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POTENTIAL ESTIMATES IN ORLICZ SPACES

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We study estimates of the form

$$M_0^{-1}\left(\int M_0(u)\,d\mu(x)\right) \le CM^{-1}\left(\int M([1-\Delta]^m u)\,d\nu(x)\right)$$

for $u(x) \in C^\infty(\mathbf{R}^n)$, where $M_0(t)$, $M(t)$ are convex functions and μ, ν are measures. We apply this inequality to the study of boundary value problems for quasilinear partial differential equations.

1. Introduction. In recent years there has been considerable interest in inequalities of the form

$$(1.1)\qquad \left(\int |u(x)|^q\,d\mu(x)\right)^{1/q} \le C\left(\int |(1-\Delta)^m u|^p\,d\nu(x)\right)^{1/p}, \qquad u \in C^\infty(\mathbf{R}^n)$$

(cf. [3–24] and the references quoted in them). Such inequalities have widespread applications to both linear and non-linear problems. We outline one here. Suppose one is interested in solving the Dirichlet problem

$$(1.2)\qquad (1-\Delta)^m u = f(x,u) \quad \text{in } \Omega \subset \mathbf{R}^n, \qquad u(x) = 0 \quad \text{on } \partial\Omega$$

(here $\partial\Omega$ is the boundary of the domain Ω). One can solve (1.2) by topological methods if one can show that

$$(1.3)\qquad \int |(1-\Delta)^m u(x)|^p\,d\nu(x) < \infty$$

implies

$$(1.4)\qquad \int |f(x,u(x))|^p\,d\nu(x) < \infty.$$

If $f(x,t)$ satisfies

$$(1.5)\qquad |f(x,t)| \le \sum V_k(x)|t|^{\alpha_k}$$

and the inequalities

$$(1.6)\qquad \left(\int V_k(x)^p |u(x)|^{\alpha_k p}\,d\nu(x)\right)^{1/\alpha_k p} \le C_k\left(\int |(1-\Delta)^m u(x)|^p\,d\nu(x)\right)^{1/p}$$

hold, then (1.3) will imply (1.4). Inequality (1.6) is precisely of the form (1.1) if we take $q = \alpha_k p$ and $d\mu = V_k^p\, d\nu$. This method was used by the author in [**23**, **24**]. If instead of (1.5), $f(x,t)$ satisfied

$$|f(x,t)| \le \sum V_k(x) M_k(t) \tag{1.7}$$

where the $M_j(t)$ are convex functions, we would need inequalities of the form

$$\left(\int V_k(x)^p M_k(u)^p\, d\nu(x)\right)^{1/p} \le c M_k \left\{ \left(\int |(1-\Delta)^m u(x)|^p\, d\nu(x)\right)^{1/p}\right\}. \tag{1.8}$$

This is a special case of the inequality we consider in this paper.

There are several approaches one can apply to prove such inequalities. We have chosen the capacity method as described in [**1**, **2**, **8**, **14**, **15**, **16**]. It is quite clear that other methods can be applied as well.

Expressions of the form

$$\rho_\nu(u, M) = \int M(u)\, d\nu$$

appear in the study of Orlicz spaces (cf. [**13**]). Some of the techniques used in the study of such spaces are useful in dealing with the problems outlined here. The functions $M_0(t)$, $M(t)$ considered by us are not required to have all of the properties of N-functions as described in [**13**].

Now we describe some of the results of the paper. First we find sufficient conditions for

$$M_0^{-1}\{\rho_\mu(u, M_0)\} \le C M^{-1}\{\rho_\nu([1-\Delta]^m u, M)\} \tag{1.9}$$

to hold for all $u \in C^\infty(\mathbf{R}^n)$. Our main hypothesis is

$$|||J_{2m,\mu}\chi_e|||_{2m,t,e,\nu,M} \le C\mu(e)/M_0^{-1}[\mu(e)], \qquad e \subset \mathbf{R}^n, t > 0, \tag{1.10}$$

where the left hand side is an Orlicz type norm not only depending on m, ν, M but also on the set e and a parameter $t > 0$. $J_{2m,\mu}$ is the Bessel potential of order $2m$ with respect to the measure μ. For the special case of inequality (1.1), hypothesis (1.10) is implied by

$$\left\| \frac{dx}{d\nu} J_{2m,\mu}\chi_e \right\|_{p',\nu} \le C\mu(e)^{1/q'}, \qquad e \subset \mathbf{R}^n. \tag{1.11}$$

For the case $1 \le p \le q$ we show that (1.11) implies (1.1) provided ν is sufficiently regular (i.e., satisfies (2.8)). This generalizes results of Adams [1, 2] and Kerman-Sawyer [10, 11] who considered the case $d\nu = dx$.

