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A SEMILINEAR WAVE EQUATION WITH NONMONOTONE NONLINEARITY

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We prove that a semilinear wave equation in which the range of the derivative of the nonlinearity includes an eigenvalue of infinite multiplicity has a solution. The solution is obtained through an iteration scheme which provides a priori estimates.

1. Introduction. Here we study the nonlinear wave equation

$$(1.1)\qquad u_{tt} - u_{xx} + \lambda u = cq(x,t) + r(x,t) + g(u), \quad (x,t) \in [0,\pi] \times \mathbf{R},$$

$$(1.2)\qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,t) = u(x,t+2\pi), \qquad (x,t) \in [0,\pi] \times \mathbf{R},$$

where $\lambda \in \mathbf{R} - \{k^2 - j^2 \colon k = 1,2,3,\ldots, j = 0,1,2,\ldots\}$ and g is a function of class C^1 such that

$$(1.3)\qquad \lim_{|u|\to\infty} g'(u) = 0.$$

A main difficulty in studying (1.1)–(1.2) arises when $-\lambda \in g'(\mathbf{R})$. This causes compactness arguments to fail because 0 is an eigenvalue of $u_{tt} - u_{xx}$, (1.2) of infinite multiplicity. Recent studies on (1.1)–(1.2) either: (i) assume that $g(u) - \lambda u$ is monotone (see **[B-N]**, **[R]**, **[W]**), or (ii) assume enough symmetry on g, q, and r so that the kernel of $u_{tt} - u_{xx}$, (1.2) reduces to $\{0\}$ (see **[Co]**), or (iii) use dichotomy on whether the Palais-Smale condition holds proving existence for values of $cq+r$ which cannot be given explictly (see **[H]**, **[W]**). Our main result (Theorem A below) does not fall in any of the above three classes.

Let $\Omega = [0,\pi] \times [0,2\pi]$. Let H^1, L^2, and L^∞ denote the Sobolev spaces $H^1(\Omega)$, $L^2(\Omega)$, and $L^\infty(\Omega)$ respectively. We let $\| \ \|_1$, $\| \ \|$, and $\| \ \|_\infty$ denote the norms in H^1, L^2, and L^∞ respectively. Let

$$N = \left\{ u \in L^2 \colon u = \sum_{k=1}^{\infty} (a_k \sin(kx)\sin(kt) + b_k \sin(kx)\cos(kt)) \right\}.$$

Let $N^\perp \subsetneqq L^2$ denote the orthogonal complement to N in L^2. Let P denote the orthogonal projection onto N and Q the othogonal projection onto $N^{\perp}$. Let φ be a solution to

$$(1.4)\qquad \begin{gathered}\varphi_{tt} - \varphi_{xx} + \lambda\varphi = q(x,t), \qquad (x,t) \in [0,\pi]\times[0,2\pi],\\ \varphi(0,t) = \varphi(\pi,t) = 0, \quad \varphi(x,t) = \varphi(x,t+2\pi).\end{gathered}$$

Throughout the rest of this paper we will assume that

$$(1.5)\qquad m\{(x,t)\in\gamma; |\varphi(x,t)| < \delta\} \to 0 \quad \text{as } \delta \to 0$$

uniformly in γ, where γ is any characteristic of the operator $\partial_{tt} - \partial_{xx}$. Our main result is

THEOREM A. *If $(P(r))_t \in L^2$, $\varphi_t \in L^\infty$, and* (1.5) *holds, then there exists c_0 such that for $|c| > c_0$ the problem* (1.1)–(1.2) *has a weak solution $u \in H^1 \cap L^\infty$.*

Our proof is based on an iteration argument that resembles the proof of the inverse function theorem. We give (see §3) estimates in $H^1 \cap L^\infty$ which show the convergence of the scheme. Even though our arguments and main result suggest applicability of the standard inverse function theorem, our hypotheses are not enough to guarantee it.

