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ON THE DIOPHANTINE EQUATION $1 = \sum 1/n_i + 1/\prod n_i$ AND A CLASS OF HOMOLOGICALLY TRIVIAL COMPLEX SURFACE SINGULARITIES

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Let $n_1, \ldots, n_N$ be integers ≥ 2, and let $x \in X$ be an isolated two-dimensional complex singularity whose dual intersection graph is a star with central weight 1 and with weights n_i on the arms. Then X is locally the cone on a homology 3-sphere if and only if $\sum n_i^{-1} + \prod n_i^{-1} = 1$. All such unit fraction expressions for 1 are given for $N \leq 7$, and properties of such sequences $\{n_i\}$ are discussed in general.

In this paper we will establish a correspondence between two-dimensional complex singularities whose local fundamental group is perfect and whose dual intersection graph is a star, and solutions in integers to the equation

$$1 = \sum_{i=1}^{N} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{N} n_i}. \tag{1}$$

Next we will discuss techniques for finding solutions to (1). We have found all solutions (there are a total of 42) for $N \leq 7$, and many further examples for larger N. Our techniques involve both elementary number theoretic methods and computer-aided searches. These in turn give rise to several unanswered questions in the theory of Egyptian fractions, the most general being as follows (Professor Erdös offers \$100 for a solution): Let $n_0, n_1, \ldots, n_k$ be positive integers, relatively prime in pairs, with $n_i \geq 2$ for $i > 0$. Under what conditions do there exist integers $n_{k+1}, \ldots, n_N$, all ≥ 2, such that

$$n_0 = \sum_{i=1}^{N} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{N} n_i}? \tag{2}$$

It might be remarked that no solution to (2) is known for $n_0 > 1$.

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1. Geometric preliminaries. Let $(X, \mathscr{O}_X)$ be a reduced, locally irreducible two-dimensional complex space with an isolated singular point x. A standard technique, due to Milnor, for investigating the topological structure of X in a neighborhood of x is as follows. Embed a small neighborhood U of x in a complex polydisc $\Delta^N \subset \mathbf{C}^N$ with x corresponding to the origin. Intersect $U \subset \Delta^N$ with a small sphere S_ε^{2N-1}, centered at the origin and of radius ε. For all sufficiently small ε, $S_\varepsilon^{2N-1} \cap U$ is a smooth three-dimensional manifold M^3, independent of ε, and $B_\varepsilon^{2N} \cap U$ is topologically the cone on $M^3 = \partial(B_\varepsilon^{2N} \cap U)$, where B_ε^{2N} denotes the ball of radius ε. The singularity $x \in X$ is called *homologically trivial* if M^3 is a homology sphere—that is, if $H_1(M^3, \mathbf{Z}) = H_2(M^3, \mathbf{Z}) = 0$. Singular points with this property are the natural analogues in dimension 2 of the higher dimensional singularities of Brieskorn (**[6]**, **[18]**), which are locally homeomorphic to the disc. They are especially interesting from a global point of view because a compact surface X each of whose singular points is homologically trivial has all the global topological properties of a smooth manifold. For example, in **[2]** and **[5]** we construct examples of singular complex surfaces X of the homotopy type of the complex projective plane $\mathbf{CP}^2$. Since Poincaré duality must hold for such a space X, it follows that each singularity of X is homologically trivial.

To investigate the topological structure of M^3 (hence of the neighborhood U of x in X) it is usually sufficient to calculate the fundamental group $\pi_1(M^3)$, called the *local fundamental group* of $x \in X$. A canonical problem in the classification theory of complex surface singularities is that of finding all singular points for which the local fundamental group satisfies some common group-theoretical criterion. For example, Brieskorn **[7]** finds all singularities whose local fundamental group is cyclic, while Orlik **[23]** and Wagreich **[26]** similarly investigate the cases where $\pi_1(M^3)$ is nilpotent and solvable, respectively. Since by the theorem of Hurwicz $H_1(M^3) = \pi_1(M^3)/(\text{commutator subgroup})$, it follows that a singularity $x \in X$ is homologically trivial exactly when the local fundamental group is *perfect*—that is, when every element of $\pi_1(M^3)$ is the product of elements of the form $\alpha\beta\alpha^{-1}\beta^{-1}$. For this reason we have also called such singular points *perfect singularities*.

Now the local fundamental group π_1 of a complex surface singularity can be calculated via a resolution of singularities. (This idea is due to Mumford **[22]**.) Suppose without loss of generality that x is the only singular point of the complex surface X. Let $p\colon \tilde{X} \to X$ be a proper holomorphic mapping of a complex manifold $\tilde{X}$ onto X such that p restricted to $p^{-1}(X - \{x\})$ is a biholomorphism of complex manifolds. Then $p^{-1}(x)$ is a closed complex curve on $\tilde{X}$ called the *exceptional curve of* p. Let $p^{-1}(x) = \bigcup_{i=1}^{n} C_i$ with C_i irreducible. If each C_i is non-singular and if C_i meets C_j (if at all) transversally in a single point, while $C_i \cap C_j \cap C_k = \varnothing$ for all distinct indices i, j, and k, then p is called a *normal resolution* of the singularity of X at x. Every singular point admits a unique normal resolution $p\colon \tilde{X} \to X$ which is *minimal* in the sense that if $p'\colon \tilde{X}' \to X$ is any other normal resolution then p' factors through p. (See Laufer **[19]**, e.g., for an exposition of the method of Hirzebruch **[17]** for finding such resolutions.) The minimal normal resolution is characterized among all normal resolutions by the property that if any component C_i of the exceptional curve is rational (homeomorphic to the two-sphere) and has self-intersection number $C_i^2 = -1$, then C_i meets at least three other components C_j.

The exceptional curve $C = \bigcup_{i=1}^{n} C_i$ of a normal resolution $p\colon \tilde{X} \to X$ is usually represented by its *weighted dual graph* Γ_p. Γ_p is the graph on n vertices $\gamma_1, \dots, \gamma_n$ with $\{\gamma_i, \gamma_j\}$ an edge of Γ_p if and only if C_i meets C_j and with the positive integral weight $n_i = -C_i^2$ assigned to the vertex γ_i. (The sign of C_i^2 is often reversed, as here, so as to correspond to the conventions of Lie algebra theory in the special case of rational double points.) If each component $C_1, \dots, C_n$ is rational and if the dual intersection graph Γ_p has no cycles, then the local fundamental group π_1 of $x \in X$ is the group on n generators $\gamma_1, \dots, \gamma_n$ with the n relations

$$(3) \qquad \prod_{j=1}^{n} \gamma_j^{(C_i \cdot C_j)} = 1, \qquad i = 1, \dots, n,$$

and $\gamma_i\gamma_j = \gamma_j\gamma_i$ if C_i meets C_j, where $C_i \cdot C_j$ is the intersection number (Mumford **[22]**). From this it follows that in this case the first homology group $H_1(M^3)$ is the Abelian group generated by $\gamma_1, \dots, \gamma_n$ with the relations (3). That is, $H_1(M^3)$ is the cokernal of the mapping $\phi_\Gamma\colon \mathbf{Z}^n \to \mathbf{Z}^n$ given by the matrix $(-C_i \cdot C_j)$, $i, j = 1, \dots, n$. Since this matrix is positive definite for any exceptional curve C (a fact first noted by Du Val in **[10]**), this means that in the case under view $H_1(M^3)$ is a finite Abelian group whose order is the determinant of the matrix $(-C_i \cdot C_j)$, regarded as a symmetric bilinear form on the integers. Thus $H_1(M^3) = 0$ if and only if $\det(-C_i \cdot C_j) = 1$, in the case in which all C_i are rational and the graph Γ_p is acyclic.

On the other hand, if any of the curves C_i has positive genus, or if Γ_p contains any cycles, then the exceptional curve $C = \bigcup_{i=1}^{n} C_i$ contains non-trivial 1-cycles which lift to non-trivial 1-cycles in $H_1(M^3)$ (indeed, to elements of $H_1(M^3)$ of infinite order). If $x \in X$ is homologically trivial no such cycles can exist. Thus we have established the following characterization of homologically trivial two-dimensional singular points.