In practice inequality (1.11) is not very useful. It requires an inequality to hold for arbitrary (closed) subsets e of $\mathbf{R}^n$. Even if we can restrict the shape of the sets e, it would be virtually impossible in general to verify whether or not (1.11) holds. For this reason we replaced (1.11) by

$$(1.12) \qquad \|(dx/d\nu)^q J_{\alpha,\mu} 1\|_{p'/q,\nu} < \infty$$

where α satisfies

$$(1.13) \qquad 0 < \alpha/nq = 2m/n - 1/q'.$$

The expression (1.12) involves one calculation of integrals of given quantities. Using this expression we obtain the inequality

$$(1.14) \qquad \|Vu\|_q \le cM_{\alpha,q,p',\nu}(V)\|(1-\Delta)^m u\|_{p,\nu}$$

where α satisfies (1.13) and

$$(1.15) \qquad M_{\alpha,q,t,\nu}(V) = \|(dx/d\nu)^q J_\alpha |V|^q\|_{t/q,\nu}^{1/q}$$

is a norm. This is convenient in some situations. For instance, if $d\mu_k = V_k^q\, dx$ and each satisfies (1.12), it is clear that $d\mu = (\sum V_k)^q\, dx$ will also satisfy it. On the other hand, it is not so clear that the same is true of (1.11). Moreover, (1.12) implies (1.1) even when $q < p$, but (1.11) does not.

All of the inequalities mentioned have counterparts when one replaces $(1-\Delta)$ by $-\Delta$. We give sufficient conditions for the inequality

$$(1.16) \qquad M_0^{-1}[\rho_\mu(u, M_0)] \le CM^{-1}[\rho_\nu(\Delta^m u, M)]$$

to hold for all $u \in C_0^\infty$. We show that this inequality holds if we replace the Bessel potential $J_{s,\mu}$ with the Riesz potential $I_{s,\mu}$ in the hypotheses. In particular we show that

$$(1.17) \qquad \|u\|_{q,\mu} \le c\|(dx/d\nu)^q I_{\alpha,\mu} 1\|_{p'/q,\nu}^{1/q} \|\Delta^m u\|_{p,\nu}$$

holds when $1 \le p, q$.

In dealing with inequality (1.9) we introduced a capacity depending on a parameter:

$$(1.18) \qquad c_{s,t,\nu,M}(e) = \inf_v \{\rho_\nu(V,M), J_s v \ge t \text{ on } e\}.$$

The reason for this is that we do not have the homogeneity properties of the L^p spaces. The corresponding capacity estimate that we need is

$$(1.19)\quad \int_0^\infty c_{2m,t,\nu,M}(\{x \in R^n : |u(x)| > t\}) M(t)^{-1}\, dM(t) \le C\rho_\nu([1-\Delta]^m u, M), \qquad u \in C_0^\infty.$$

In dealing with inequality (1.16) we define the corresponding capacity by replacing J_s by I_s in (1.18). The inequality corresponding to (1.19) replaces $1-\Delta$ by $-\Delta$.

2. The inequalities. For a function $u(x) \in C^m(\mathbf{R}^n)$, we let $D^m u(x)$ denote the vector, the components of which consist of all derivatives of u of order m. By $|D^m u(x)|$ we shall denote the sum of the absolute values of all such components. We let $M_0(t)$, $M(t)$ be continuous, even functions, with $M(t)$ convex and $M_0(t)$ strictly increasing in $|t|$. Also we assume

$$\text{(a)}\qquad M(2t) \le C_1 M(t), \qquad t > 0,$$

$$\text{(b)}\qquad M_0^{-1}\left\{\int_0^\infty M_0(f(t))\, dM_0(t)\right\} \le C_2 M^{-1}\left\{\int_0^\infty M(f(t))\, dM(t)\right\}$$

holds for all non-increasing functions $f(t) \ge 0$.

$$\text{(c)}\qquad M_0(t) \to \infty, \quad M(t) \to \infty \text{ as } t \to \infty,$$

$$\text{(d)}\qquad M(0) = 0.$$

If we take $M_0(t) = t^q$, $M(t) = t^p$, then (b) becomes

$$\left(\int_0^\infty f(t)^q\, dt^q\right)^{1/q} \le c\left(\int_0^\infty f(t)^p\, dt^p\right)^{1/p}.$$

This holds for non-increasing $f(t)$ provided $1 \le p \le q$. We define

$$(2.1)\qquad \rho_\nu(u, M) = \int M(u)\, d\nu$$

and we are interested in determining conditions on $M_0(t)$, $M(t)$, ν, μ so that the inequality

$$(2.2)\qquad M_0^{-1}\{\rho_\mu(u, M_0)\} \le CM^{-1}\{\rho_\nu([1-\Delta]^m u, M)\}$$

holds for all $u \in C^\infty(\mathbf{R}^n)$. For this we shall assume also that

(e) $$\rho_\nu(T_n v, M) \le C\rho_\nu(v, M), \qquad v \in C(\mathbf{R}^n)$$

where

$$T_n v(x) = \sup_{\delta \le 1} \delta^{-n} \int_{|x-y| \le \delta} v(y)\, dy.$$

When $M(t) = t^p$ with $1 < p < \infty$, it is well known that (e) holds for the case $d\nu = dx$ (cf., e.g., **[26]**). For other cases necessary and sufficient conditions can be found in **[17, 19]**.