2. Preliminaries and notations. A direct calculation shows that if $u \in N$, then $u_{tt} - u_{xx} = 0$ in the sense of distributions. Let $Au = u_{tt} - u_{xx}$. We recall (see **[B-N]**) that A subject to condition (1.2) is selfadjoint, the range of A is closed in L^2, and $R(A) = N^{\perp}$. The eigenvalues of A subject to (1.2) form the set $\{k^2 - j^2 : k = 1,2,3,\dots, j = 0,1,2,\dots\}$. The corresponding eigenfunctions are $\sin(kx)\sin(jt)$ and $\sin(kx)\cos(jt)$. The operator A^{-1} is compact from $N^{\perp}$ into $N^{\perp}$, and there exists a real number c such that

$$\begin{aligned}\|A^{-1}f\|_\infty &\le c\|f\| \quad \text{for all } f \in R(A),\\ \|A^{-1}f\|_1 &\le c\|f\| \quad \text{for all } f \in R(A).\end{aligned}$$

Using Fourier series it is easy to show that if $u \in H^1$, $v \in N \cap H^1$ then

$$(2.1)\qquad \iint (P(u))_t v_t = \iint u_t v_t,$$

$$(2.2)\qquad \int_0^{2\pi} (v)^2(x,s)\,ds \le \int\int_\Omega (v)^2\,dx\,dt \quad \text{for all } x \in [0,\pi].$$

Let A_1 denote the operator defined by

$$A_1\left(\sum_{\substack{k=1\\j=0\\k\neq j}}^{\infty}(a_{kj}\sin(kx)\sin(jt)+b_{kj}\sin(kx)\cos(jt))\right)$$
$$=\sum_{\substack{k=1\\j=0\\k\neq j}}^{\infty}\left[\frac{a_{kj}}{k^2-j^2+\lambda}\sin(kx)\sin(jt)+\frac{b_{kj}}{k^2-j^2+\lambda}\sin(kx)\cos(jt)\right].$$

It is easy to see that if $f\in L^2$, then $w=A_1(f)$ is a weak solution to $w_{tt}-w_{xx}+\lambda w=f$. An elementary Fourier series argument shows that if $w=A_1(Q(f))$, then

$$\int_0^{2\pi}((w_x)^2(x,t)+(w_t)^2(x,t))dt\le b_0^2\|Q(f)\|^2 \qquad \text{for all } x\in[0,\pi]. \tag{2.3}$$

where $b_0=\max\{(2/\pi)(k^2+j^2)/(k^2-j^2+\lambda)^2\colon k\neq j,\ k=1,2,3,\dots,$ $j=1,2,3,\dots\}$. In particular

$$\|A_1(Q(f))\|_1\le b_0\pi^{1/2}\|Q(f)\|. \tag{2.4}$$

Also it is easy to show that

$$\|A(Q(f))\|_\infty\le b_1\|Q(f)\| \tag{2.5}$$

where

$$b_1=\left(\sum_{\substack{k=1\\j=0\\k\neq j}}^{\infty}\frac{2}{(k^2-j^2+\lambda)^2}\right)^{1/2}$$

We can rewrite (1.1) as the following:

$$u_{tt}-u_{xx}+\lambda u=(c/R)Rq(x,t)+r(x,t)+g(u) \tag{2.6}$$

where

$R=\max\{2(2)^{1/2}dK\lambda,\ 2(2)^{1/2},16d^4\|g'\|_\infty^3\lambda,\ 16\|g'\|_\infty^3/\lambda\}/(\|\varphi_t\|_\infty+2^{1/2})$,

$K=b_0\pi^{1/2}+b_1$ and $d>0$ is a constant such that $\|u\|_{L^4}\le d\|u\|_1$ and $\|u\|\le d\|u\|_1$ for all $u\in H^1$ (the existence of d follows the fact that the embeddings $H^1 \to L^4$ and $H^1 \to L^2$ are continuous (see [A])). Let $w = Ru/c$ and $\beta = R/c$. Then (2.6) becomes

$$(2.7) \qquad w_{tt} - w_{xx} + \lambda w = Rq(x,t) + \beta(r(x,t) + g(w/\beta)).$$