1. PROPOSITION. *Let x be an isolated singular point of a complex surface X, and let $p\colon \tilde{X} \to X$ be the minimal normal resolution of the singularity of X at x, with exceptional curve $p^{-1}(x) = \bigcup_{i=1}^{n} C_i$. Then $x \in X$ is homologically trivial if and only if*

(a) *each component C_i is rational,*

(b) *the dual graph Γ_p contains no cycles, and*

(c) *the bilinear form $(-C_i \cdot C_j)$ is unimodular.*

Conversely, up to homeomorphism type all homologically trivial complex surface singularities can be constructed from certain positive definite symmetric integral unimodular bilinear forms ϕ as follows. Let $\phi\colon \mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}$ be such a form. Suppose furthermore that there exists a basis $\gamma_1, \dots, \gamma_n$ of $\mathbf{Z}^n$ for which

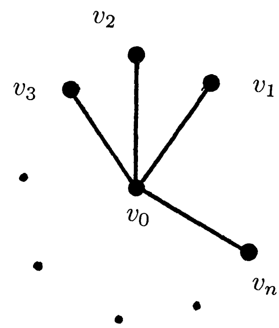
(α) $\phi(\gamma_i, \gamma_j) = -1$ or 0 $\forall i \neq j$,

(β) if $\phi(\gamma_i, \gamma_i) = 1$ then $\phi(\gamma_i, \gamma_j) = -1$ for at least three indices j, and

(γ) there does not exist a sequence $\gamma_{i_1}, \dots, \gamma_{i_k}$, $k > 1$, of distinct basis vectors for which $\phi(\gamma_{i_j}, \gamma_{i_{j+1}}) = -1$ for $j = 1, \dots, k-1$, and also $\phi(\gamma_{i_k}, \gamma_{i_1}) = -1$.

For $i = 1, \dots, n$, denote by $\tilde{U}_i$ the complex line bundle on the Riemann sphere $\mathbf{CP}^1$ with Chern class equal to $-\phi(\gamma_i, \gamma_i)$. Denote by C_i the base curve of $\tilde{U}_i$, let $U_i \subset \tilde{U}_i$ be the unit disc bundle, and plumb the spaces U_i together as follows. For each $i \neq j$, choose points P_{ij} on C_i and P_{ji} on C_j, such that $P_{ij} \neq P_{ik}$ $\forall j \neq k$, $\forall i$. Let D_{ij}, D_{ji} be discs in C_i, C_j, respectively, centered at P_{ij} and P_{ji} and sufficiently small that $\overline{D}_{ij} \cap \overline{D}_{ik} = \varnothing$ $\forall j \neq k$, $\forall i$. For each i, j, identify the fibres of U_i over D_i with the cross sections of the part of U_j sitting over D_j. The result is an open two-dimensional complex manifold $\tilde{X}$ containing the curve $C = \bigcup_{i=1}^{n} C_i$, with the identifications $P_{ij} = P_{ji}$, as a negatively embedded divisor (in the sense of Grauert **[15]**). Thus the topological space $X = \tilde{X}/C$ admits the structure of a normal complex space with a singular point x such that the "blowing down" map $p\colon \tilde{X} \to X$ is a normal resolution of singularities with exceptional curve C. Furthermore, by work of Neumann [**23**], every homologically trivial surface singularity $y \in Y$ admitting a resolution whose dual intersection form is equivalent to the form ϕ, is locally homeomorphic to $x \in X$ (but not necessarily, of course, having the same complex structure). Thus we have a correspondence between oriented homeomorphism equivalence classes of homologically trivial singularities and equivalence classes of integral symmetric positive definite unimodular forms satisfying the conditions (α), (β), and (γ) above. We want to exploit this correspondence to find all homologically trivial surface singularities of an especially interesting type.

2. Definition. A graph Γ is called a *star*, or a *star-like graph*, if it contains one vertex v_0 which meets every other vertex v_i, and with no other edges except those of the form $\{v_0, v_i\}$:



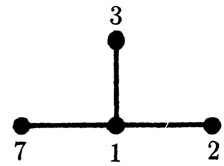
A positive definite integral bilinear form ϕ is *star-like* if there is a basis $\gamma_1, \ldots, \gamma_n$ of $\mathbf{Z}^n$ satisfying (α) and (β) above and whose associated graph Γ_ϕ is a star. Here Γ_ϕ is the graph on vertices $v_1, \ldots, v_n$ with edges $\{v_i, v_j\}$ whenever $\phi(\gamma_i, \gamma_j) = -1$. A complex surface singularity is said to *have star-like graph* if the dual intersection graph Γ_p of the minimal normal resolution of singularities is a star.

Note. A "star" is thus a special case of a "star-shaped graph", as examined, for example, in [**24**].

3. Example. Let $X \subset \mathbf{C}^3$ be defined by the vanishing of the function

$$f(x, y, z) = x^2 + y^3 + z^7,$$

which has an isolated singular point at the origin. This is a "minimally elliptic singularity of type *Cu*" (Laufer's classification [**20**]). Blowing up the origin produces a resolution (but not a normal one) whose exceptional curve is a rational curve $\overline{C}$ with a simple cusp and with self-intersection -1. Resolving the singularity of $\overline{C}$ gives the minimal normal resolution, whose weighted dual graph is the star



The local fundamental group π_1 is the group on four generators γ_0, γ_1, γ_2, γ_3 with relations

$$\begin{aligned} \gamma_0\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1} &= 1, \\ \gamma_0^{-1}\gamma_1^2 &= 1, \\ \gamma_0^{-1}\gamma_1^3 &= 1, \\ \gamma_0^{-1}\gamma_1^7 &= 1, \end{aligned}$$

together with $\gamma_0\gamma_i = \gamma_i\gamma_0$ for $i = 1, 2, 3$. To confirm that π_1 is perfect, we need only check that

$$\det \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ -1 & 0 & 0 & 7 \end{bmatrix} = 1.$$

In fact, π_1 is an infinite perfect group which is an extension of the finite simple group PSL(2,7), as is seen by mapping γ_0 to 1, γ_1 to $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, γ_2 to $\left[\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right]$, and γ_3 to $\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$. This singularity corresponds to the Diophantine equation

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{2 \cdot 3 \cdot 7}$$

under the association which we will now establish.

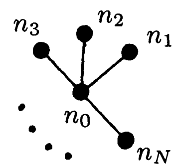
The following can be obtained as a special case of a theorem of Seifert [27]. Since the proof in our case is both short and transparent, we will give it in full.

4. THEOREM. *There is a one-to-one correspondence between oriented homeomorphism equivalence classes of homologically trivial complex surface singularities with star-like graphs, and solutions in integers* $N \geq 3$, $n_0 \geq 1$, $n_i \geq 2$ *to the equation*

$$n_0 = \sum_{i=1}^{N} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{N} n_i}. \tag{2}$$

Proof. Let $x \in X$ be a singularity whose weighted dual intersection graph is

(4)



Since by assumption this represents the minimal normal resolution, $n_i \geq 2\ \forall i > 0$, and $N \geq 3$ if $n_0 = 1$. We compute the determinant of the corresponding matrix

$$\phi = [\phi_{ij}] = \begin{bmatrix} n_0 & -1 & -1 & \cdot & \cdot & \cdot & -1 \\ -1 & n_1 & & & & & \\ -1 & & n_2 & & \bigcirc & & \\ \cdot & & & \cdot & & & \\ \cdot & & \bigcirc & & \cdot & & \\ \cdot & & & & & \cdot & \\ -1 & & & & & & n_N \end{bmatrix}$$

as follows:

$$\det \phi = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=0}^{N} \phi_{i\sigma(i)}$$

where S_n is the symmetric group and $\mathrm{sgn}(\sigma)$ is the number of transpositions in the decomposition of σ into the product of 2-cycles. In our case $\prod_{i=0}^{N} \phi_{i\sigma(i)} = 0$ except when σ is the identity or when σ interchanges 1 and some index $i > 1$ and fixes all other indices, in which case $\prod_{j=0}^{N} \phi_{j\sigma(j)} = (-1) \prod_{j \neq 0, i} n_j$. Thus

$$\det \phi = n_0 \prod_{i=1}^{N} n_i - \sum_{i=1}^{N} \prod_{j \neq i} n_j.$$

Dividing by $\prod_{i=1}^{N} n_i$ gives the equation

$$n_0 = \sum_{i=1}^{N} \frac{1}{n_i} + \frac{\det \phi}{\prod_{i=1}^{N} n_i} \tag{6}$$

In the case of homologically trivial points, $\det \phi = 1$ and we have the equation (2). Finally, it is clear that there is no solution to (2) for $n_0 > 1$ and $N \leq 2$, so $N \geq 3$ in every case.