In stating our hypotheses we shall make use of the operator

$$J_s f = (1 - \Delta)^{-s/2} f.$$

It is well known that

(2.3) $$J_s f(x) = \int_{\mathbf{R}^n} G_s(x - y) f(y)\, dy$$

where the function $G_s(x) \in C^\infty(\mathbf{R}^n - \{0\})$ and satisfies

(2.4) $$\begin{aligned} C_3|x|^{s-n} &\le G_s(x) \le C_4|x|^{s-n}, && |x| < 1, \\ C_5|x|^\gamma e^{-|x|} &\le G_s(x) \le C_6|x|^\gamma e^{-|x|}, && |x| > 1, \end{aligned}$$

where $\gamma = (s - n - 1)/2$ (cf. **[3]**). Let

$$(u, v) = \int_{\mathbf{R}^n} u(x)\overline{v(x)}\, dx.$$

For any set $e \subset \mathbf{R}^n$ define

(2.5) $$|||u|||_{s,t,e,\nu,M} = \sup_{\substack{J_s v \ge t \\ \text{on } e}} \frac{(u, v)}{tM^{-1}[M(t)^{-1}\rho_\nu(v, M)]}$$

where the supremum is taken over all v such that $J_s v(x) \ge t$ for $x \in e$.

Our first result is

THEOREM 2.1. *Assume that there is a constant C_7 independent of t and e such that*

(2.6) $$|||J_{2m,\mu}\chi_e|||_{2m,t,e,\nu,M} \le C_7\mu(e)/M_0^{-1}[\mu(e)], \qquad e \subset \mathbf{R}^n,\ t > 0,$$

where χ_e is the characteristic function of e and

(2.7) $$J_{2m,\mu}\chi_e(x) = \int_e G_{2m}(x - y)\, d\mu(y).$$

Assume also that ν is absolutely continuous with respect to Lebesgue measure and satisfies

$$\sum_{k=1}^{2m} J_k(d\nu/dx) \le C\, d\nu/dx. \tag{2.8}$$

Under the assumptions given above, there is a constant C such that (2.2) *holds for all $u \in C_0^\infty$.*

As an application of Theorem 2.1 let us consider the case $M_0(t) = t^q$, $M(t) = t^p$ with $1 \le p \le q$. They satisfy the hypotheses of Theorem 2.1. In this case

$$tM^{-1}[M(t)^{-1}\rho_\nu(v, M)] = \|v\|_{p,\nu} = \left(\int |v(x)|^p\, d\nu(x)\right)^{1/p}.$$

Thus if $dx \ll d\nu$,

$$|||w|||_{2m,t,e,\nu,M} = \sup_{\substack{J_{2m}v \ge t \\ \text{on } e}} \frac{(w, v)}{\|v\|_{p,\nu}} \le \left\| \frac{dx}{d\nu} w \right\|_{p',\nu}.$$

This means that (2.6) is implied by

$$\left\| \frac{dx}{d\nu} J_{2m,\mu}\chi_e \right\|_{p',\nu} \le C_7 \mu(e)^{1/q'}, \qquad e \subset \mathbf{R}^n. \tag{2.9}$$

Hence we have

THEOREM 2.2. *If $1 \le p \le q$, then a sufficient condition for* (1.1) *to be true is that* (2.8) *and* (2.9) *hold.*

Theorem 2.2 was proved by D. R. Adams **[2]** and Kerman-Sawyer **[10, 11]** for the case $d\nu = dx$, $p > 1$. The latter result requires (2.9) to hold only for the sets e which are dyadic cubes.

Inequality (2.6) is rather difficult to verify in practice, since C_7 is essentially the supremum of a ratio over all subsets e of $\mathbf{R}^n$. For practical purposes it is much better to give a single expression which requires a single calculation. This is given by

COROLLARY 2.3. *For any $p, q \ge 1$,*

$$\|u\|_{q,\mu} \le C\|(dx/d\nu)^q J_{\alpha,\mu} 1\|_{p'/q,\nu}^{1/q} \|(1-\Delta)^m u\|_{p,\nu} \tag{2.10}$$

where

$$0 < \alpha/nq \le 2m/n - 1/q'. \tag{2.11}$$

Proof. Let $f = (1-\Delta)^m u$. Then $u = J_{2m}f$. Thus

$$\begin{aligned}\|J_{2m}f\|_{q,\mu} &= \left(\int\left(\int G_{2m}(x-y)f(y)\,dy\right)^q d\mu(x)\right)^{1/q}\\ &\le \int\left(\int G_{2m}(x-y)^q\,d\mu(x)\right)^{1/q} |f(y)|\,(dy/d\nu)\,d\nu\\ &\le \left(\int\left(\int G_\alpha(x-y)\,d\mu(x)\right)^{p'/q}(dy/d\nu)^{p'}\,d\nu\right)^{1/p'}\|f\|_{p,\nu}\end{aligned}$$

since

$$G_{2m}(x)^q \le CG_\alpha(x)$$

when α satisfies (2.11). This gives (2.10). □

When $d\mu = |V(x)|^q\,dx$, this reduces to

COROLLARY 2.4. *For any* $p, q \ge 1$

$$\|Vu\|_q \le C\|(dx/d\nu)^q J_\alpha|V|^q\|_{p'/q,\nu}^{1/q}\|(1-\Delta)^m u\|_{p,\nu} \tag{2.12}$$

provided α *satisfies* (2.11).

