrices $U, V \in \mathbf{M}_{np}$ and unit vectors $\xi, \eta \in (\mathbf{C}^n)^n$ such that

$$\xi_0 = (I_n \otimes U)\xi', \quad \xi' = \xi \oplus 0 \in (\mathbf{C}^n)^n \oplus (\mathbf{C}^n)^{np-n},$$
$$\eta_0 = (I_n \otimes V)\eta', \quad \eta' = \eta \oplus 0 \in (\mathbf{C}^n)^n \oplus (\mathbf{C}^n)^{np-n}.$$

It follows that

$$\omega_{\xi_0 \eta_0} = \overline{V} \omega_{\xi' \eta'} U^{\mathrm{tr}}$$

since $(\xi_0)_j = \sum_{l=1}^{np} U_{jl}\xi'_l$, and $(\eta_0)_i = \sum_{k=1}^{np} V_{ik}\eta'_k$ imply that

$$(\omega_{\xi_0\eta_0})_{ij} = \sum_{k,l=1}^{np} U_{jl}\overline{V}_{ik}\omega_{\xi'_l,\eta'_k}.$$

But as elements of $\mathbf{M}_{n^*}$, we have that $\omega_{\xi'_j,\eta'_i} = \omega_{\xi_j,\eta_i}$ for $i, j \leq n$, and otherwise they are zero. Thus

$$\omega_{\xi',\eta'} = E^*\omega_{\xi,\eta}E$$

where $E = [I_n\ O_{p-n}]$, and we have from (3.1),

$$F = \tilde{v} \times \overline{V}E^*\omega_{\xi,\eta}EU^{\text{tr}} = EV^*\tilde{v}UE^* \times \omega_{\xi,\eta} = v \times \omega_{\xi,\eta},$$

where $v = EV^*\tilde{v}UE^* \in \mathbf{M}_n(V)$ satisfies $\|v\| \leq \|\tilde{v}\| < 1$.

Our next task is to perturb $\xi, \eta \in (\mathbf{C}^n)^n$ so that the resulting linear function $\omega_{\xi,\eta} = [\omega_{\xi_j,\eta_i}]$ is non-singular, i.e., so that both $\xi$ and $\eta$ are cyclic for $\mathbf{M}_n \otimes I$. Letting $e_i$ be the canonical basis for $\mathbf{C}^n$, we have that $e = (e_1, \ldots, e_n)$ is cyclic for $\mathbf{M}_n \otimes I$. Given $t$ with $0 \leq t < 1$, we let $v_t = (1-t)^{-2}v$ and

$$\xi_t = (1-t)\xi \oplus te, \qquad \eta_t = (1-t)\eta \oplus te,$$

which are vectors in $(\mathbf{C}^n)^{2n}$. We then have that

$$F = (1-t)^2(v_t \times \omega_{\xi,\eta}) + t^2(0 \times \omega_{e,e}) = (v_t \oplus 0) \times \omega_{\xi_t,\eta_t}.$$

Fixing $t$, we have from Lemma 3.1 that there exist unitaries $U, V \in \mathbf{M}_{2n}$ such that $(I \otimes U)\xi_t = \xi' \oplus 0$ and $(I \otimes V)\eta_t = \eta' \oplus 0$, and thus as above

$$F = w \times \omega_{\xi',\eta'}$$

where $\|w\| \leq \|v_t\|$. For small $t$, we will have that $\|\xi_t\|$, $\|\eta_t\|$ are close to 1 and $\|v_t\|$ is close to $\|v\|$. It thus suffices to show that we can choose arbitrarily small $t > 0$ with $\xi', \eta'$ cyclic for $\mathbf{M}_n \otimes I_n$, since even after we normalize $\xi'$ and $\eta'$ to be unit vectors, absorbing the constants into $w$, we will still have that $\|w\| < 1$. $\xi'$ will be cyclic for $\mathbf{M}_n \otimes I_n$ if and only if the $n^2$ vectors $\varepsilon_{ij} \otimes I_{2n}(\xi' \oplus 0)$ span an $n^2$ dimensional space, or since $I_n \otimes U$ commutes with the action of $\mathbf{M}_n \otimes I_{2n}$, if and only if the vectors $\varepsilon_{ij} \otimes I_{2n}\xi_t$ are linearly independent in $(\mathbf{C}^n)^{2n}$. Using exterior products, this is equivalent to

$$R(t) = (\varepsilon_{11} \otimes I_{2n}\xi_t) \wedge \cdots \wedge (\varepsilon_{nn} \otimes I_{2n}\xi_t) \neq 0.$$