Conversely, if the positive integers $n_0, n_1, \ldots, n_N$ satisfy (2) with $N \geq 3$, and $n_i \geq 2\ \forall i > 0$, we must first show that the graph (4) in fact corresponds to the exceptional curve in a resolution of some complex singularity. For this it is sufficient to show that the matrix (5) is positive definite (Grauert [15]). We compute the characteristic polynomial f_ϕ of ϕ:

$$
\begin{aligned}
f_\phi(x) &= \det(xI - \phi) \\
&= (x-n_0)\sum_{i=1}^{N}(x-n_i) - \sum_{i=1}^{N}\prod_{j\neq 1}(x-n_j) \\
&= (x-n_0)\sum_{k=0}^{N}(-1)^{N-k}\left(\sum_{i_0<\cdots<i_{N-k}}\prod_{j=1}^{N-k} n_{i_j}\right)x^k \\
&\quad -\sum_{i=1}^{N}\left[\sum_{k=0}^{N-1}(-1)^{N-1-k}\left(\sum_{\substack{j_i<\cdots<j_{n-k-1}\\ j_l\neq i}}\prod_{l=1}^{N-k} n_{j_l}\right)x^k\right] \\
&= \sum_{k=0}^{N+1}(-1)^{N+1-k}\left(\sum_{i_1<\cdots<i_{N+1-k}}\prod_{j=1}^{N+1-k} n_{i_j}\right. \\
&\qquad\left. +n_0\sum_{i_1<\cdots<i_{N-k}}\prod_{j=1}^{N+1-k} - \sum_{i=1}^{N}\sum_{\substack{j_1<\cdots<j_{N-1-k}\\ j_l\neq i}}\prod_{l=1}^{N-k} n_{j_l}\right)x^k \\
&= \sum_{k=0}^{N+1}(-1)^{N+1-k}\left[\sum_{i_1<\cdots<i_{N+1-k}}\prod_{j=1}^{N+1-k} n_{i_j}\right. \\
&\qquad\left. + \sum_{i_1<\cdots<i_{N-k}}\left(n_0\prod_{j=1}^{n+1-k} n_{i_j} - \sum_{j=1}^{N+1-k}\prod_{l\neq j} n_{i_l}\right)\right]x^k \\
&= \sum_{k=0}^{N+1}(-1)^{n+1-k}\left[\sum_{i_1<\cdots<i_{n+1-k}}\prod_{j=1}^{N+1-k} n_{i_j}\right. \\
&\qquad\left. + \sum_{i_1<\cdots<i_{N-k}}\prod_{j=1}^{n+1-k} n_{i_j}\left(n_0 - \sum_{j=1}^{N+1-k}\frac{1}{n_{i_j}}\right)\right]x^k.
\end{aligned}
$$

Since for all indices k,

$$
n_0 - \sum_{j=1}^{N+1-k}\frac{1}{n_{i_j}} \geq n_0 - \sum_{i=1}^{N}\frac{1}{n_i} = \frac{1}{\prod_{i=1}^{N} n_i} > 0,
$$

the coefficients of the characteristic polynomial strictly alternate in sign. Thus all the roots of f_ϕ are positive, so the bilinear form ϕ is positive definite as claimed.

Let $x \in X$ be the singularity constructed from the graph (4) as described above. That the graph (4) corresponds to the minimal normal resolution is again guaranteed by the conditions $N \geq 3$ and $n_i \geq 2$ $\forall i \neq 0$. That x is topologically trivial (i.e., that $\det \phi = 1$) is seen by comparing the general result (6) with our assumption (2). Finally, uniqueness up to homotopy type follows from the uniqueness of our constructions, as discussed above, and uniqueness up to oriented homeomorphism type follows from the results of Neumann.

2. The equation $n_0 = \sum 1/n_i + 1/\prod n_i$. The general theory of expressing rational numbers as sums of reciprocals of positive integers goes back to the Rhind papyrus of ancient Egypt, hence the term "Egyptian fraction" for an expression of the form

$$P/Q = \sum 1/n_i.$$

The reader is referred to section D11 of **[16]** and section 4 of **[11]** for discussions of some of the many unsolved problems in this area, as well as for extensive bibliographies. (Cf. especially **[1]**, **[3]**, **[4]** and **[8]**.) Our equation (2) is the special case in which $Q = 1$ and the last integer n_{N+1} is the product of all the previous ones. It is our hope that the connections with complex geometry exposed here will provide motivation for a further study of this special case. In this section we will develop the properties of solutions to (2) from first principles. Our immediate goal is to find all solutions for small N, thus generating, via Theorem 4, an interesting family of complex surface singularities.

5. THEOREM. *The following is the complete list of solutions to the equation*

$$n_0 = \sum_{i=1}^{N} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{N} n_i} \tag{2}$$

for $N \leq 6$. Thus (excluding the first 3 examples, which correspond to "resolutions" of the non-singular point) there are exactly 13 (up to homeomorphism) homologically trivial complex surface singularities *whose associated dual intersection graphs are stars with 6 or fewer arms.*

$N=0:\quad 1=\frac{1}{\text{empty product}},$

$N=1:\quad 1=\frac{1}{2}+\frac{1}{2},$

$N=2:\quad 1=\frac{1}{2}+\frac{1}{3}+\frac{1}{2\cdot 3},$

$N=3:\quad 1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{2\cdot 3\cdot 7},$

$N=4:\quad 1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{2\cdot 3\cdot 7\cdot 43},$

$N=5:\quad 1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1807}+\frac{1}{2\cdot 3\cdot 7\cdot 43\cdot 1807},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{47}+\frac{1}{395}+\frac{1}{2\cdot 3\cdot 7\cdot 47\cdot 395},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{11}+\frac{1}{23}+\frac{1}{31}+\frac{1}{2\cdot 3\cdot 11\cdot 23\cdot 31},$

$N=6:\quad 1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1807}+\frac{1}{3263443}+\frac{1}{\prod n_i},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1823}+\frac{1}{193667}+\frac{1}{\prod n_i},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{47}+\frac{1}{395}+\frac{1}{779731}+\frac{1}{\prod n_i},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{47}+\frac{1}{403}+\frac{1}{19403}+\frac{1}{\prod n_i},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{47}+\frac{1}{415}+\frac{1}{8111}+\frac{1}{\prod n_i},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{47}+\frac{1}{583}+\frac{1}{1223}+\frac{1}{\prod n_i},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{55}+\frac{1}{179}+\frac{1}{24323}+\frac{1}{\prod n_i},$

$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{11}+\frac{1}{23}+\frac{1}{31}+\frac{1}{47059}+\frac{1}{\prod n_i}.$

For $N=7$, the complete list of solutions is given in the appendix. For $N=8$ there are many solutions which begin with $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}$ and $\frac{1}{2}+\frac{1}{3}+\frac{1}{11}$. In addition to these there are only two other solutions:

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{13}+\frac{1}{25}+\frac{1}{29}+\frac{1}{67}+\frac{1}{2981}+\frac{1}{11294561851}+\frac{1}{\prod n_i},\quad\text{and}$$

$$1=\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{17}+\frac{1}{157}+\frac{1}{961}+\frac{1}{4398619}+\frac{1}{\prod n_i}.$$

This last example is the smallest which does not begin with $\frac{1}{2}+\frac{1}{3}$. We do not know any solution which does not begin with $\frac{1}{2}$.

In order to verify that our lists are complete, and to explain how they were obtained, we will first develop some of the properties of the sequences $n_1,\dots,n_N$ of integers which appear in the solutions. First, it is clear from the equation (2), or from its equivalent form

$$n_0\prod_{i=1}^{N} n_i=\sum_{i=1}^{N}\prod_{j\neq i} n_j+1,$$

that n_{i_1} and n_{i_2} are relatively prime for all distinct indices $i_1, i_2 > 0$. From this it is easy to check the following.

6. PROPOSITION. *If $n_0 \geq 2$ there is no solution in integers $n_i \geq 2$ to the equation*

$$n_0 = \sum_{i=1}^{N} \frac{1}{n_j} + \frac{1}{\prod_{i=1}^{N} n_i} \tag{2}$$

for $N < 59$.

Proof. Since the n_i's are relatively prime in pairs, $n_i \geq p_i\ \forall i$, where p_i is the ith prime number. Thus

$$\sum_{i=1}^{N} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{N} n_i} \leq \sum_{i=1}^{N} \frac{1}{p_i} + \frac{1}{\prod_{i=1}^{N} p_i}.$$

But the right-hand side is less than 2 for the first 58 primes, so no solution to (2) can exist for $n_0 \geq 2$ and $N < 59$, as claimed.

REMARK. Similarly, by examining the rate of divergence of the series $\sum_{i=1}^{\infty} 1/p_i$ we can estimate the minimum number $N(n_0)$ of terms needed for a solution to (2) for $n_0 = 3, 4, \ldots$. The fact that we need at least 58 terms to get close even to 2 shows that we are unlikely to find a solution for $n_0 \geq 2$ either by a lucky guess or by an unsophisticated search. In particular, if we want solutions for small N we may restrict our attention to the case $n_0 = 1$.

Even for $n_0 = 1$, some properties of the n_i are apparent from consideration of size alone. For instance, there is no solution to $\sum 1/n_i + 1/\prod n_i = 1$ with none of the n_i prime. For if $n_i = r_i s_i$, with $2 \leq r_i \leq s_i$ for all i, then, since the r_i are distinct, we have

$$\sum_{i=1}^{N} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{N} n_i} \leq \sum_{i=1}^{N} \frac{1}{r_i^2} + \frac{1}{(\prod_{i=1}^{N} r_i)^2} < \sum_{i=2}^{N+2} \frac{1}{i^2} < \frac{\pi^2}{6} - 1 < 1$$

(cf. **[13]**). We will want, then, to keep track of just how close to 1 various expressions of the form $\sum 1/n_i$ may be.