The expression

$$M_{\alpha,q,t,\nu}(V) = \|(dx/d\nu)^q J_\alpha|V|^q\|_{t/q,\nu}^{1/q} \tag{2.13}$$

is a norm, and (2.12) states that

$$\|Vu\|_q \le CM_{\alpha,q,p',\nu}(V)\|(1-\Delta)^m u\|_{p,\nu} \tag{2.14}$$

where α satisfies (2.11). In a sense, $M_{\alpha,q,p',\nu}(V)$ is the smallest norm for which (2.14) holds. Note that Corollaries 2.3 and 2.4 do not require $p \le q$ nor do they need assumptions (2.8) or (e).

As another example of functions $M_0(t)$, $M(t)$ satisfying (b), let $M_0(t) = \phi(t)^q$, $M(t) = \phi(t)^p$, where $1 \le p \le q$ and ϕ satisfies the conditions imposed on $M(t)$ at the beginning of the section. Then

$$M_0^{-1}(\tau) = \phi^{-1}(\tau^{1/q}), \qquad M^{-1}(\tau) = \phi^{-1}(\tau^{1/p}).$$

Thus the left hand side of (b) equals

$$\phi^{-1}\left[\left(\int_0^\infty \phi(f(t))^q\,d\phi(t)^q\right)^{1/q}\right] \le \phi^{-1}\left[\left(\int_0^\infty \phi(f(t))^p\,d\phi(t)^p\right)^{1/p}\right].$$

This equals the right hand side. Thus (b) holds in this case as well.

In place of (2.2) one can consider inequalities of the form

$$M_0^{-1}[\rho_\mu(u, M_0)] \le CM^{-1}[\rho_\nu(\Delta^m u, M)]. \tag{2.15}$$

The methods used to prove Theorem 2.1 apply equally well here. In this case we replace the operator J_s with

$$I_s f(x) = c_s \int |x-y|^{s-n} f(y)\,dy \tag{2.16}$$

where

$$c_s = \Gamma(\tfrac{1}{2}(n-s))/2^s \pi^{n/2}\Gamma(s/2)$$

and we replace assumption (e) with

$$\rho_\nu(Tv, M) \le C\rho_\nu(v, M), \qquad v \in C(\mathbf{R}^n) \tag{e$'$}$$

where

$$Tv(x) = \sup_\delta \delta^{-n} \int_{|x-y|<\delta} v(y)\,dy$$

is the Hardy-Littlewood maximal function (modulo a constant). We have

THEOREM 2.5. *Inequality* (2.15) *holds under the hypotheses of Theorem* 2.1 *if we replace* J_s *by* I_s *in* (2.5)–(2.8) *and replace hypothesis* (e) *by* (e$'$).

The counterpart of Theorem 2.2 is

THEOREM 2.6. *Assume that* $1 \le p \le q$ *and that*

$$\|(dx/d\nu) I_{2m,\mu}\chi_e\|_{p',\nu} \le C\mu(e)^{1/q'}, \qquad e \subset \mathbf{R}^n. \tag{2.17}$$

Then

$$\|u\|_{q,\mu} \le C\|\Delta^m u\|_{p,\nu}, \qquad u \in C_0^\infty. \tag{2.18}$$

Corresponding to Corollary 2.3 we have

COROLLARY 2.7. *For any* $p, q \ge 1$

$$\|u\|_{q,\mu} \le C\|(dx/d\nu)^q I_{\alpha,\mu}1\|_{p'/q,\nu}^{1/q} \|\Delta^m u\|_{p,\nu} \tag{2.19}$$

where

$$\alpha/nq = 2m/n - 1/q'. \tag{2.20}$$

In place of Corollary 2.4 we have

COROLLARY 2.8. *For any* $p, q \geq 1$

$$\|Vu\|_q \leq C\|(dx/d\nu)^q I_\alpha |V|^q\|_{p'/q,\nu}^{1/q} \|\Delta^m u\|_{p,\nu} \tag{2.21}$$

where α *satisfies* (2.18).

Note that (2.8) is not required in Theorems 2.5, 2.6 and Corollaries 2.7, 2.8. It is also not required that $p \leq q$.