Inductively we define $w_0 = 0$, w_1 as the solution to

$$(2.8) \qquad \begin{aligned} &(w_1)_{tt} - (w_1)_{xx} + \lambda w_1 = \beta(r(x,t) + g(R\varphi/\beta)), \\ &w_1(0,t) = w_1(\pi,t) = 0, \qquad w_1(x,t) = w_1(x,t+2\pi), \end{aligned}$$

and w_{n+1} as the solution to

$$(2.9) \quad \begin{aligned} &(w_{n+1})_{tt} - (w_{n+1})_{xx} + \lambda w_{n+1} = \beta(r(x,t) + g((R\varphi + w_n)/\beta)), \\ &w_{n+1}(0,t) = w_{n+1}(\pi,t) = 0, \qquad w_{n+1}(x,t) = w_{n+1}(x,t+2n). \end{aligned}$$

3. Estimates. For the sake of simplicity we will assume throughout the rest of this paper that $\lambda > 0$, and $c > 0$. The case $\lambda < 0$, or $c < 0$ requires only obvious modifications.

LEMMA 3.1. *Let* $\{w_n\}_n$ *be defined by* (2.8) *and* (2.9). *Under the assumption of Theorem* A, *there exists* $\beta_1 > 0$ *such that if* $\beta \in (0, \beta_1)$ *then for all* $n = 1, 2, 3, \ldots$ *we have*

$$(3.1) \qquad \|w_n\|_1 + \|w_n\|_\infty \le 1.$$

Proof. Since for each characteristic γ, $m\{(x,t) \in \gamma : |\varphi(x,t)| < \delta\} \to 0$ as $\delta \to 0$ uniformly in γ then there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ then

$$(3.2) \qquad \begin{aligned} &m\{(x,t) \in \gamma : |\varphi(x,t)| < \delta\} \\ &\quad < (\lambda/(32\pi R(\|g'\|_\infty + 1)(\|\varphi_t\|_\infty + 2^{1/2})))^2 \end{aligned}$$

for any characteristic γ. Because of (1.3) there exists $L \ge 0$ such that for all $u \in \mathbf{R}$

$$(3.3) \qquad |g(u)| \le (|u|/64\pi KR(\|\varphi\|_\infty + 1)) + L,$$

and there exists $M > 0$ such that if $|u| > M$ then

$$(3.4) \qquad |g'(u)| < \lambda/(64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2})).$$

Now we define β_1 by

$$(3.5) \quad \begin{aligned} \beta_1 = \min\{&1/(32(M + (L+1)\pi K + K\|r\|)), \\ &\lambda/(32\pi(\|r_t\| + 1)), R\delta_0/M\}. \end{aligned}$$

Next we prove (3.1) by induction. First we observe that

$$\begin{aligned}(3.6)\quad \|g(R\varphi/\beta)\|^2 &= \iint_\Omega (g(R\varphi/\beta))^2 \\ &\le 2\iint_\Omega L^2 + 2\left(\iint_\Omega (R\varphi/\beta)^2\right)/(64\pi KR(\|\varphi\|_\infty + 1))^2 \\ &\le 4\pi^2 L^2 + (1/32K\beta)^2.\end{aligned}$$