7. DEFINITION. Let $n_1, \ldots, n_N$ be positive integers with $\sum_{i=1}^{N} 1/n_i < 1$. Define the *sequence of remainders of* $(n_1, \ldots, n_N)$ to be the sequence $R_0, R_1, \ldots, R_N$ defined by

$$R_k = \left(\prod_{i=1}^{k} n_i\right)\left(1 - \sum_{i=1}^{k} n_i\right) = \prod_{i=1}^{k} n_i - \sum_{i=1}^{k} \prod_{j \neq i} n_i.$$

That is, R_k is the unique positive integer satisfying

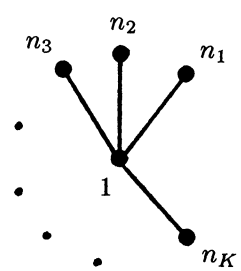
$$1 = \sum_{i=1}^{k} \frac{1}{n_i} + \frac{R_k}{\prod_{i=1}^{k} n_i}.$$

8. REMARK. It is trivial to check that the sequence $R_0, R_1, \ldots, R_n$ is also determined recursively by putting $R_0 = 1$ and

$$R_k = n_k R_{k-1} - \prod_{i=1}^{k-1} n_i \tag{7}$$

for $k = 1, \ldots, N$. Thus if $R_N = 1$ then each R_k is relatively prime to $n_1, \ldots, n_k$, for if a prime P were to divide both R_k and $\prod_{i=1}^{k} n_i$, then by repeated applications of (1), P divides each of $R_{k+1}, \ldots, R_N = 1$, an absurdity.

We also note that by the proof of Theorem 4, R_k is the determinant of the bilinear form associated to the star



9. LEMMA. *Let $n_1, \ldots, n_N$ be positive integers with $\sum_{i=1}^{N} 1/n_i < 1$. Let $R_0, \ldots, R_N$ be the sequence of remainders, and for fixed $k < N$ put*

$$\overline{R}_l = n_l R_k - \prod_{i=1}^{k} n_i$$

for $l = k+1, \ldots, N$. Then the positive integers $\overline{R}_{k+1}, \ldots, \overline{R}_N$ satisfy the relation

$$\prod_{l=k+1}^{N} \overline{R}_l = \sum_{r=0}^{N-k-2} (N-k-1-r) \left(\prod_{i=1}^{k} n_i \right)^{N-k-r} \times \sum_{k+1 \le j_1 < \cdots < j_r \le N} \overline{R}_{j_1} \cdots \overline{R}_{j_r} + R_k^{N-k-1} R_N.$$

Proof. This is a direct computation. Substituting

$$n_l = \left(\overline{R}_l + \prod_{i=1}^{k} n_i\right) / R_k$$

into the equation

$$1 = \sum_{i=1}^{k} \frac{1}{n_i} + \sum_{l=k+1}^{N} \frac{1}{n_l} + \frac{R_N}{\prod_{j=1}^{N} n_j}$$

gives

$$1 = \sum_{i=1}^{k} \frac{1}{n_i} + \sum_{l=k+1}^{N} \frac{R_k}{\overline{R}_l + \prod_{i=1}^{k} n_i} + \frac{R_k^{N-k} R_N}{\prod_{i=1}^{k} n_i \prod_{l=k+1}^{N} (\overline{R}_l + \prod_{i=1}^{k} n_i)},$$

or

$$\frac{R_k}{\prod_{i=1}^{k} n_i} = \sum_{l=k+1}^{N} \frac{R_k}{\overline{R}_l + \prod_{i=1}^{k} n_i} + \frac{R_k^{N-k} R_N}{\prod_{i=1}^{k} n_i \prod_{l=k+1}^{N} (\overline{R}_l + \prod_{i=1}^{k} n_i)}.$$

Dividing by R_k and clearing denominators yields

$$\prod_{l \neq k+1}^{N} \left(\overline{R}_l + \prod_{i=1}^{k} n_i\right) = \prod_{i=1}^{k} n_i \sum_{l=k+1}^{N} \prod_{j \neq l} \left(\overline{R}_j + \prod_{i=1}^{k} n_i\right) + R_k^{N-k-1} R_N.$$

That is,

$$\begin{aligned} \sum_{r=0}^{N-k} \left(\prod_{i=1}^{k} n_i\right)^{N-k-r} & \sum_{k+1 \leq l_1 < \cdots < l_r \leq N} \overline{R}_{l_1} \cdots \overline{R}_{l_r} \\ &= \prod_{i=1}^{k} n_i \sum_{l=k+1}^{N} \sum_{r=0}^{N-k-1} \left(\prod_{i=1}^{k} n_i\right)^{N-k-r-1} \\ &\quad \times \sum_{k+1 \leq j_1 < \cdots < j_r \leq N} \overline{R}_{j_1} \cdots \overline{R}_{j_r} + R_k^{N-k-1} R_N \\ &= \sum_{r=0}^{N-k-1} (N-k-r) \left(\prod_{i=1}^{k} n_i\right)^{N-k-4} \\ &\quad \times \sum_{k+1 \leq l_1 < \cdots < l_r \leq N} \overline{R}_{l_1} \cdots \overline{R}_{l_r} + R_k^{N-k-1} R_N. \end{aligned}$$

Comparing like powers of $\prod_{i=1}^{k} n_i$ now gives the desired result.

10. Corollary. *Let $n_1, \dots, n_N$, $R_0, \dots, R_N$ be as above. Then* $\forall k$,

$$\left(\prod_{i=1}^{k} n_i\right)^2 + R_k R_{k+2} = R_{k+1}\overline{R}_{k+2}, \tag{8}$$

where $\overline{R}_{k+2} = n_{k+2}R_k - \prod_{i=1}^{k} n_i$.

Proof. Take $N = k+2$ in Lemma 9.

We can now give criteria for extending a partial sequence $\sum_{i=1}^{k} 1/n_i$ to a solution $1 = \sum_{i=1}^{N} 1/n_i + 1/\prod_{i=1}^{N} n_i$, for small $N - k$.

11. Definition. Let $n_1, \dots, n_k$ be positive integers with $\sum_{i=1}^{k} 1/n_i < 1$. Then the sequence $(n_1, \dots, n_k)$ *can be completed in m steps* if there exist positive integers $n_{k+1}, \dots, n_{k+m}$ such that $\sum_{i=1}^{k+m} 1/n_i + 1/\prod_{i=1}^{k+m} n_i = 1$.

12. Proposition. *Let $n_1, \dots, n_k$, $R_0, \dots, R_k$ be as above. Then $(n_1, \dots, n_k)$ can be completed in m steps, for $m = 0, 1, 2$, or 3, if and only if the following criteria are met, respectively:*

$m = 0 : R_k = 1$.

$m = 1 : \prod_{i=1}^{k} n_i \equiv -1 \bmod R_k$.

$m = 2 : (\prod_{i=1}^{k} n_i)^2 + R_k$ *admits a factor F congruent to* $-\prod_{i=1}^{k} n_i \bmod R_k$.

$m = 3 :$ *There exist integers X and F such that*

$$\left(\prod_{i=1}^{k} n_i\right)^2 X^2 + R_k X - \prod_{i=1}^{k} n_i \equiv 0 \mod F \quad \textit{and}$$

$$F \equiv -X \prod_{i=1}^{k} n_i \mod \left(R_k X - \prod_{i=1}^{k} n_i\right).$$

Proof. The criteria for $m = 0$ and $m = 1$ are obvious (for $m = 1$, if $\prod_{i=1}^{k} n_i \equiv -1 \bmod R_k$, put $n_{k+1} = (\prod_{i=1}^{k} n_i + 1)/R_k$). For $n = 2$, if n_{k+1} and n_{k+2} exist to complete the sequence, then by Corollary 9 we have $(\prod_{i=1}^{k} n_i)^2 + R_k \cdot 1 = R_{k+1}\overline{R}_{k+2}$, where $R_{k+1} = n_{k+1}R_k - \prod_{i=1}^{k} n_i$ and $\overline{R}_{k+2} = n_{k+2}R_k - \prod_{i=1}^{k} n_i$ as in that corollary. Conversely, if $(\prod_{i=1}^{k} n_i)^2 + R_k = F \cdot G$ with F (hence also G) $\equiv -\prod_{i=1}^{k} n_i \bmod R_k$, put $n_{k+1} = (F + \prod_{i=1}^{k} n_i)/R_k$ and $n_{k+2} = (G + \prod_{i=1}^{k} n_i)/R_k$ to complete the sequence.