Given a basis $d_k$ for $(\mathbf{C}^n)^{2n}$, the exterior products

$$d_K = d_{k_1} \wedge \cdots \wedge d_{k_{n^2}}$$

for $n^2$-tuples $K = (k_1, \ldots k_{n^2})$ with $k_1 < \cdots < k_{n^2}$, form a basis for $\bigwedge^{n^2}(\mathbf{C}^n)^{2n}$. Thus we have unique coefficients $P_K(t)$ with

$$R(t) = \sum_K P_K(t) d_K.$$

Letting $t$ vary in $\mathbf{R}$, the $P_K(t)$ are polynomials in $t$. Since the vectors $(\varepsilon_{ij} \otimes I_n)e$ $(1 \leq i, j \leq n^2)$ are linearly independent, the same is true for the vectors $0 \oplus (\varepsilon_{ij} \otimes I_n)e$ $(1 \leq i, j \leq n^2)$, and

$$\begin{aligned} R(1) &= (\varepsilon_{11} \otimes I_{2n}(0 \oplus e)) \wedge \cdots \wedge (\varepsilon_{nn} \otimes I_{2n}(0 \oplus e)) \\ &= (0 \oplus (\varepsilon_{11} \otimes I_n)e) \wedge \cdots \wedge (0 \oplus (\varepsilon_{nn} \otimes I_n)e) \neq 0. \end{aligned}$$

Thus there is a coefficient $K$ with $P_K \neq 0$. It must have only finitely many zeros, and thus we have the desired situation. The same argument applies to $\eta$. □

THEOREM 3.7 **[14]** **[9]**. *Suppose that $V \subseteq W$ are operator spaces, and that $A$ is an injective $C^*$-algebra. Then any complete contraction $\varphi\colon V \to A$ has a completely contractive extension $\psi\colon W \to A$.*

*Proof*. First let us suppose that $A = \mathbf{M}_n$. We wish to show that the restriction map

$$\rho\colon \mathscr{M}(W, \mathbf{M}_n) \to \mathscr{M}(V, \mathbf{M}_n)$$

sends the closed unit ball of $\mathscr{M}(W, \mathbf{M}_n)$ onto that of $\mathscr{M}(V, \mathbf{M}_n)$. It suffices to show that the inclusion map of the preduals

$$V \otimes_{\mathscr{M}} \mathbf{M}_{n^*} \hookrightarrow W \otimes_{\mathscr{M}} \mathbf{M}_{n^*} \tag{3.5}$$

is isometric. It is clear that this map is norm decreasing. Given $F \in V \otimes_{\mathscr{M}} \mathbf{M}_{n^*}$, we denote its norms in the latter and in $W \otimes_{\mathscr{M}} \mathbf{M}_{n^*}$ by $\|F\|_V$ and $\|F\|_W$, respectively, let us suppose that $\|F\|_W < 1$. From Lemma 3.6 we may choose elements $w \in \mathbf{M}_n(W)$ and $g \in \mathbf{M}_n(\mathbf{M}_{n^*})$ such that $\|w\| < 1$, $\|g\| = 1$, and $g$ is non-singular with $F = w \times g$. Since $g$ is non-singular, the map

$$\theta_g\colon \mathbf{M}_n(W) \to W \otimes \mathbf{M}_{n^*}\colon w = \sum w_{ij} \otimes \varepsilon_{ij} \mapsto w \times g = \sum w_{ij} \otimes g_{ij}$$

is an isomorphism. Thus since $\theta_g(w) = F \in V \otimes \mathbf{M}_{n^*} = \theta_g(\mathbf{M}_n(V))$, we must have that $w \in \mathbf{M}_n(V)$. Noting that $\mathbf{M}_n(V) \hookrightarrow \mathbf{M}_n(W)$ is assumed isometric, we conclude that $\|F\|_V \leq \|w\|\|g\| < 1$.