We write $w_1 = v_1 + z_1$ with $v_1 \in N$ and $z_1 \in N^\perp$. From (2.4), (2.5), and (2.7) it follows that if $\beta \in (0, \beta_1)$ then

$$(3.7)\qquad \|z_1\|_1 + \|z_1\|_\infty \le 1/4.$$

Projecting (2.8) into N, differentiating with respect to t, multiplying by $(v_1)_t$, and integrating over Ω we have

$$\begin{aligned}\lambda \iint_\Omega ((v_1)_t)^2\,dx\,dt &= \beta \iint_\Omega (P(r + g(R\varphi/\beta)))_t (v_1)_t\,dx\,dt \\ &= \beta \iint_\Omega (r + g(R\varphi/\beta))_t (v_1)_t\,dx\,dt \\ &= \beta \iint_\Omega r_t (v_1)_t + \iint_\Omega g'(R\varphi/\beta)(R\varphi_t)(v_1)_t \\ &= \beta\|r_t\|\cdot\|(v_1)_t\| + R\|\varphi_t\|_\infty \|(v_1)_t\| \left(\iint_\Omega (g'(R\varphi/\beta))^2\right)^{1/2}.\end{aligned}$$

Therefore

$$\|(v_1)_t\| \le \left(\beta\|r_t\| + R\|\varphi_t\|_\infty \left(\iint_\Omega (g'(R\varphi/\beta))^2\right)^{1/2}\right)/\lambda.$$

In order to estimate $\iint_\Omega (g'(R\varphi/\beta))^2$ we define $s_\beta = \{(x,t)\colon |R\varphi(x,t)| \le M\beta\}$ and $c_\beta = \Omega - s_\beta$. Since $M\beta/R < \delta_0$ we have

$$m(s_\beta) < (\lambda/(32\pi R(\|g'\|_\infty + 1)(\|\varphi_t\|_\infty + 2^{1/2})))^2$$

(see (3.2) and (3.5)); then

$$\begin{aligned}\left(\iint_\Omega (g'(R\varphi/\beta))^2\right)^{1/2} &= \left[\iint_{s_\beta} (g'(R\varphi/\beta))^2 + \iint_{c_\beta} (g'(R\varphi/\beta))^2\right]^{1/2} \\ &\le [\|g'\|_\infty^2 m(s_\beta) + 2\pi^2\lambda^2/(64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2]^{1/2} \\ &\le [2\lambda^2/(32R(\|\varphi_t\|_\infty + 2^{1/2}))^2]^{1/2} \\ &\le \lambda/(16R(\|\varphi_t\|_\infty + 2^{1/2})).\end{aligned}$$

Hence if $\beta \in (0, \beta_1)$ then $\|(v_1)_t\| \leq 1/8$. Since $\|v_t\| = \|v_x\|$ for all $v \in N \cap H^1$, we have $\|v_1\|_1 \leq 1/4$. Because of (2.2) we have

$$|v_1(x, t)| = \left|\int_0^t (v_1)_t(x, s)\, ds\right| \leq \left(\int_0^t ((v_1)_t)^2(x, s)\, ds\right)^{1/2} \left(\int_0^t ds\right)^{1/2}$$
$$\leq (2\pi)^{1/2}\|(v_1)_t\| \leq (2\pi)^{1/2}/8 \leq 1/3.$$

Therefore

$$\|v_1\|_1 + \|v_1\|_\infty \leq 7/12. \tag{3.8}$$

Combining (3.7) and (3.8) we have

$$\|w_1\|_1 + \|w_1\|_\infty \leq 1.$$

Suppose now that $\|w_n\|_1 + \|w_n\|_\infty \leq 1$. We write $w_{n+1} = v_{n+1} + z_{n+1}$ with $v_{n+1} \in N$ and $z_{n+1} \in N^\perp$. Again from (2.4), (2.5) and (2.7) we have

$$\|z_{n+1}\|_1 + \|z_{n+1}\|_\infty \leq \beta K(\|r\| + \|g((R\varphi + w_n)/\beta)\|).$$