The criterion for $m = 3$ follows from that of $m = 2$ by putting $X = n_{k+1}$, $R_{k+1} = XR_k - \prod_{i=1}^{k} n_i$, and $F = R_{k+2}$.

REMARK. In practice the criterion for $m = 3$ is not very useful since it does not tell us how to find the integers X and F. But for $m = 2$ the criterion is very easy to apply, provided that we can find the factors of the integer $(\prod_{i=1}^{N-2} n_i)^2 + R_{N-2}$, which may be quite large.

13. EXAMPLE. Find all completions of the sequence 2, 3, 7, 47 in two or fewer steps.

Solution. We have

$$1 = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{47} + \tfrac{5}{2\cdot 3\cdot 7\cdot 47}.$$

Since $2 \cdot 3 \cdot 7 \cdot 47 \equiv -1 \bmod 5$, a one step solution exists, namely $n_5 = (2 \cdot 3 \cdot 7 \cdot 47)/5 = 395$. That is,

$$1 = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{47} + \tfrac{1}{395} + \tfrac{1}{2\cdot 3\cdot 7\cdot 47\cdot 395}$$

is a solution to our equation (1).

For $m = 2$ we have $(2 \cdot 3 \cdot 7 \cdot 47)^2 + 5 = 3896681 = 1 \cdot 41 \cdot 101 \cdot 941$, and each of these factors is congruent to $-1974 \bmod 5$. Thus we have the four distinct solutions

(i) $n_5 = \frac{1974+1}{5}$, $n_6 = \frac{1974+41\cdot 101\cdot 941}{5}$,
(ii) $n_5 = \frac{1974+41}{5}$, $n_6 = \frac{1974+101\cdot 941}{5}$,
(iii) $n_5 = \frac{1974+101}{5}$, $n_6 = \frac{1974+41\cdot 941}{5}$,
(iv) $n_5 = \frac{1974+941}{5}$, $n_6 = \frac{1974+41\cdot 101}{5}$.

These are lines 3 through 6 of the list of solutions for $N = 6$ in Theorem 5 above.

In view of the prominence of the expression $(\prod_{i=1}^{k} n_i)^2 + R_k$, our sequences can be further analyzed via the theory of quadratic residues.

14. LEMMA. *Let $n_1, \dots, n_N$ be positive integers with $\sum_{i=1}^{N} 1/n_i < 1$, and let $R_0, \dots, R_N$ be the sequence of remainders. Suppose that n_1 is even and $R_N = 1$. Then $\forall k = 1, \dots, N$, R_k is odd and coprime to R_{k-1}. Furthermore, the Jacobi (or Legendre) symbols of the R_k's satisfy*

$$(9)\qquad \left(\frac{R_{k+1}}{R_k}\right) = \begin{cases} \left(\frac{R_k}{R_{k-1}}\right) & \text{if } R_k \equiv 1 \bmod 4, \text{ or} \\ & \text{if } R_k \equiv R_{k-1} \equiv 3 \bmod 4, \\ -\left(\frac{R_k}{R_{k-1}}\right) & \text{if } R_k \equiv 3 \bmod 4 \text{ and } R_{k-1} \equiv 1 \bmod 4. \end{cases}$$

Proof. Since n_1 is even, R_k is odd for all k by Remark 8. Similarly if R_{k-1} and R_k had a common prime factor P, then also P divides $\prod_{i=1}^{k-1} n_i = n_k R_{k-1} - R_k$, again prohibited by Remark 8. Finally, by

Corollary 10, for each k, $(\prod_{i=1}^{k-1} n_i)^2 + R_{k-1}R_{k+1} \equiv 0 \bmod R_k$. Thus $((R_{k+1}R_{k-1})/R_k) = (-1/R_k)$. The result (9) now follows by considering the various cases in which R_k and R_{k-1} are congruent to 1 or $-1 \bmod 4$, and using the familiar properties of the Jacobi symbol.

15. Proposition. *Let* $1 = \sum_{i=1}^N 1/n_i + 1/\prod_{i=1}^N n_i$ *be a solution to our basic equation* (1). *For* $j = 0, 1, 2, 3$, *denote by* A_j *the number of terms* n_i *which are congruent to* $j \bmod 4$. *Then*

Case 1. *If* $A_2 = 1$ *then* $A_1 \equiv 0 \bmod 4$.

Case 2. *If* $A_0 = 1$ *then* $A_3 \equiv 3 \bmod 4$.

Case 3. *If* $A_0 = A_2 = 0$, *then* $N \equiv 0 \bmod 4$.

Proof. First we note that since the n_i are relatively prime in pairs, $A_0 + A_2 \leq 1$, so the three cases listed exhaust all possibilities.

Case 1. Assume without loss of generality that $n_1 \equiv 2 \bmod 4$, that $n_2, \dots, n_{1+A_3} \equiv 3 \bmod 4$, and that $n_{1+A_3+1}, \dots, n_{1+A_3+A_1} = n_N \equiv 1 \bmod 4$. Then from the relations $R_{k+1} = n_{k+1}R_k - \prod_{i=1}^k n_i$, for $R_0, \dots, R_N$ the sequence of remainders, it is easy to check inductively that $R_k \equiv 1 \bmod 4$ for $k = 0, 1, \dots, 1+A_3$, and $R_{1+A_3+k} \equiv (-1)^k \bmod 4$ for $k = 1, \dots, A_1$. Thus in particular $R_N = 1 \Rightarrow A_1$ is even.

Now use formula (9) of Lemma 14 to show that for $k = 1, \dots, 1+A_3$ the Jacobi symbol (R_k/R_{k-1}) is equal to 1, while for indices greater than $1 + A_3$ the sequence $(R_{1+A_3+k}/R_{1+A_3+k-1})$ follows the pattern 1, -1, -1, 1, repeated in groups of 4. Thus, since $(R_{1+A_3+A_1}/R_{1+A_3+A_1-1}) = (1/R_{N-1}) = 1$, the fact that A_1 is even shows that in fact $A_1 \equiv 0 \bmod 4$, as claimed.

Case 2 is proved similarly, but it requires division into subcases, according to the congruence mod 4 of A_1, to check all the details. Case 3 can be verified from the recursion formulas $R_{k+1} = n_{k+1}R_k - \prod_{i=1}^k n_i$ alone, by considering each subcase separately, without appealing to quadratic residue theory.

16. *Application.* (a) For $N < 12$, there is no solution in odd integers n_i to the equation $1 = \sum_{i=1}^N 1/n_i + 1/\prod_{i=1}^N n_i$.

(b) For $N \leq 5$, if $1 = \sum_{i=1}^N 1/n_i + 1/\prod_{i=1}^N n_i$, with $n_1 \equiv 2 \bmod 4$, then $n_i \equiv 3 \bmod 4$ $\forall i > 1$.

Proof. (a) For the first 8 odd primes we have

$$\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} + \tfrac{1}{19} + \tfrac{1}{23} + \frac{1}{\prod_{i=1}^{8} n_i} < 1.$$

Hence, since $\prod_{i=1}^N 1/n_i + 1/\prod_{i=1}^N n_i$ strictly decreases as any n_i increases, there cannot be a solution for $N \le 8$. By Theorem 14, case 3, then, N is at least 12 if all the n_i's are odd.

(b) If $n_1 \equiv 2 \bmod 4$ and $n_2 \equiv 1 \bmod 4$, then by case 1 of the Theorem, n_3, n_4, and n_5 must also be $\equiv 1 \bmod 4$. But, again taking the smallest possibilities,

$$\tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{9} + \tfrac{1}{13} + \tfrac{1}{17} + \frac{1}{\prod_{i=1}^5 n_i} < 1,$$

so no such solution exists.

REMARK. In fact, for $N \le 6$ every solution has $n_1 = 2$ and $n_i \equiv 3 \bmod 4\ \forall i > 1$. For $N = 7$, however, we have the examples

$$1 = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{11} + \tfrac{1}{17} + \tfrac{1}{101} + \tfrac{1}{149} + \tfrac{1}{3109} + \frac{1}{\prod n_i}, \quad \text{and}$$
$$1 = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{13} + \tfrac{1}{25} + \tfrac{1}{29} + \tfrac{1}{67} + \tfrac{1}{2991} + \frac{1}{\prod n_i},$$

each of which has exactly 4 terms congruent to 1 mod 4.

3. Search techniques. We will now describe a general search method for finding all solutions for fixed N to the equation $1 = \sum_{i=1}^N 1/n_i + 1/\prod_{i=1}^N n_i$.