Given an arbitrary injective $C^*$-algebra $A \subseteq \mathscr{B}(H)$, there is a complete contraction $\Phi\colon \mathscr{B}(H) \to A$. Since we may compose this with a map of $W$ into $\mathscr{B}(H)$, it suffices to consider the case $A = \mathscr{B}(H)$. If $e$

is a projection with $\dim eH = n$, then $e\mathscr{B}(H)e \cong \mathbf{M}_n$. Thus letting $\mathscr{L}$ be the family of finite dimensional spaces $L \subseteq H$, and letting $e_L$ be the projection onto $L$, we may choose for each $L \in \mathscr{L}$ a completely contractive extension $\psi_L: W \to e_L\mathscr{B}(H)e_L$ of $e_L\varphi e_L: V \to e_L\mathscr{B}(H)e_L$. $\mathscr{L}$ is a directed set under inclusion, hence $\{\psi_L: L \in \mathscr{F}\}$ may be regarded as a net in the unit ball of $\mathscr{M}(W, \mathscr{B}(H))$. Since the latter is compact in the topology of point-weak* convergence, this net has a cluster point $\psi$. It is quickly seen that this is the desired extension of $\varphi$. □

**4. The Haagerup tensor product and bimodules.** Given matricially normed spaces $V$ and $W$, the *Haagerup tensor product* $V \otimes_h W$ is a matricially normed space, which is defined as follows (see **[4]**). Given $v \in \mathbf{M}_{n,p}(V)$ and $w \in \mathbf{M}_{p,n}(W)$, we define

$$v \odot w \in \mathbf{M}_n(V \otimes W)$$

by "matrix multiplication":

$$(v \odot w)_{ij} = \sum_k v_{ik} \otimes w_{kj}.$$

The *Haagerup norm* $\| \ \|_h$ on $\mathbf{M}_n(V \otimes W)$ is given by

$$\|u\|_h = \inf\left\{\sum_\nu \|v^\nu\|\|w^\nu\| : u = \sum_\nu v^\nu \odot w^\nu\right\},$$

where the sums are finite, and we choose $v^\nu \in \mathbf{M}_{n,p_\nu}(V)$, $w^\nu \in \mathbf{M}_{p_\nu,n}$.

LEMMA 4.1. *Given matricially normed spaces $V$ and $W$, the Haagerup norms determine a matricial norm structure on $V \otimes W$.*

*Proof*. Subadditivity is trivial. Let us suppose that $u \in \mathbf{M}_n(V \otimes W)$. If

$$u \oplus 0_q = \sum_\nu v^\nu \odot w^\nu, \qquad (v^\nu \in \mathbf{M}_{n+q,p_\nu}(V), w^\nu \in \mathbf{M}_{p_\nu,n+q}(W),$$

then letting $[I_n \ O]$,

$$u = \sum_\nu Ev^\nu \odot w^\nu E^*,$$

where $\|Ev^\nu\| \leq \|v^\nu\|$, and $\|w^\nu E^*\| \leq \|w^\nu\|$. It follows that $\|u\|_h \leq \|u \oplus 0\|_h$. The reverse inequality is trivial. Given $\alpha \in \mathbf{M}_n$, and $u = \sum_\nu v^\nu \odot w^\nu$, we have that $\alpha u = \sum_\nu (\alpha v^\nu) \odot w^\nu$, and thus $\|\alpha u\|_h \leq \|\alpha\| \sum_\nu \|v^\nu\|\|w^\nu\|$. It follows that $\|\alpha u\|_h \leq \|\alpha\|\|u\|_h$. Similarly, $\|u\alpha\| \leq \|u\|\|\alpha\|$.

Finally we have to show that the Haagerup norms indeed have no null space. Given $0 \neq u \in V \otimes W$, let us suppose that $f \in V^*$ and $g \in W^*$ satisfy $\|f\| = \|g\| = 1$. Then given any decomposition $u = \sum_\nu v^\nu \odot w^\nu$, $v^\nu \in \mathbf{M}_{1,p_\nu}(V)$, $w^\nu \in \mathbf{M}_{p_\nu,1}(W)$, we have that

$$|f \otimes g(u)| \leq \sum_\nu |f_{1,p_\nu}(v^\nu) \odot g_{p_\nu,1}(w^\nu)| \leq \sum_\nu \|v^\nu\| \|w^\nu\|,$$

since we have that $\|f\|_{cb} = \|f\|$ and $\|g\|_{cb} = \|g\|$. Thus $0 \neq \|u\|_\lambda \leq \|u\|_h$. In general given $0 \neq u \in \mathbf{M}_n(V \otimes W)$, let us assume that $u_{ij} \neq 0$. Then letting