In order to estimate $\|g((R\varphi + w_n)/\beta)\|$ we observe that

$$\iint_\Omega (g((R\varphi + w_n)/\beta))^2$$
$$\leq 2\iint_\Omega L^2 + 2\left(\iint_\Omega ((R\varphi + w_n)/\beta)^2\right)/(64\pi RK(\|\varphi\|_\infty + 1))^2$$
$$\leq 4\pi^2L^2 + 8\pi^2[(R\|\varphi\|_\infty + 1)/(64\pi RK(\|\varphi\|_\infty + 1))]^2/\beta^2.$$

Therefore if $\beta \in (0, \beta_1)$ then

$$\|z_{n+1}\|_1 + \|z_{n+1}\|_\infty \leq 1/4. \tag{3.9}$$

Now projecting (2.9) into N, differentiating with respect to t, multiplying by $(v_{n+1})_t$ and integrating over Ω we have

$$\begin{aligned}\lambda \iint_\Omega ((v_{n+1})_t)^2 \quad (3.10)\\ = \beta\Big(\iint_\Omega (r_t)(v_{n+1})_t \\ + \iint_\Omega g'((R\varphi + w_n)/\beta)((R\varphi + w)_n/\beta)_t(v_{n+1})_t\Big)\\ \leq \beta\|r_t\| \cdot \|(v_{n+1})_t\| \\ + (2\pi R\|\varphi_t\|_\infty + 2^{1/2})\left[\iint_\Omega (g'((R\varphi + w_n)/\beta))^2((v_{n+1})_t)^2\right]^{1/2}.\end{aligned}$$

Now we consider

$$I = \iint_\Omega (g'((R\varphi + w_n)/\beta))^2((v_{n+1})_t)^2 = \int_0^{2\pi}\int_0^{\pi} (g'((R\varphi + w_n)/\beta))^2((v_{n+1})_t)^2\,dx\,dt. \tag{3.11}$$

Without loss of generality we can assume that $v_{n+1} = h(x-t)$ or $(v_{n+1})_t = -h'(x-t)$. Because the integrand in (3.11) is 2π periodic in t we have

$$I = \int_0^{\pi}\int_x^{2\pi+x} (g'((R\varphi + w_n)(x,t)/\beta))^2(h'(x-t))^2\,dt\,dx.$$

By defining $\eta = x$, $\zeta = -x + t$, $\gamma_\zeta = \{(s, s+\zeta)\colon s \in [0,\pi]\}$ and $A_\beta = \{(x,t) \in \Omega\colon |R\varphi(x,t)| \le M\beta + 1\}$ we have

(3.12)

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^{\pi} (g'((R\varphi(\eta,\eta+\zeta) + w_n(\eta,\eta+\zeta))/\beta))^2(h'(-\zeta))^2\,d\eta\,d\zeta \\ &= \int_0^{2\pi}(h'(-\zeta))^2\left(\int_0^{\pi}(g'((R\varphi(\eta,\eta+\zeta) + w_n(\eta,\eta+\zeta))/\beta))^2 d\eta\right) d\zeta \\ &= \int_0^{2\pi}(h'(-\zeta))^2\left[\int_{\gamma_\zeta\cap A_\beta}(g'((R\varphi(\eta,\eta+\zeta) + w_n(\eta,\eta+\zeta))/\beta))^2\,d\eta \right. \\ &\qquad \left. + \int_{\gamma_\zeta-(\gamma_\zeta\cap A_\beta)}(g'((R\varphi(\eta,\eta+\zeta) + w_n(\eta,\eta+\zeta))/\beta))^2 d\eta\right] d\zeta \\ &\le \|(v_{n+1})_t\|^2[(\|g'\|_\infty\lambda/(32\pi(\|g'\|_\infty + 1)))^2 \\ &\qquad + (\lambda\pi^{1/2}/64\pi^2)^2]/(2R(\|\varphi_t\|_\infty + 2^{1/2}))^2. \end{aligned}$$

Hence if $\beta \in (0, \beta_1)$ then

$$\|(v_{n+1})_t\| \le 1/8.$$

Imitating the argument in (3.8) we have

$$\|v_{n+1}\|_1 + \|v_{n+1}\|_\infty \le 7/12. \tag{3.13}$$

Combining (3.9) and (3.13) we have

$$\|w_{n+1}\|_1 + \|w_{n+1}\|_\infty \le 1,$$

which proves the lemma.