3. Variable capacity. In proving Theorem 2.1 we shall prove a capacity inequality of the type derived in [**1, 14, 15**]. However, we lack homogeneity and are forced to use a capacity depending on a parameter. We define

$$c_{s,t,\nu,M}(e) = \inf_v \{\rho_\nu(v, M), J_s v \geq t \text{ on } e\}. \tag{3.1}$$

For $u \in C^\infty(\mathbf{R}^n)$, let

$$N_t = N_t(u) = \{x \in R^n : |u(x)| > t\}.$$

Set

$$c_{s,\nu,M}(t) = c_{s,t,\nu,M}(N_t).$$

We shall prove

THEOREM 3.1. *Under the hypotheses of* §2

$$\begin{aligned}\int_0^\infty c_{2m,\nu,M}(t) M(t)^{-1}\, dM(t) \\ \leq C\rho_\nu([1-\Delta]^m u, M), \qquad u \in C_0^\infty.\end{aligned} \tag{3.2}$$

Before proving Theorem 3.1, we show how it implies Theorem 2.1. For $u \in C^\infty$, let $f(t) = M_0^{-1}(\mu(N_t))$. Then the left hand side of (2.2) equals

$$\begin{aligned} M_0^{-1}&\left\{\int_0^\infty \mu(N_t)\, dM_0(t)\right\} \\ &= M_0^{-1}\left\{\int_0^\infty M_0(f(t))\, dM_0(t)\right\} \\ &\leq C_2 M^{-1}\left\{\int_0^\infty M(f(t))\, dM(t)\right\}\end{aligned} \tag{3.3}$$

by (b) of §2. Now by (2.5)

$$\begin{aligned}\mu(e) &\leq t^{-1}\int [J_{2m}v]\chi_e\, d\mu \leq t^{-1}\int v J_{2m,\mu}\chi_e\, dx \\ &\leq M^{-1}[M(t)^{-1}\rho_\nu(v, M)]\||J_{2m,\mu}\chi_e\||_{2m,t,e,\nu,M}\end{aligned}$$

provided $J_{2m}v \geq t$ on e. Thus by (2.6)

$$\mu(e) \leq M^{-1}[M(t)^{-1}\rho_\nu(v, M)]C_7\mu(e)/M_0^{-1}[\mu(e)].$$

Hence

$$M_0^{-1}[\mu(e)] \leq C_7 M^{-1}[M(t)^{-1}\rho_\nu(v, M)].$$

This implies that

$$\begin{aligned} M[f(t)] = M[M_0^{-1}\{\mu(N_t)\}] &\leq M[C_7 M^{-1}\{M(t)^{-1}\rho_\nu(v, M)\}] \\ &\leq C' M(t)^{-1}\rho_\nu(v, M) \end{aligned}$$

by (a) of §2. Since this is true for every v such that $J_{2m}v \geq t$ on N_t, we have

$$M[f(t)] \leq C' M(t)^{-1} c_{2m,\nu,M}(t).$$

Thus the right hand side of (3.3) is bounded by

$$C_2 M^{-1}\{C' \int_0^\infty c_{2,m,\nu,M}(t) M(t)^{-1}\, dM(t)\}.$$

The assumptions on M imply

$$M^{-1}(Cu) \leq \max(C, 1) M^{-1}(u). \tag{3.4}$$

If we now apply Theorem 3.1 we obtain (2.2). □

Inequality (3.4) is a consequence of

$$M(\theta t) \leq \theta M(t), \qquad 0 \leq \theta \leq 1 \tag{3.5}$$

which follows from the convexity of M and (d) of §2.

4. The capacity inequality. In proving Theorem 3.1 we shall make use of some elementary lemmas. We shall make use of the following function for $r < s$

$$\begin{aligned} G_{s,r}(x) &= G_{s-r}(x), & |x| \leq 1 \\ &= G_s(x), & |x| > 1. \end{aligned}$$

Our need for these functors stems from the fact that

$$|D^r G_s(x)| \leq C G_{s,r}(x). \tag{4.1}$$

This can be verified by noting that

$$\partial G_s(x)/\partial x_j = c x_j G_{s-2}(x)$$

and applying (2.4). If we introduce

$$J_{s,r} f(x) = \int G_{s,r}(x - y) f(y)\, dy$$

we can state

LEMMA 4.1. *If* $0 \le k \le r < s$, *then*

$$|D^k J_s f| \le c J_s f^{1-(k/r)} J_{s,r} f^{k/r}, \qquad f \ge 0.$$

Proof. We note that

$$G_{s,k}(x) \le c G_s(x)^{1-(k/r)} G_{s,r}(x)^{k/r} \tag{4.2}$$

by (2.4). If we now make use of (4.1), we have

$$\begin{aligned} |D^k J_s f(x)| &\le c \int G_{s,k}(x-y) f(y)\, dy \\ &\le c \left(\int G_s(x-y) f(y)\, dy \right)^{1-(k/r)} \left(\int G_{s,r}(x-y) f(y)\, dy \right)^{k/r} \\ &= c J_s f^{1-(k/r)} J_{s,r} f^{k/r}. \end{aligned}$$