$$E_i = [0 \cdots \underset{\substack{\uparrow \\ i}}{1} \cdots 0]$$

we have that

$$0 \neq \|u_{ij}\|_h = \|E_i u E_j^*\| \leq \|u\|_h. \qquad \square$$

In many respects the Haagerup tensor product is analogous to the projective tensor product for normed vector spaces (see §2). In particular, it is natural to define a bilinear map $\varphi\colon V \times W \to X$ to be *completely bounded* if the corresponding linear map $\varphi\colon V \otimes_h W \to X$ is completely bounded (see **[2]**). A convincing argument for this point of view was suggested to us by B. E. Johnson. He remarked that the theory of matricially normed spaces might be simplified if one instead considered normed modules over the infinite matrix algebra. We verify below that the corresponding functor transforms the Haagerup tensor product of two matricially normed spaces into the corresponding projective normed bimodule tensor product.

We define $\mathscr{F}$ to be the *-algebra of complex matrices with countably many rows and columns, having only finitely many non-zero entries. We may regard $\mathscr{F}$ as the inductive limit of the system of matrix algebras:

$$\mathbf{M}_1 \subseteq \mathbf{M}_2 \subseteq \cdots$$

where the connecting maps are given by $\alpha \to \alpha \oplus 0_1$. We let $e^n_{ij}$ denote the matrix units in $\mathbf{M}_n$, and $1_n$ the identity for $\mathbf{M}_n$. Since the usual operator norms are compatible, we may regard $\mathscr{F}$ as a normed *-algebra.

Given a normed algebra $\mathscr{A}$, a *normed $\mathscr{A}$-bimodule $V$* is an $\mathscr{A}$-bimodule with a norm satisfying $\|\alpha v\| \leq \|\alpha\| \|v\|$ and $\|v\alpha\| \leq \|v\| \|\alpha\|$ for $v \in V$ and $\alpha \in \mathscr{A}$. Given any matricially normed space $V$, we may identify $\mathscr{V} = V \otimes \mathscr{F}$ with the direct limit of the system of normed spaces

$$V \otimes \mathbf{M}_1 \subseteq V \otimes \mathbf{M}_2 \subseteq \cdots .$$

By condition $M_1$, the matricial norms are compatible, and thus determine a norm on $\mathscr{V} = V \otimes \mathscr{F}$. Regarding $\mathscr{V}$ as an $\mathscr{F}$-bimodule, it follows from $M_2$ that $\mathscr{V}$ is a normed $\mathscr{F}$-bimodule.

We say that an $\mathscr{F}$-bimodule $\mathscr{V}$ is *non-degenerate* if for each $v \in \mathscr{V}$, there is an $n \in \mathbf{N}$ such that $1_n v = v 1_n = v$. Given a non-degenerate $\mathscr{F}$-bimodule $\mathscr{V}$, we let $V = e_{11}\mathscr{V}e_{11}$. We have that $V \neq \{0\}$. To see this suppose that $0 \neq v \in \mathscr{V}$. Letting $n$ be such that $v = 1_n v 1_n$, it follows that there exist $i, j$ with $v' = e_{ii} v e_{jj} \neq 0$. Letting $v'' = e_{1i} v' e_{j1}$, we have that $v' = e_{i1} v'' e_{1j}$, and thus $v''$ is a non-zero element of $V$. It is a simple matter to verify that the map

$$v \mapsto \sum e_{1i} v e_{j1} \otimes e_{ij}$$

determines a bimodule isomorphism $\mathscr{V} \cong V \otimes \mathscr{F}$. If we assume that $\mathscr{V}$ is a non-degenerate normed $\mathscr{F}$-module, then we may let $1_n \mathscr{V} 1_n \cong V \otimes \mathbf{M}_n$ have the relative norm. $V$ is thereby a matricially normed space, and we see that the construction of the previous paragraph essentially gives all of the non-degenerate normed $\mathscr{F}$-bimodules.