LEMMA 3.2. *If $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le \tau(a_n + a_{n-1}), \qquad n = 2, 3, 4, \ldots,$$

then

$$a_{n+1} \leq c_{n+1} k\tau^{[n/2]},$$

where $k = \max\{a_1, a_2\}$, *and* c_n *is the nth Fibonacci number of the sequence defined by* $c_{n+1} = c_n + c_{n-1}$, $c_2 = 2$ *and* $c_1 = 0$, *and* $[x]$ *denotes the largest integer less than or equal to* x. *In particular the series* $\sum a_n$ *converges if* τ *is small enough.*

Proof. We prove the lemma by induction. For $n = 2$ we have

$$a_3 \leq \tau(a_2 + a_1) \leq 2\tau k = (c_2 + c_1)\tau k = c_3 k\tau^{[2/2]}.$$

Suppose that

$$a_n \leq c_n k\tau^{[(n-1)/2]}.$$

If n is even we have

$$\begin{aligned} a_{n+1} &\leq \tau(a_n + a_{n-1}) \leq \tau(c_n k\tau^{[(n-1)/2]} + c_{n-1} k\tau^{[(n-2)/2]}) \\ &\leq \tau k(c_n \tau^{(n-2)/2} + c_{n-1}\tau^{(n-2)/2}) \\ &\leq (c_n + c_{n-1})k\tau^{n/2} \leq c_{n+1} k\tau^{[n/2]}. \end{aligned}$$

Similarly if n is odd we have

$$\begin{aligned} a_{n+1} &\leq \tau(c_n k\tau^{(n-1)/2} + c_{n-1} k\tau^{(n-3)/2}) \\ &= c_n k\tau\tau^{(n-1)/2} + c_{n-1} k\tau^{(n-1)/2} \leq (c_n + c_{n-1})k\tau^{[n/2]}, \end{aligned}$$

which proves Lemma 3.2.

4. Proof of Theorem A. For $n = 1, 2, 3, \ldots$ we write $w_n = v_n + z_n$ where $v_n \in N$ and $z_n \in N^{\perp}$. Let $n > 2$. From the Sobolev imbedding theorem and (2.4) and (2.5) we have

$$\begin{aligned} (4.1)\quad \|z_{n+1} - z_n\|^2 &\leq d^2 \|z_{n+1} - z_n\|_1^2 \\ &\leq (dK\beta)^2 \iint_\Omega (g((R\varphi + w_n)/\beta) - g((R\varphi + w_{n-1})/\beta))^2 \\ &= (dK)^2 \iint_\Omega (g'((R\varphi + \zeta)/\beta))^2 (w_{n-1} - w_n)^2 \\ &\leq 2(dK)^2 \left[\iint_\Omega (g'((R\varphi + \zeta)/\beta))^2 (v_n - v_{n-1})^2 \right. \\ &\qquad \left. + \iint_\Omega (g'((R\varphi + \zeta)/\beta))^2 (z_n - z_{n-1})^2 \right] \end{aligned}$$

where $\zeta \in [w_n(x,t), w_{n-1}(x,t)] \cup [w_{n-1}(x,t), w_n(x,t)]$. Since v_n and $v_{n-1} \in N$, imitating the argument in (3.11)–(3.12) we see that

$$\begin{aligned} &\iint_{A_\beta} (g'((R\varphi + \zeta)/\beta))^2 (v_n - v_{n-1})^2 \\ &\qquad \leq (\lambda \|v_n - v_{n-1}\| / 32\pi R(\|\varphi_t\|_\infty + 2^{1/2}))^2. \end{aligned}$$