□

LEMMA 4.2. *Let* $\{t_j\}_{-\infty}^{\infty}$ *be a sequence of positive numbers such that*

$$t_j \to 0 \quad \text{as } j \to -\infty, \qquad t_j \to \infty \quad \text{as } j \to \infty, \tag{4.3}$$

$$t_{j-1} \le \theta t_j, \quad t_j \le C_8 t_{j-1}, \qquad 0 < \theta < 1. \tag{4.4}$$

Let $\varphi(t) \in C^\infty(\mathbf{R})$ *be such that*

$$\begin{aligned} \varphi(t) &= 0, \qquad t < \delta, \\ &= 1, \qquad t > 1 - \delta, \end{aligned}$$

where δ *is some positive quantity. Let* $F_j(t)$ *be the* C^∞ *function*

$$F_j(t) = t_j \varphi[(t - t_{j-1})/(t_j - t_{j-1})]. \tag{4.5}$$

Then

$$F_j(t) = t_j, \qquad t \ge t_j \tag{4.6}$$

and

$$t^{k-1} |F_j^{(k)}(t)| \le C_9, \qquad k = 0, 1, \ldots, m. \tag{4.7}$$

Proof. Inequality (4.6) is obvious. To prove (4.7) note that for $t \ge t_j$, (4.6) gives

$$F_j(t) = t_j \le t.$$

In the interval $I_j = [t_{j-1}, t_j]$,

$$F_j(t) \le t_j \le C_8 t_{j-1} \le C_8 t.$$

Since $F_j(t) = 0$ for $t < t_{j-1}$, we see that (4.7) holds for $k = 0$ with $C_9 = \max(C_8, 1)$. By (4.5)

$$F_j^{(k)}(t) = t_j \varphi^{(k)}[(t - t_{j-1})/(t_j - t_{j-1})]/(t_j - t_{j-1})^k.$$

Thus for $k \le m$

$$\begin{aligned} |F_j^{(k)}(t)| &\le C_{10} t_j^k/(t_j - t_{j-1})^k \quad \text{in } I_j \\ &= 0 \quad \text{outside } I_j. \end{aligned}$$

Consequently in I_j we have by (4.4)

$$t^{k-1}|F_j^{(k)}(t)| \le C_{10} t_j^k/(t_j - t_{j-1})^k \le C_{10}/(1-\theta)^k.$$

Thus we can take $C_9 = C_{10}/(1-\theta)^m$, and (4.7) holds. □

LEMMA 4.3. *If $1 \le r < s$, then*

$$|D^r F_j(J_s f)| \le C J_{s,r} f, \qquad f \ge 0, \tag{4.8}$$

where the constant C does not depend on j or f.

Proof. Let $u = J_s f$. We have

$$D^r F_j(u) = \sum_{k=1}^{r} \sum_{\sum i_q = r} F_j^{(k)}(u) D^{i_1} u \cdots D^{i_k} u. \tag{4.9}$$

Thus by (4.7)

$$|D^r F_j(u)| \le C_9 \sum \sum |u|^{1-k} |D^{i_1} u| \cdots |D^{i_k} u|.$$

Moreover, by Lemma 4.1

$$|D^i J_s f| \le C J_s f^{1-(i/r)} J_{s,r} f^{i/r}. \tag{4.10}$$

Substituting this into the expression above, we obtain (4.8). □

LEMMA 4.4. *For $0 \le r < s$,*

$$M(J_{s,r} f) \le C J_{s,r} M(f), \qquad f \ge 0. \tag{4.11}$$

Proof. We have

$$J_{s,r} f = c \int G_{s,r}(x-y) f(y)\, dy \Big/ \int G_{s,r}(x-y)\, dy.$$

Thus by Jensen's inequality (cf., e.g., **[13]**)

$$\begin{aligned} M(J_{s,r} f) &\le c' \int G_{s,r}(x-y) M(f(y))\, dy \Big/ \int G_{s,r}(x-y)\, dy \\ &= c'' J_s M(f). \end{aligned}$$

LEMMA 4.5. *For* $0 \leq r < s \leq 2m$

$$\rho_\nu(J_{s,r}f, M) \leq C\rho_\nu(f, M). \tag{4.12}$$

Proof. By Lemma 4.4 and (2.8)

$$\begin{aligned}\int M(J_{s,r}f)\,d\nu &\leq c\int J_{s,r}M(f)\,d\nu \\ &= c\int\int G_{s,r}(x-y)M(f(y))\,dy\,d\nu(x) \\ &= c\int\left[\int G_{s,r}(x-y)\,d\nu(x)\right]M(f(y))\,dy \\ &\leq C'\int (d\nu/dy)M(f(y))\,dy = C'\rho_\nu(f, M).\end{aligned}$$