If $\varphi \colon V \to W$ is a linear map, then the corresponding map $\varphi \otimes \mathrm{id} \colon V \otimes \mathscr{F} \to W \otimes \mathscr{F}$ is an $\mathscr{F}$-bimodule map, and the $\mathscr{F}$-bimodule maps $\Phi : V \otimes \mathscr{F} \to W \otimes \mathscr{F}$ are precisely those of the form $\varphi \otimes 1$. This is a consequence of the fact that $\Phi(e_{ij} v e_{kl}) = e_{ij}\Phi(v)e_{kl}$. Under this construction, *the complete contractions $\varphi$ correspond exactly to the contractive $\mathscr{F}$-bimodule maps $\Phi : V \otimes \mathscr{F} \to W \otimes \mathscr{F}$.*

Given non-degenerate normed $\mathscr{F}$-bimodules $\mathscr{V}$ and $\mathscr{W}$ we shall notationally identify the algebraic tensor product $\mathscr{V} \otimes \mathscr{W} = V \otimes \mathscr{F} \otimes W \otimes \mathscr{F}$ with $V \otimes W \otimes \mathscr{F} \otimes \mathscr{F}$. The projective tensor product $\mathscr{V} \hat{\otimes} \mathscr{W}$ (see §2—we do not complete) is then a normed $\mathscr{F}$-bimodule under the *external* operations $\alpha(v \otimes w) = \alpha v \otimes w$, and $(v \otimes w)\alpha = v \otimes w\alpha$. Letting

$$J = J(\mathscr{V} \otimes \mathscr{W}) = \left\{ \sum_i v_i \alpha_i \otimes w_i - v_i \otimes \alpha_i w_i \colon v_i \in \mathscr{V}, w_i \in \mathscr{W}, \alpha_i \in \mathscr{F} \right\},$$

we define the *projective $\mathscr{F}$-tensor product* by

$$\mathscr{V} \hat{\otimes}_{\mathscr{F}} \mathscr{W} = (\mathscr{V} \hat{\otimes} \mathscr{W})/\overline{J},$$

where the bar indicates the norm closure. Since $\overline{J}$ is a closed (external) $\mathscr{F}$-bimodule, the quotient is a normed non-degenerate $\mathscr{F}$-bimodule.

Letting $\mathscr{V} = V \otimes \mathscr{F}$ and $\mathscr{W} = W \otimes \mathscr{F}$ be two non-degenerate $\mathscr{F}$-bimodules, we define an external bimodule map

$$\Lambda : V \otimes \mathscr{F} \otimes W \otimes \mathscr{F} \to V \otimes W \otimes \mathscr{F}$$

by letting

$$\Lambda(v \otimes \alpha \otimes w \otimes \beta) = v \otimes w \otimes \alpha\beta.$$

In general, given $v \in \mathscr{V}$, $w \in \mathscr{W}$, we let $v \odot w = \Lambda(v \otimes w)$.

THEOREM 4.2. $J(\mathscr{V} \otimes \mathscr{W}) = \ker \Lambda$.

*Proof*. Since we have that

$$\Lambda(v \otimes \alpha\gamma \otimes w \otimes \beta) = \Lambda(v \otimes \alpha \otimes w \otimes \gamma\beta),$$

it is evident that $J = J(\mathscr{V} \otimes \mathscr{W}) \subseteq \ker \Lambda$.

Conversely if we are given $\Lambda(u) = 0$, where $u = \sum_{i=1}^{p} u_i \otimes \alpha_i \otimes \beta_i$ $(u_i \in V \otimes W)$, it suffices to show that $u \equiv 0 \pmod{J}$. We may choose $n \in \mathbf{N}$ with $\alpha_i, \beta_i \in \mathbf{M}_n$ and we define $E_{ij} \in \mathscr{F}$ $(1 \le k, l \le p)$ by

$$E_{ij} = i \begin{bmatrix} 0 & & \overset{j}{0} & & \\ & & \cdot & & \\ & & \cdot & & \\ 0 & \cdot \quad \cdot & 1_n & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \\ & & & & \end{bmatrix}$$

Letting $t = \sum_i u_i \otimes \alpha_i E_{1i}$, $\beta = \sum_j E_{j1}\beta_j$ we have that

$$t\beta = \begin{bmatrix} u_1 \otimes \alpha_1 & u_2 \otimes \alpha_2 & \dots \\ 0 & 0 & \dots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \beta_1 & 0 & \dots \\ \beta_2 & 0 & \dots \\ & \dots & \end{bmatrix} = \sum_i u_i \otimes \alpha_i\beta_i = 0.$$