Thus

$$
\begin{aligned}
(4.2)\qquad & 2(dK)^2 \iint_\Omega (g'((R\varphi+\zeta)/\beta))^2(v_n - v_{n-1})^2 \\
&\le 2(dK)^2 \left[\iint_{A_\beta} (g'((R\varphi+\zeta)/\beta))^2(v_n - v_{n-1})^2 \right. \\
&\qquad \left. + \iint_{\Omega - A_\beta} (g'((R\varphi+\zeta)/\beta))^2(v_n - v_{n-1})^2 \right] \\
&\le 2(dK)^2[(\lambda/32\pi R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \\
&\qquad + (\lambda/64\pi R(\|\varphi_t\|_\infty + 2^{1/2}))^2]\|v_n - v_{n-1}\|^2 \\
&\le 2(dK)^2(\lambda/16R(\|\varphi_t\|_\infty + 2^{1/2}))^2\|v_n - v_{n-1}\|^2.
\end{aligned}
$$

where we also have used that $|(R\varphi+\zeta)/\beta| > M$ for $(x,t) \in \Omega - A_\beta$ (see (3.4)). On the other hand we have

$$
\begin{aligned}
(4.3)\quad & 2(dK)^2 \iint_\Omega (g'((R\varphi+\zeta)/\beta))^2(z_n - z_{n-1})^2 \\
&\le 2(dK)^2 \left[\iint_{A_\beta} (g'((R\varphi+\zeta)/\beta))^2(z_n - z_{n-1})^2 \right. \\
&\qquad \left. + \iint_{\Omega - A_\beta} (g'((R\varphi+\zeta)/\beta))^2(z_n - z_{n-1})^2 \right] \\
&\le 2(dK)^2 \left[\|g'\|_\infty^2 \iint_{A_\beta} (z_n - z_{n-1})^2 \right. \\
&\qquad \left. +(\lambda/64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2\|z_n - z_{n-1}\|^2 \right] \\
&\le 2(dK)^2 \left[\|g'\|_\infty^2 \iint_\Omega (\chi_{A_\beta}(z_n - z_{n-1})^2 \right. \\
&\qquad \left. +(\lambda/64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2\|z_n - z_{n-1}\|^2 \right] \\
&\le 2(dK)^2 \left[\|g'\|_\infty^2 \left(\iint_\Omega (\chi_{A_\beta})^2\right)^{1/2} \left(\iint_\Omega (z_n - z_{n-1})^4\right)^{1/2} \right. \\
&\qquad \left. +(\lambda/64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2\|z_n - z_{n-1}\|^2 \right] \\
&\le 2(dK)^2[\|g'\|_\infty^2 \lambda d^2 \|z_n - z_{n-1}\|^2/32\pi R(\|g'\|_\infty + 1)(\|\varphi_t\|_\infty + 2^{1/2}) \\
&\qquad + (\lambda/64\pi^2 R(\|\varphi_t\|_\infty + 2^{1/2}))^2\|z_n - z_{n-1}\|^2],
\end{aligned}
$$

where we have used that by the Sobolev imbedding theorem (see **[A]**) $\|z_n - z_{n-1}\|_{L^4} \leq d\|z_n - z_{n-1}\|_1$. Also since g' is bounded,

$$\|\beta(g(\varphi + w_n)/\beta) - g((\varphi + w_{n-1})/\beta)\| \leq \|g'\|_\infty \|w_{n-1} - w_{n-2}\|_1.$$

This, (4.2) and (4.3) give

$$\begin{aligned}(4.4)\quad & 2(dK)^2 \iint_\Omega (g'((R\varphi + \zeta)/\beta))^2 (z_n - z_{n-1})^2 \\ & \leq 2(dK)^2 \frac{\|g'\|_\infty^4 \lambda (dK)^2 \|w_{n-1} - w_{n-2}\|^2}{32\pi R(\|g'\|_\infty + 1)(\|\varphi_t\|_\infty + 2^{1/2})} \\ & \leq 2(dK)^2 [\|g'\|_\infty^3 \lambda (dK)^2 \|w_{n-1} - w_{n-2}\|^2 / 32R(\|\varphi_t\|_\infty + 2^{1/2}) \\ & \qquad + (\lambda/64\pi(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|z_n - z_{n-1}\|^2].\end{aligned}$$