This gives (4.12). □

We shall need to prove an inequality corresponding to Lemma 4.1 for the case $r = s$. For this purpose we make the following definitions. We let

$$\begin{aligned}M_s f(x) &= \int_{|x-y|<1} |x-y|^{s-n} f(y)\,dy, \\ T_s f(x) &= \sup_{\delta\leq 1} \delta^{-s}\int_{|x-y|<\delta} |x-y|^{s-n} f(y)\,dy, \\ J_{s,s} f(x) &= J_s f(x) + T_n f(x) + f(x).\end{aligned}$$

We have

LEMMA 4.6. *For* $0 < s \leq n$ *there is a constant* C_s *such that*

$$T_s f(x) \leq C_s T_n f(x), \qquad x \in \mathbf{R}^n. \tag{4.13}$$

Proof. For $\delta \leq 1$,

$$\begin{aligned}\int_{|x-y|<\delta} |x-y|^{s-n} f(y)\,dy &= \sum_{k=0}^{\infty}\int_{\delta 2^{-k-1}<|x-y|<\delta 2^{-k}} \\ &\leq \sum_{k=0}^{\infty}(\delta 2^{-k-1})^{s-n}(\delta 2^{-k})^n T_n f(x) \\ &\leq \delta^s 2^{n-s} T_n f(x)\sum_{k=0}^{\infty} 2^{-sk}.\end{aligned}$$

Thus we can take $C_s = 2^n/(2^s - 1)$. □

COROLLARY 4.7. *For* $0 \le k < s$

$$|D^k J_s f| \le C J_s f^{1-(k/s)} J_{s,s} f^{k/s}, \qquad f \ge 0. \tag{4.14}$$

Proof. We have

$$|D^r J_s f| \le C \int G_{s,r}(x-y) f(y)\, dy \le C' M_{s-r} f(x) + C J_s f(x).$$

Now for $\delta \le 1$,

$$\begin{aligned} M_{s-r} f(x) &= \int_{|x-y|<\delta} |x-y|^{s-r-n} f(y)\, dy + \int_{\delta<|x-y|<1} \\ &\le \delta^{s-r} T_{s-r} f(x) + \delta^{-r} M_s f(x). \end{aligned}$$

Take $\delta^s = M_s f(x)/C_s T_n f(x)$. It is ≤ 1 by Corollary 4.7. Then

$$M_{s-r} f(x) \le C M_s f(x)^{(s-r)/s} T_n f(x)^{r/s} \tag{4.15}$$

by Lemma 4.6. Since $M_s f \le J_s f$, we have

$$|D^r J_s f| \le C J_s f^{(s-r)/s} [T_n f^{r/s} + J_s f^{r/s}].$$

This gives the lemma. □

Lemma 4.8 is similar to an inequality of Hedberg **[9]** (cf. also **[20]**).

LEMMA 4.9. *For* $f \ge 0$

$$|[(1-\Delta)^m - 1] F_j(J_{2m} f)| \le C \sum_{r=0}^{2m} J_{2m,r} f \tag{4.16}$$

where the constant is independent of j and f.

Proof. Let $u = J_{2m} f$. Then

$$[(1-\Delta)^m - 1] F_j(u) \tag{4.17}$$

equals

$$F_j'(u)[(1-\Delta)^m - 1]u \tag{4.18}$$

plus terms of the form (4.9) with $k > 1$ and $1 \le r \le 2m$. Taking $s = 2m$ in Lemma 4.3, we see that for $r < 2m$, the sum of the absolute values of these terms is bounded by the right hand side of (4.16) with $r = 0$ and $r = 2m$ missing. For those terms in which $r = 2m$, we make use of Lemma 4.8. From that lemma we see that (4.10) holds even when $r = s$ provided we define $J_{s,s}$ as above. Applying this to

(4.9) we obtain $CJ_{2m,2m}f$ as a bound for such terms. Finally we note that (4.18) equals

$$F_j(u)(f-u)$$

which is bounded by $C(f+u)$. This gives (4.16). □

Now we give the

Proof of Theorem 3.1. For $u \in C_0^\infty$ fixed, let $f = (1-\Delta)^m u$, $v = J_{2m}|f|$. Then $|u| \leq v$. Let the sequence $\{t_i\}$ satisfy the hypotheses of Lemma 4.2. Then the left hand side of (3.2) is bounded by

$$\begin{aligned}\sum_{j=-\infty}^{\infty} c_{2m,\nu,M}(t_j)\ln[M(t_{j+1})/M(t_j)] &\leq C\sum_{j=-\infty}^{\infty} c_{2m,\nu,M}(t_j)\\ &\leq C\sum_{j=-\infty}^{\infty}\rho_\nu([1-\Delta]^m F_j(v), M)\\ &\leq C'\sum_{j=-\infty}^{\infty}[\rho_\nu([(1-\Delta)^m-1]F_j(v), M)+\rho_\nu(F_j(v), M).\end{aligned}$$

Here we made use of (4.4), (4.6) and (a) of §2. Note that

$$\begin{aligned}(4.2)\qquad M(g+h) &= M(2[(g+h)/2])\\ &\leq C_1 M([g+h]/2) \leq \tfrac{1}{2}C_1(M(g)+M(h)).\end{aligned}$$