It follows that if $\rho$ is the range projection of $\beta$, i.e., the minimal projection with $\beta = \rho\beta$, then $t\rho = 0$. To see this, let $\chi_{(0,\infty)}$ be the characteristic function of the set $(0,\infty)$. Then $\rho = p(\beta\beta^*)$ where

$$p(X) = \alpha_1 X + \alpha_2 X^2 + \cdots + \alpha_r X^r$$

is a real polynomial such that $p|_{sp\beta\beta^*} = \chi_{(0,\infty)}|_{sp\beta\beta^*}$. We conclude that

$$\begin{aligned} u = \sum_i u_i \otimes \alpha_i \otimes \beta_i &= \sum_{i,j} u_i \otimes \alpha_i E_{1i} \otimes E_{j1}\beta_j \\ &= t \otimes \beta \equiv t\rho \otimes \beta = 0 \pmod{J}. \end{aligned}$$

□

We define a linear map $\theta\colon V \otimes W \otimes e_{11} \to \mathscr{V} \otimes \mathscr{W}$ by

$$\theta(v \otimes w \otimes e_{11}) = v \otimes w \otimes e_{11} \otimes e_{11}.$$

This extends uniquely to an $\mathscr{F}$-bimodule map $\theta\colon V \otimes W \otimes \mathscr{F} \to \mathscr{V} \otimes \mathscr{W}$ via the formula

$$\theta(v \otimes w \otimes e_{ij}) = v \otimes w \otimes e_{i1} \otimes e_{1j}$$

(recall that we let $\mathscr{V} \otimes \mathscr{W}$ have the external module operations). It follows that for all $u \in V \otimes W \otimes \mathscr{F}$, $\Lambda(\theta(u)) = u$. Letting $\pi\colon \mathscr{V} \otimes W \to \mathscr{V} \otimes_{\mathscr{F}} \mathscr{W}$ be the quotient map we have that

$$\pi(\theta(V \otimes W \otimes e_{11})) = e_{11}(\mathscr{V} \otimes_{\mathscr{F}} \mathscr{W})e_{11}$$

since if we are given $v \in V$, $w \in W$,

$$\begin{aligned} e_{11}(v \otimes w \otimes \alpha \otimes \beta)e_{11} &= v \otimes w \otimes e_{11}\alpha \otimes \beta e_{11} \\ &= v \otimes w \otimes e_{11}\alpha\beta e_{11} \otimes e_{11} \pmod{\overline{J}}, \end{aligned}$$

where $e_{11}\alpha\beta e_{11} \in \mathbf{C}e_{11}$. Since $\pi$ is a bimodule map, it follows that $\pi$ maps $\theta(V \otimes W \otimes \mathscr{F})$ surjectively onto $\mathscr{V} \otimes_{\mathscr{F}} \mathscr{W}$. Letting $u \in V \otimes W \otimes \mathscr{F}$, we find that

$$\begin{aligned} \|\pi\theta(u)\| &= \inf\left\{\sum \|v_i\|\|w_i\| : \theta(u) - \sum v_i \otimes w_i \in \overline{J}, v_i \in \mathscr{V}, w_i \in \mathscr{W}\right\} \\ &= \inf\left\{\sum \|v_i\|\|w_i\| : \theta(u) - \sum v_i \otimes w_i \in J, v_i \in \mathscr{V}, w_i \in \mathscr{W}\right\} \\ &= \inf\left\{\sum \|v_i\|\,\|w_i\| : \Lambda(\theta(u)) = \Lambda\left(\sum v_i \otimes w_i\right),\ v_i \in \mathscr{V},\ w_i \in \mathscr{W}\right\} \\ &= \inf\left\{\sum \|v_i\|\|w_i\| : u = \sum v_i \odot w_i, v_i \in \mathscr{V}, w_i \in \mathscr{W}\right\}. \end{aligned}$$

Given operator spaces $V$ and $W$, this coincides with the Haagerup matricial tensor product norms. Thus in that case we have

$$(V \otimes_h W) \otimes \mathscr{F} \cong \mathscr{V} \hat{\otimes}_{\mathscr{F}} \mathscr{W}. \tag{4.1}$$