Combining (4.2) and (4.4) we have

$$\begin{aligned}(4.5)\qquad \|z_{n+1} - z_n\| \leq\ & 2^{1/2} dK\lambda \|w_n - w_{n-1}\| \\ & + 2^{1/2}(dK)^2 \|g'\|_\infty^{3/2} \|w_{n-1} - w_{n-2}\| \\ & \times (\lambda/32R(\|\varphi_t\|_\infty + 2^{1/2}))^{1/2}.\end{aligned}$$

Also

$$\begin{aligned}\lambda^2 \|v_{n+1} - v_n\|^2 & \leq \iint_\Omega (g'((\varphi + \zeta)/\beta))^2 (w_n - w_{n-1})^2 \\ & \leq 2 \iint_\Omega (g'((\varphi + \zeta)/\beta))^2 [(v_n - v_{n-1})^2 + (z_n - z_{n-1})^2].\end{aligned}$$

Using now (4.2) and (4.4) we have

$$\begin{aligned}& \lambda^2 \|v_{n+1} - v_n\|^2 \\ & \quad \leq 2(\lambda/16R(\|\varphi_t\|_\infty + 2^{1/2}))^2 \|v_n - v_{n-1}\|^2 \\ & \qquad + 2[\|g'\|_\infty^3 \lambda (dK)^2 \|w_{n-1} - w_{n-2}\|^2 / 32R(\|\varphi_t\|_\infty + 2^{1/2}) \\ & \qquad\qquad + (\lambda \|z_n - z_{n-1}\| / 64R(\|\varphi_t\|_\infty + 2^{1/2}))^2].\end{aligned}$$

Hence

$$\begin{aligned}(4.6)\quad & \|v_{n+1} - v_n\| \\ & \quad \leq 2^{1/2} \|w_n - w_{n-1}\| / 8R(\|\varphi_t\|_\infty + 2^{1/2}) \\ & \qquad + 2^{1/2} \|g'\|_\infty^{3/2} \|w_{n-1} - w_{n-2}\| / (32\lambda R(\|\varphi_t\|_\infty + 2^{1/2}))^{1/2}.\end{aligned}$$

Combining (4.5) and (4.6) and using the definition of R we have

$$\|w_{n+1} - w_n\| \leq (\|w_n - w_{n-1}\| + \|w_{n-1} - w_{n-2}\|)/8.$$

Hence by Lemma 3.2 we have

$$\|w_{n+1} - w_n\| \leq k 2^n (1/8)^{[n/2]}, \qquad n = 2, 3, 4, \ldots.$$

Hence $\sum_{n=3}^{\infty} \|w_n - w_{n-1}\|$ converges. Thus the sequence $\{w_n = (w_n - w_{n-1}) + (w_{n-1} - w_{n-2}) + \cdots + (w_2 - w_1) + (w_1 - w_0)\}$ converges in L^2 to some $w \in L^2$. Since $\{w_n\}$ is bounded in $H^1 \cap L^\infty$ we see that w also belongs to $H^1 \cap L^\infty$. Hence by (2.11) we see that $w + R\varphi$ is a solution to (1.2), (2.7). Hence $u = c(w + R\varphi)$ is a solution to (1.1)–(1.2) which proves the Theorem.

REMARK. Double checking the proofs it is easily seen that Theorem A also holds when the limits in (1.3) are allowed to be in some interval of the form $(-s, s)$, with s depending on the distance from λ to $\{k^2 - j^2 : k = 1, 2, 3, \ldots, j = 0, 1, 2, 3, \ldots\}$.

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