Since only derivatives appear in $(1-\Delta)^m - 1$, the support of the function (4.14) is contained in the set

$$G_j = \{x \in \mathbf{R}^n : t_{j-1} < v(x) < t_j\}.$$

Thus by Lemmas 4.9, 4.5 and (e)

$$\begin{aligned}\rho_\nu([(1-\Delta)^m-1]F_j(v), M) &\leq C\sum_{r=0}^{2m}\int_{G_j} M(J_{2m,r}|f|)\,d\nu\\ &\leq C'\int_{G_j} M(f)\,d\nu\end{aligned}$$

where the constants are independent of j and f. Hence

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\rho_\nu([(1-\Delta)^m-1]F_j(v), M) &\leq C\sum_{j=-\infty}^{\infty}\int_{G_j} M(f)\,d\nu\\ &= C\int M(f)\,d\nu.\end{aligned}$$

On the other hand, the support of $F_j(v)$ is contained in

$$\bigcup_{k=j}^{\infty} G_k$$

and it is bounded by t_j. Thus

$$\rho_\nu(F_j(v), M) \le \sum_{k=j}^{\infty} \int_{G_k} M(t_j)\, d\nu = M(t_j) \sum_{k=j}^{\infty} \nu(G_k).$$

Consequently,

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \rho_\nu(F_j(u), M) &\le \sum_{j=-\infty}^{\infty} \sum_{k=j}^{\infty} M(t_j)\nu(G_k) \\
&= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k} M(t_j)\nu(G_k) \\
&\le \sum_{k=-\infty}^{\infty} \nu(G_k) M(t_k)(1 + \theta + \theta^2 + \cdots) \\
&\le C \sum_{k=-\infty}^{\infty} \nu(G_k) M(t_{k-1}) \le C \sum_{k=-\infty}^{\infty} \int_{G_k} M(v)\, d\nu \\
&= C \int M(J_{2m}|f|)\, d\nu \le C' \int M(f)\, d\nu.
\end{aligned}$$

Here we made use of (4.4), (4.5), (3.5) and Lemma 4.5. This gives (3.2). □

5. The homogeneous inequalities. Now we turn to the proof of inequality (2.15). For this purpose we need a counterpart of Theorem 3.1. In place of (3.1) we define

$$\tilde{c}_{s,t,\nu,M}(e) = \inf_{v}\{\rho_\nu(v, M), I_s v \ge t \text{ on } e\} \tag{5.1}$$

and

$$\tilde{c}_{s,\nu,M}(t) = \tilde{c}_{s,t,\nu,M}(N_t)$$

where $N_t = N_t(u)$ is the set defined in §3. Corresponding to Theorem 3.1 we have

THEOREM 5.1. *Under the hypotheses of Theorem 2.5 there is a constant C such that*

$$\int_0^{\infty} \tilde{c}_{2m,\nu,M}(t) M(t)^{-1}\, dM(t) \le C\rho_\nu(\Delta^m u, M), \qquad u \in C_0^{\infty}. \tag{5.2}$$

In proving Theorem 5.1 we replace Lemma 4.8 with

LEMMA 5.2. *If* $0 \le k < s$, *then*

$$|D^k I_s f| \le C I_s f^{1-(k/s)} T f^{k/s}, \qquad f \ge 0. \tag{5.3}$$

This lemma is due to Hedberg **[9]** (cf. also **[20]**). The proof is similar to that of Lemma 4.8. Replacing Lemma 4.9 we have

LEMMA 5.3. *There is a constant independent of j and f such that*

$$|\Delta^m F_j(I_{2m} f)| \le C(f + Tf), \qquad f \ge 0. \tag{5.4}$$

Proof. Let $u = I_{2m} f$. Then $\Delta^m F_j(u)$ equals

$$F_j'(u)\Delta^m u \tag{5.5}$$

plus terms of the form (4.9) with $k > 1$ and $\sum i_q = 2m$. If we now apply (4.7) and (5.3), we see that these terms are bounded by CTf. Since $f = \Delta^m u$, we obtain (5.4). □

We can now give the

Proof of Theorem 5.1. Following the proof of Theorem 3.1 we see that the left hand side of (5.2) is bounded by

$$\sum_{j=-\infty}^{\infty} \rho_\nu(\Delta^m F_j(v), M), \tag{5.6}$$

where $f = \Delta^m u$, $v = I_{2m}|f|$. Since $F_j(v)$ is constant outside G_j, we see that (5.6) is bounded by

$$C \sum_{j=-\infty}^{\infty} \int_{G_j} M(|f| + T|f|)\, d\nu \tag{5.7}$$

in view of Lemma 5.3. If we now make use of (4.20), we see that (5.7) is bounded by

$$\int [M(f) + M(T|f|)]\, d\nu.$$

This is where assumption (e′) is used to show that (5.2) holds. □

The proof that (5.2) implies (2.15) is similar to the proof of (2.2) and is omitted.

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