We conclude this section with a problem. Paulsen and Smith have recently proved that if $V \subseteq V_1$ and $W$ are operator spaces, then the inclusion $V \otimes_h W \hookrightarrow V_1 \otimes_h W$ is completely isometric **[10]**. This came as quite a surprise since the projective product for normed vector spaces does not have this property. In fact a Banach space $W$ has the property that $V \hat{\otimes} W \hookrightarrow V_1 \hat{\otimes} W$ is an isometry for all pairs $V \subseteq V_1$ if and only if $W$ is isometric to $L^1(X, \mu)$ for some measure space $(X, \mu)$ (see **[5]**). The inclusion property for $\otimes_h$ seems analogous to that given for $\otimes_{\mathscr{M}}$ in §3. In fact it follows from **[4]** that if $V$ and $W$ are operator spaces, and $u \in \mathbf{M}_{nn}(V \otimes_h W)$ satisfies $\|u\| < 1$, then there exist a

$p \in \mathbf{N}$, $v \in \mathbf{M}_{np}(V)$, $w \in \mathbf{M}_{pn}(W)$ with $\|v\|$, $\|w\| < 1$ and $u = v \odot w$. In this case, however, we do not know if we may take the components of $w$ to be linearly independent. That fact would give an elementary proof of the Paulsen and Smith result. Since it seems unlikely one could maintain equality with the components of $w$ independent, we conjecture that it is rather some perturbation version of this argument that will apply.

**5. The completely bounded tensor product and bimodules.** Let us suppose that $V$ and $W$ are matricially normed spaces. Extending the discussion of §3, we define the *completely bounded seminorm* on $V \otimes W$ by

$$\|u\|_{\mathscr{M}} = \inf\left\{\sum_{\nu} \|v^{\nu}\|\|w^{\nu}\| : u = \sum_{\nu} v^{\nu} \times w^{\nu}\right\},$$

where the sums are finite, and we choose $v^{\nu} \in \mathbf{M}_{n^{\nu}}(V)$, $w^{\nu} \in \mathbf{M}_{n^{\nu}}(W)$. If $W^*$ is an operator space, one may use a generalization of Corollary 3.4 to show that this is a norm, but in general this is not the case. We define the *completely bounded tensor product* $V \otimes_{\mathscr{M}} W$ to be $(V \otimes W)/N$, where $N$ is the null space of $\| \ \|_{\mathscr{M}}$.

As in §4, we let $\mathscr{V} = V \otimes \mathscr{F}$ and $\mathscr{W} = W \otimes \mathscr{F}$. We regard $\mathscr{V}$ (resp. $\mathscr{W}$) as a *right (resp. left) normed module over the algebraic tensor product* $\mathscr{F} \otimes \mathscr{F}$ by letting

$$v(\alpha \otimes \beta) = \alpha^{\mathrm{tr}} v \beta, \qquad (\alpha \otimes \beta) w = \alpha w \beta^{\mathrm{tr}}.$$

We define the corresponding *projective* $\mathscr{F} \otimes \mathscr{F}$ *tensor product* by

$$\mathscr{V} \hat{\otimes}_{\mathscr{F} \otimes \mathscr{F}} \mathscr{W} = \mathscr{V} \hat{\otimes} \mathscr{W} / \overline{K},$$

where

$$\begin{aligned} K &= K(\mathscr{V} \otimes \mathscr{W}) \\ &= \left\{\sum_i v_i \gamma_i \otimes w_i - v_i \otimes \gamma_i w_i : v_i \in \mathscr{V}, w_i \in \mathscr{W}, \gamma_i \in \mathscr{F} \otimes \mathscr{F}\right\}, \end{aligned}$$

and we denote the quotient map by $\pi$. It should be noted that this tensor product is a normed vector space, but that it does not have a natural bimodule structure.

We have a natural bilinear form $\langle\, ,\, \rangle : \mathscr{F} \times \mathscr{F} \to \mathbf{C}$ defined by

$$\langle \alpha, \beta \rangle = \sum \alpha_{ij} \beta_{ij}.$$

We define a linear map $\Theta : \mathscr{V} \otimes \mathscr{W} \to V \otimes W$ by letting

$$\Theta(v \otimes \alpha \otimes w \otimes \beta) = \langle \alpha, \beta \rangle v \otimes w.$$

Since for all $\alpha, \beta, \gamma, \delta \in \mathscr{F}$

$$\langle \delta\alpha\varepsilon, \beta\rangle = \langle \alpha, \delta^{\mathrm{tr}}\beta\varepsilon^{\mathrm{tr}}\rangle,$$

we have that for all $\gamma \in \mathscr{F} \otimes \mathscr{F}$ and $v \in \mathscr{V}$, $w \in \mathscr{W}$

$$\Theta(v\gamma \otimes w) = \Theta(v \otimes \gamma w). \tag{5.1}$$

Defining $v \times w = \sum v_{ij} \otimes w_{ij}$ for $v \in \mathscr{V}$ and $w \in \mathscr{W}$, we have $\Theta(v \otimes w) = v \times w$.