17. LEMMA. *Let* $1 = \sum_{i=1}^N 1/n_i + 1/\prod_{i=1}^N n_i$ *be a solution to* (1) *with* $n_1 < n_2 < \cdots < n_N$ *and with remainders* $R_0, \ldots, R_N = 1$. *Then* $\forall k = 1, \ldots, N-2$,

$$\frac{\prod_{i=1}^k n_i}{R_k} < n_{k+1} < (N-k)\frac{\prod_{i=1}^k n_i}{R_k}.$$

Proof. Since

$$\sum_{i=1}^k \frac{1}{n_i} + \frac{1}{n_{k+1}} < 1, \quad n_{k+1} > \left(1 - \sum_{i=1}^k \frac{1}{n_i}\right)^{-1} = \frac{\prod_{i=1}^k n_i}{R_k}.$$

Also, since $(N-k) \ge 2$ and $n_{k+j} > n_{k+1}$ for $j = 2, \ldots, N-k$,

$$\frac{R_k}{\prod_{i=1}^k n_i} = \sum_{j=1}^{N-k} \frac{1}{n_{k+j}} + \frac{1}{\prod_{i=1}^k n_i \prod_{j=1}^{N-k} n_{k+j}} < \frac{(N-k)}{n_{k+1}} + \frac{1}{(n_{k+1})^{N-k} \prod_{i=1}^k n_i}$$

or

$$R_k(n_{K=1})^{N-k} - (N-k)\left(\prod_{i=1}^{k} n_i\right)(n_{k+1})^{N-k-1} < 1.$$

Since the left-hand side is an integer, we have

$$R_k(n_{k+1})^{N-k} - (N-k)\left(\prod_{i=1}^{k} n_i\right)(n_{k+1})^{N-k-1} \leq 0;$$

that is

$$R_k n_{k+1} - (N-k)\prod_{i=1}^{k} n_i \leq 0. \tag{10}$$

If equality holds, then both R_k and n_{k+1} cannot be relatively prime to $n_1, \ldots, n_k$, contrary to Remark 8. Thus strict inequality holds and

$$n_{k+1} < (N-k)\frac{\prod_{i=1}^{k} n_i}{R_k} \tag{11}$$

as claimed.

REMARK. If $k = 0$ in (10) and equality holds, then $n_1 = N$, $R_1 = N-1$, and if $N-1 \geq 2$, (10) again shows that $n_2 < (N-1)n_1/(N-1) = n_1$, contrary to assumption. Thus (11) holds also in the case $k = 0$, except for the single example $1 = 1/2 + 1/3 + 1/(2 \cdot 3)$.

Assembling all our results, we can now inaugurate a search for solutions. For fixed $N > 3$ we construct a tree of possibilities for $n_1 < n_2 < \cdots < n_{N-2}$ recursively as follows. The possibilities for n_1 are $2, 3, \ldots, N-1$. For each possible choice of $n_1, \ldots, n_k$, $1 < k \leq N-3$, the possibilities for n_{k+1} are those numbers in the interval from the minimum of n_k and $[\prod_{i=1}^{k} n_i/R_k] + 1$ to $[(N-k)\prod_{i=1}^{k} n_i/R_k]$ which are coprime to each of $n_1, \ldots, n_k$, where $[\ ]$ is the least integer function. To find actual solutions among these possibilities, for each choice of $n_1, \ldots, n_{N-2}$, put $C \doteq C(n_1, \ldots, n_{N-2}) = (\prod_{i=1}^{N-2} n_i)^2 + R_{N-2}$. By Proposition 12 a solution

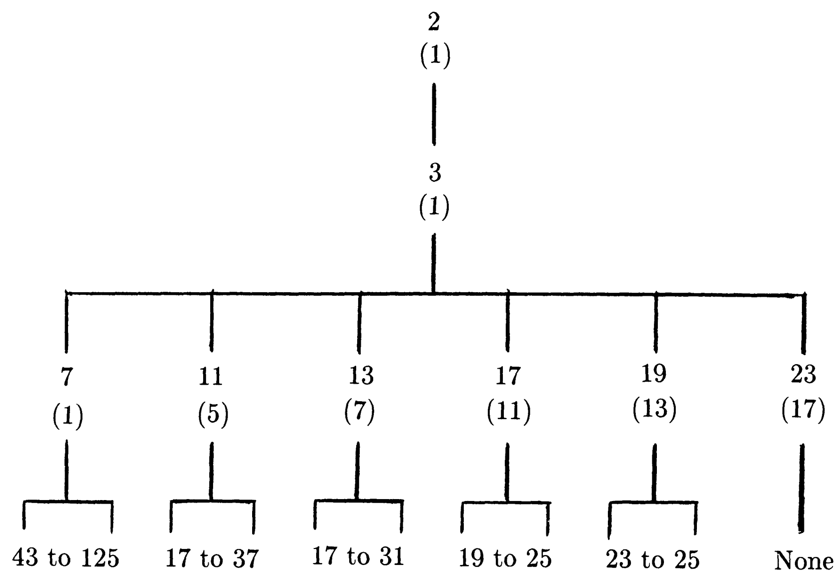
$$1 = \sum_{i=1}^{N-2} \frac{1}{n_i} + \frac{1}{n_{N-1}} + \frac{1}{n_N} + \frac{1}{\prod_{i=1}^{N} n_i}$$

exists if and only if C admits a factor $F \equiv -\prod_{i=1}^{N-2} n_i \bmod R_{N-2}$. Thus if D is the least positive residue of $-\prod_{i=1}^{N-2} n_i \bmod R_{N-2}$, we need only check to see if $D + mR_{N-2}$ divides C for $m = 0, 1, \ldots, [\prod_{i=1}^{N-2} n_i + D/R_{N-2}]$ (note that since $R_{N-2} < \prod_{i=1}^{N-2} n_i$, $\prod_{i=1}^{N-2} n_i = [\sqrt{C}]$). If $F = D + mR_{N-2}$ does divide C, then we have the solution

$$n_{N-1} = \frac{F + \prod_{i=1}^{N-2} n_i}{R_{N-2}}, \qquad n_N = \frac{\frac{C}{F} + \prod_{i=1}^{N-2} n_i}{R_{N-2}},$$

otherwise there is no solution.

We will illustrate the technique by finding all solutions for $N = 6$, thus providing a proof of Theorem 5 above. Here is the "(2,3) branch" of the tree of possibilities of depth $N - 2 = 4$, with remainders R_k shown in parentheses. The candidates for n_4 in the bottom row are all integers in the indicated ranges which are coprime to 2,3, and n_3.



For each of these possibilities we check the integer $(\prod_{i=1}^{4} n_i)^2 + R_4$ for factors congruent to $-\prod_{i=1}^{4} n_i$, as discussed above. Such factors exist only for $(2, 3, 7, 43)$, $(2, 3, 7, 47)$, $(2, 3, 7, 55)$, and $(2, 3, 11, 23)$. Taking all appropriate factorizations produces (see Example 13 above) the list of eight solutions given above in Theorem 5 for $N = 6$. The remaining branches of the tree can be checked similarly to confirm that there are no other solutions for $N = 6$, and the cases $N < 6$ are very easily computed as well to complete the proof of Theorem 5.

REMARK. To prune the tree we can also use the relations of Proposition 15. Namely, if A'_j now denotes the number of integers $n_1, \ldots, n_{N-2}$ which are congruent to $j \bmod 4$, and if F again denotes the factor of $(\prod_{i=1}^{N-2} n_i)^2 + R_{N-2}$ that we seek, then

Case 1. If $A_2' = 1$ and if
$A_1' \equiv 1 \bmod 4$, then there is no solution;
$A_1' \equiv 2 \bmod 4$, then $F \equiv 3 \bmod 4$;
$A_1' \equiv 0 \bmod 4$, then $F \equiv 1 \bmod 4$;
$A_1' \equiv 3 \bmod 4$, then no restriction on F.

Case 2. If $A_0 = 1$, and if
$A_3' \equiv 0 \bmod 4$, there is no solution;
$A_3' \equiv 1 \bmod 4$, then $F \equiv 3 \bmod 4$;
$A_3' \equiv 3 \bmod 4$, then $F \equiv 1 \bmod 4$;
$A_3' \equiv 2 \bmod 4$, then no restriction on F.

Case 3. If $A_0' + A_2' = 0$, and if $A_1' \equiv 1$ or 2 and $A_3' \equiv 0$ or $1 \bmod 4$, then there is no solution (and for certain other combinations of congruences for A_1' and A_3', there are various restrictions on F).

Thus in some cases we can eliminate the branch $(n_1, \dots, n_{N-2})$ altogether, while in some others, in the last step we need check only one number in every $4R_{N-2}$ for possible factors F. And, of course, we also need not check those candidates for F which have factors in common with any of $n_1, \dots n_{N-2}$. For instance if we apply these criteria to the last tier of the tree above, we achieve this shortened list of possibilities:

$$\begin{array}{ccc}
2 & 2 & 2 \\
3 & 3 & 3 \\
7 & 11 & 19 \\
43, 47, 55, 59, 67, 71, 79, 83, 95, 103, 107, 115 & 19, 23 & 23
\end{array}$$

and F must be $\equiv 1 \bmod 4$. Thus we have reduced the total number of computations by about 75%.