THEOREM 5.1. $K(\mathscr{V} \otimes \mathscr{W}) = \ker\Theta$.

*Proof*. From (5.1), the inclusion $K(\mathscr{V} \otimes \mathscr{W}) \subseteq \ker\Theta$ is immediate. Conversely, let us suppose that $u = \sum_k v_k \otimes \alpha^k \otimes w_k \otimes \beta^k \in \ker\Theta$, where $v_k \in V$, $w_k \in W$. Then

$$0 = \Theta(u) = \sum_k \langle \alpha^k, \beta^k\rangle v_k \otimes w_k.$$

Since $\varepsilon_{ij} = \varepsilon_{i1}\varepsilon_{11}\varepsilon_{1j}$, we have *modulo* $K$,

$$\begin{aligned}
u &= \sum_{i,j,k} \alpha^k_{ij} v_k \otimes \varepsilon_{ij} \otimes w_k \otimes \beta^k \\
&= \sum_{i,j,k} \alpha^k_{ij}[(v_k \otimes \varepsilon_{11})(\varepsilon_{1i} \otimes \varepsilon_{1j})] \otimes [w_k \otimes \beta^k] \\
&= \sum_{i,j,k} \alpha^k_{ij}[(v_k \otimes \varepsilon_{11})] \otimes [(\varepsilon_{1i} \otimes \varepsilon_{1j})(w_k \otimes \beta^k)] \\
&= \sum_{i,j,k} \alpha^k_{ij}[(v_k \otimes \varepsilon_{11})] \otimes [(w_k \otimes \beta^k_{ij}\varepsilon_{11})] \\
&= \sum_k \langle \alpha^k, \beta^k\rangle v_k \otimes \varepsilon_{11} \otimes w_k \otimes \varepsilon_{11} = 0,
\end{aligned}$$

since the map $V \otimes W \to \mathscr{V} \otimes \mathscr{W} : v \otimes w \mapsto v \otimes \varepsilon_{11} \otimes w \otimes \varepsilon_{11}$ is linear. Thus $u \in K(\mathscr{V} \otimes \mathscr{W})$. □

We define $\psi : V \otimes W \to \mathscr{V} \hat{\otimes} \mathscr{W}$ by $\psi(v \otimes w) = v \otimes \varepsilon_{11} \otimes w \otimes \varepsilon_{11}$. It is evident from the above calculation that $\pi \circ \psi : V \otimes W \to \mathscr{V} \hat{\otimes}_{\mathscr{F}\otimes\mathscr{F}} \mathscr{W}$ is surjective. The quotient norm is given by

$$\begin{aligned}
\|\pi\psi(u)\| &= \inf\left\{\sum \|v_i\|\|w_i\| : \psi(u) - \sum v_i \otimes w_i \in \overline{K}, v_i \in \mathscr{V}, w_i \in \mathscr{W}\right\} \\
&= \inf\left\{\sum \|v_i\|\|w_i\| : \psi(u) - \sum v_i \otimes w_i \in K, v_i \in \mathscr{V}, w_i \in \mathscr{W}\right\} \\
&= \inf\left\{\sum \|v_i\|\|w_i\| : \Theta\psi(u) = \Theta\left(\sum v_i \otimes w_i\right), v_i \in \mathscr{V}, w_i \in \mathscr{W}\right\} \\
&= \inf\left\{\sum \|v_i\|\|w_i\| : u = \sum v_i \times w_i, v_i \in \mathscr{V}, w_i \in \mathscr{W}\right\} \\
&= \|u\|_{\mathscr{M}}.
\end{aligned}$$

We conclude that we have a natural isometry:

$$V \otimes_{\mathcal{M}} W = \mathscr{V} \hat{\otimes}_{\mathcal{T} \otimes \mathcal{T}} \mathscr{W}. \tag{5.2}$$

*Added in proof*. It follows from [**15**, Theorem 3.3], that any complete $L^1$-matricially normed space is a quotient of $\mathscr{B}(H)_*$ for some Hilbert space $H$. This generalization of Corollary 2.3 may be used to simplify the proof of Theorem 2.4.

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