For $N > 6$ it is impossible to carry out this search procedure by hand. Already for $N = 7$ the single branch $(2, 3, 7, 43)$ has 504 children at level 5, even after pruning, some of which require on the order of 10^5 divisions to search for F.

$$\begin{aligned}
(504 = \tfrac{1}{2} \cdot 2 \cdot (2 \cdot 3 \cdot 7 \cdot 43)(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{7} - \tfrac{1}{43} + \tfrac{1}{2\cdot 3} + \tfrac{1}{2\cdot 7} + \tfrac{1}{2\cdot 43} + \tfrac{1}{3\cdot 7} \\
+ \tfrac{1}{3\cdot 43} + \tfrac{1}{7\cdot 43} - \tfrac{1}{2\cdot 3\cdot 7} - \tfrac{1}{2\cdot 3\cdot 43} - \tfrac{1}{2\cdot 7\cdot 43} \\
- \tfrac{1}{3\cdot 7\cdot 43} + \tfrac{1}{2\cdot 3\cdot 7\cdot 43}).)
\end{aligned}$$

For $N = 8$ the full problem involves on the order of 10^{13} numbers, with some final products as large as 10^{52}.

4. Some open questions. 1. First we repeat the Erdös problem mentioned in the introduction, in the form of a conjecture.

Conjecture. Given any set of mutually coprime integers $n_0, n_1, \ldots, n_k$ with $\sum_{i=1}^{k} 1/n_i < n_0$, there exist integers m, $n_{k+1}, \ldots, n_{k+m}$, all ≥ 2, such that

$$\sum_{i=1}^{k+m} \frac{1}{n_i} + \frac{1}{\prod_{i=1}^{k+m} n_i} = n_0.$$

This can be viewed as a question involving the distribution of points of the form $(\prod_{j=1}^{m} n_j, \sum_{j=1}^{m} \prod_{l \neq j} n_l)$ in the integer lattice $\mathbf{Z} \times \mathbf{Z} \subset \mathbf{R}^2$. Namely, the equation

$$1 = \sum_{i=1}^{k} \frac{1}{n_i} + \sum_{j=1}^{m} \frac{1}{n_j} + \frac{1}{\prod_{i=1}^{k} n_i \prod_{j=1}^{m} n_j}$$

is equivalent to

$$P \prod_{j=1}^{m} n_j - Q \left(\sum_{j=1}^{m} \prod_{l \neq j} n_l \right) = 1,$$

where $P = R_k = (\prod_{i=1}^{k} n_i)(1 - \sum_{i=1}^{k} 1/n_i)$ and $Q = \prod_{i=1}^{k} n_i$. Since P and Q are relatively prime, there exist integers x_0, y_0 such that $Px_0 - Qy_0 = 1$, and then all solutions to the equation $Px - Qy = 1$ have the form $x = x_0 + tQ$, $y = y_0 + tQ$, $t \in \mathbf{Z}$. Thus the question can be generalized as follows:

1′. Given any line $L = \{(x_0 + tQ, y_0 + tP) | t \in \mathbf{Z}\}$ in $\mathbf{Z} \times \mathbf{Z}$, with $Px_0 - Qy_0 = 1$, does L contain points of the form $(\prod_{j=1}^{m} n_j, \sum_{j=1}^{m} \prod_{l \neq j} n_l)$?

We note that if L contains any such point then it contains infinitely many, for a solution to $P/Q = \sum_{j=1}^{m} 1/n_j + 1/Q \prod_{j=1}^{m} n_j$ can always be extended to another solution by putting $n_{m+1} = Q \prod_{j=1}^{m} n_j + 1$.

2. If conjecture 1 is correct, for what other collections $\mathscr{C}$ of positive integers (other than $\mathscr{C} =$ all positive integers ≥ 2) is a solution to

$$n_0 = \sum_{i=1}^{k} \frac{1}{n_i} + \sum_{j=1}^{N-k} \frac{1}{n_{k+j}} + \frac{1}{\prod_{i=1}^{N} n_i}$$

possible for any n_0, $n_1, \ldots, n_k$? (Cf. **[12]**, **[14]**.) For example, can $\mathscr{C}$ be taken to be the set of prime numbers? The set of odd integers > 1?

If $\mathscr{C}$ is such a set, then certainly $\sum_{c\in\mathscr{C}} 1/c = \infty$, but this condition is far from sufficient. For instance, if $\mathscr{C}$ is the set of composite numbers then the remark preceding Definition 7 shows that there is no solution to $n_0 = \sum 1/n_i + 1/\prod n_i$ with $n_i \in \mathscr{C}\ \forall i$. Likewise, Proposition 15, Case 1, shows that for $n_0 = 1$, $n_1 = 2$, $n_2 = 5$, and $\mathscr{C} = \{3 + 4t | t = 1, 2, \ldots\}$, there is no solution to $n_0 = \sum_{i=1}^{N} 1/n_i + 1/\prod_{i=1}^{N} n_i$ with $n_i \in \mathscr{C}$ for $i = 3, \cdots, N$.

In a similar spirit, let n_1 be any integer > 2 and let s be any positive integer which does not generate the multiplicative group of integers $\operatorname{mod} n_1$ which are coprime to n_1. Then there exists an integer n_2, coprime to n_1, for which there is no solution N to the congruence $s^N \equiv -n_2^{-1} \operatorname{mod} n_1$. But then the sequence $1/n_1 + 1/n_2$ cannot be completed to a solution to $1 = \sum_{i=1}^{N} 1/n_i + 1/\prod_{i=1}^{N} n_i$ with $n_3, \ldots, n_N$ all chosen from the set $\mathscr{C} = \{s + n_1 t | t = 1, 2, \ldots\}$. To see this, clear denominators and reduce $\operatorname{mod} n_1$ to obtain the congruence

$$0 \equiv n_2 s^N + 1 \quad \operatorname{mod} n_1,$$

which by choice of s and n_2 has no solution N.

Does there exist any arithmetic sequence $\mathscr{C} = \{A + Bt | t = 1, 2, \ldots\}$, $B > 2$, with the property that every initial set $n_0, n_1, \ldots, n_k$ can be completed by elements of $\mathscr{C}$? (If so, then a slight variation of the argument of the previous paragraph shows that B is of the form $B = p^k$ for P an odd prime.)

3. The "Greedy Algorithm". Restricting our attention to the case $n_0 = 1$, an obvious procedure for searching for solutions is as follows. Given any choice of $n_1, \ldots, n_k$, choose n_{k+1} to be the smallest integer coprime to $n_1, \ldots, n_k$ for which $\sum_{i=1}^{k+1} 1/n_i < 1$, and similarly for $n_{k+2}, n_{k+3}, \ldots$. For instance, the sequence of solutions

$$2, 3, 7, 43, 1807, \ldots, \prod_{j=1}^{i} n_j + 1, \ldots$$

is found in this way, starting with the empty set of integers. (These numbers give the largest values of n_{N+1} to be found in *any* solution to $1 = \sum_{i=1}^{N} 1/n_i + 1/n_{N+1}$ (see **[9]** or **[25]**, e.g.).) We ask: For which initial sets $\{n_1, \ldots, n_k\}$ will this algorithm eventually produce a solution?

(We note parenthetically that if we do not require that the last term be the product of the first N terms, then the greedy algorithm always terminates after at most $R_k - 1$ steps. That is, given $n_1, \ldots, n_k$ with

$1 - \sum_{i=1}^{k} 1/n_i = (R_k / \prod_{i=1}^{k} n_i) > 0$, then choosing n_{k+j} = the smallest integer for which $\sum_{i=1}^{k+j} 1/n_i < 1$ produces a strictly decreasing sequence of remainders R_{k+j} and thus eventually a solution to the equation $1 = \sum_{i=1}^{N} 1/n_i + 1/n_{N+1}$. This result is contained in work of Fibonacci ([**24**]) dating back to 1202. But the greedy algorithm technique rarely produces numbers n_i which are mutually coprime so that $n_{N+1} = \prod_{i=1}^{N} n_i$. Nevertheless, it is possible that the answer to question 3 is "all sets of mutually coprime integers n_i with $\sum 1/n_i < 1$".)

4. How many solutions $(n_1, \ldots, n_N)$ are there for large N, and what is their distribution in the integer lattice $\mathbf{Z}^N$? Our search technique gives the estimate

$$\frac{\prod_{i=1}^{k} n_i}{R_k} < n_{k+1} < (N-k)\frac{\prod_{i=1}^{k} n_i}{R_k}$$

for the "bushiness" of the tree of possibilities at the $(k+1)$st step for the branch $(n_1, \ldots, n_k)$. For given, N, $n_1 < \cdots < n_k$, define the *width of the $n_1, \ldots, n_k$ branch at depth N* to be the integer $W(N; n_1, \ldots, n_k)$ = the number of sequences $(n_{k+1}, \ldots, n_N)$ which can be chosen recursively under the constraints that $\forall j = 1, \ldots, N-k$, $n_{k+j} > n_{k+j-1}$, n_{k+j} is coprime to $n_1, \ldots, n_{k+j-1}$, and

$$\frac{(\prod_{i=1}^{k+j} n_j) + 1}{R_{k+j}} \leq n_{k+j} \leq (N-k-j)\frac{(\prod_{i=1}^{k+j} n_i) + 1}{R_{k+j}}.$$

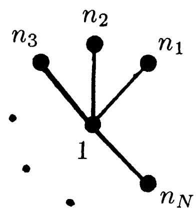
Let $S(N; n_1, \ldots, n_k)$ denote the number of solutions $(n_{k+1}, \ldots, n_N)$ to $1 = \sum_{i=1}^{N} 1/n_i + 1/\prod_{i=1}^{N} n_i$. Do there exist positive constants c_1 and c_2 such that for all pairs $(n_1, \ldots, n_k)$, $(n'_1, \ldots, n'_{k'})$ of sequences,

$$\begin{aligned} c_1 &< \liminf_{N\to\infty} \frac{S(N; n_1, \ldots, n_k)}{W(N; n_1, \ldots, n_k)} \Big/ \frac{S(N; n'_1, \ldots, n'_{k'})}{W(N; n'_1, \ldots, n'_{k'})} \\ &< \limsup_{N\to\infty} \frac{S(N; n_1, \ldots, n_k)}{W(N; n_1, \ldots, n_k)} \Big/ \frac{S(N; n'_1, \ldots, n'_{k'})}{W(N; n'_1, \ldots, n'_{k'})} < c_2? \end{aligned}$$

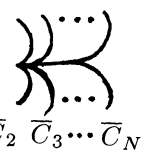
This would, of course, imply that every sequence $n_1, \ldots, n_k$ can be completed in many ways, in particular, in at least one way. It also means that the solutions are more or less randomly distributed among all "possibilities". If this is not true, which sequences $n_1, \ldots, n_k$ are relatively fertile and which are relatively barren of completions to solutions to our equation (1)? We note also that solutions to (1) are extremely sparse among *all* solutions in integers to $1 = \sum_{i=1}^{N} 1/n_i + 1/n_{N+1}$. For example, Singmaster (unpublished) calculates that for

$N = 5$ there are 3462 distinct solutions with $n_1 \leq \cdots \leq n_5 \leq n_6$. Only 3 of these have $n_6 = \prod_{i=1}^{5} n_i$ as we require.

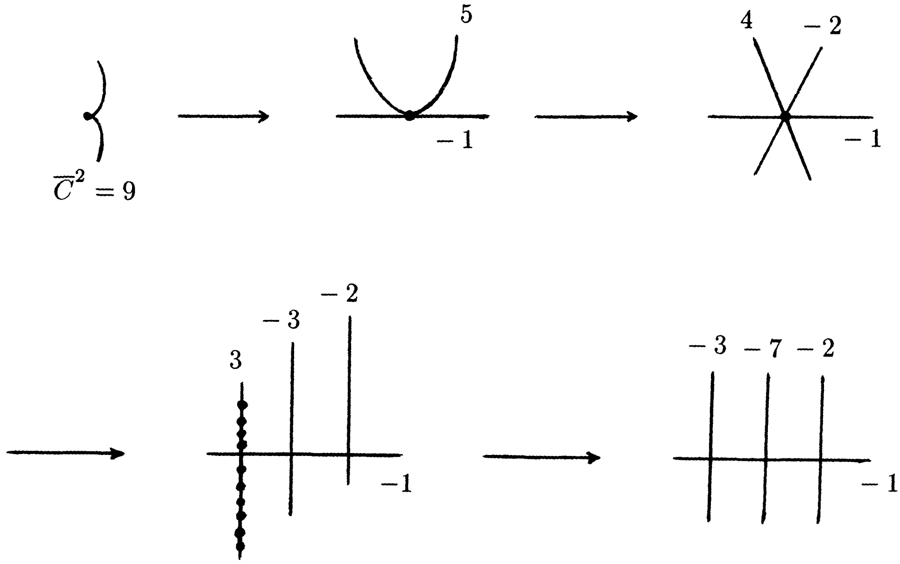
5. What can we say about the singular points $x \in X$ which correspond to our solutions? For starters, these points are "highly cusp-like" in the following sense. Let $p\colon \tilde{X} \to X$ be the minimal normal resolution of singularities, with weighted dual graph



Suppose that $n_1 = 2$ and $n_2 = 3$. Then the *minimal* (non-normal) resolution is obtained by blowing down the exceptional curves C_0, C_1 and C_2 (those with self-intersection -1, -2, and -3, respectively) to a point $\tilde{x} \in \tilde{X}$. The resulting surface $\tilde{X}$ is non-singular with negatively embedded curve $\tilde{C} = \bigcup_{i=3}^{N} \overline{C}_i$, each component of which is a rational curve with a cusp of high order, all with the same 1-dimensional tangent space at $\overline{x}$:



This observation provides a natural construction for at least some of our singular points x. Namely, on the projective plane $\mathbf{CP}^2$, find $N - 2$ rational curves $\overline{C}_i$ with cusps of the form $y^2 = x^{m_i}$ at the origin of $\mathbf{C}^2 \subset \mathbf{CP}^2$. Resolve all the other singularities of $\bigcup_{i=1}^{N} \overline{C}_i$, if any, blow up the origin, then twice blow up the point of mutual tangential intersection of the proper transform of the $\overline{C}_i$. Finally, blow up additional points as necessary to lower the self-intersections of the new curves to the desired weights $-n_i$. The perfect singularity $x \in X$ is then obtained by blowing down all the resulting curves C_i, together with the three new curves introduced in the first three blow-ups. For example, the minimal normal non-singular model of the minimally elliptic point $x^2 + y^3 + z^7 = 0$ (Example 3 above) is achieved from the curve $zy^2 = x^3$ in $\mathbf{CP}^2$ by blowing up points as pictured below.



We also note that the positive definite symmetric bilinear forms ϕ associated to these graphs are highly eccentric, in the sense that some eigenvalues are very large and some are very small. To see this, recall that the trace of ϕ is the large integer $\sum_{i=1}^{N} n_i + 1$, while the determinant is equal to 1. Thus the largest eigenvalue is larger than $(\sum_{i=1}^{N} n_i + 1)/(N+1)$, while the smallest is smaller than the $-N$th root of the largest. Renewed interest in the possible geometric interpretation of these eigenvalues was sparked by McKay's observation **[21]** of their significance in the case of rational double points. We ask, then, what properties of the singular points are exposed by an analysis of the eigenvalues of this unimodular form?

Appendix. All solutions in integers $n_1 < n_2 < \cdots < n_N$ to the equation $1 = \sum_{i=1}^{N} 1/n_i + 1/\prod_{i=1}^{N} n_i$ for $N = 7$.

2, 3, 7, 43, 1807, 3263443, 10650056950807
2, 3, 7, 43, 1807, 3263447, 2130014000915
2, 3, 7, 43, 1807, 3263591, 71480133827
2, 3, 7, 43, 1807, 3264187, 14298637519
2, 3, 7, 43, 1823, 193667, 637617223447
2, 3, 7, 43, 3263, 4051, 2558951
2, 3, 7, 43, 3559, 3667, 33816127
2, 3, 7, 47, 395, 779731, 607979652631
2, 3, 7, 47, 395, 779831, 6020772531

2, 3, 7, 47, 403, 19403, 15435513367
2, 3, 7, 47, 415, 8111, 6644612311
2, 3, 7, 47, 583, 1223, 1407479767
2, 3, 7, 55, 179, 24323, 10057317271
2, 3, 7, 67, 187, 283, 334651
2, 3, 11, 17, 101, 149, 3109
2, 3, 11, 23, 31, 47059, 2214502423
2, 3, 11, 23, 31, 47063, 442938131
2, 3, 11, 23, 31, 47095, 59897203
2, 3, 11, 23, 31, 47131, 30382063
2, 3, 11, 23, 31, 47243, 12017087
2, 3, 11, 23, 31, 47423, 6114059
2, 3, 11, 23, 31, 49759, 866923
2, 3, 11, 23, 31, 60563, 211031
2, 3, 11, 25, 29, 1097, 2753
2, 3, 11, 31, 35, 67, 369067
2, 3, 13, 25, 29, 67, 2981

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