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DEHN-SURGERY ALONG A TORUS T^2-KNOT

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A T^2-knot means a 2-torus embedded in a 4-manifold. We define *torus T^2-knots* in the 4-sphere S^4 as a generalization of torus knots in S^3. We classify them up to equivalence and study the manifolds obtained by Dehn-surgery along them.

1. Introduction. Dehn-surgery, which was introduced by Dehn [**1**], plays an important role in knot theory and 3-dimensional manifold theory. The classical Dehn-surgery is the operation of cutting off the tubular neighborhood $N = S^1 \times D^2$ of a knot in S^3 and of pasting it back via an element of $\pi_0 \operatorname{Diff} \partial N$, which is isomorphic to $\mathrm{GL}_2\mathbf{Z}$. Gluck-surgery [**2**] along a 2-knot in S^4 is a 4-dimensional version of Dehn-surgery. In this version, $N = S^2 \times D^2$ and $\pi_0 \operatorname{Diff} \partial N = (\mathbf{Z}/2)^3$. $(\mathbf{Z}/2)^2$ corresponds to the orientation reversing diffeomorphisms of S^2 and ∂D^2. Therefore Gluck-surgery yields at most one new manifold from one 2-knot and it is a homotopy 4-sphere (see [**2**]). Another 4-dimensional version is Dehn-surgery along a 2-torus embedded in S^4 [**7**], which we call a *T^2-knot* in this paper. In this version, $N = T^2 \times D^2$ and $\pi_0 \operatorname{Diff} \partial N = \mathrm{GL}_3\mathbf{Z}$. Countably many manifolds are obtained from one T^2-knot. A manifold obtained by Gluck-surgery is also obtained by Dehn-surgery along a T^2-knot (see Proposition 3.5). Dehn-surgery along an unknot is studied in [**7**], [**9**]. See also [**3**].

In this paper, we define a *torus T^2-knot* which is analogous to the torus knots in the classical knot theory, and classify them up to equivalence. Then we study the manifolds obtained by Dehn-surgeries along them.

Dehn-surgery along a torus knot is studied by Moser [**8**].

THEOREM 1.1. (*Moser* [**8**], *Propositions* 3.1, 3.2, 4.) *Assume that a Dehn-surgery of type (α, β) is performed along $k(p,q)$, the torus knot of type (p,q). Put $|\sigma| = |pq\beta - \alpha|$. The manifold obtained is denoted by M.*

(i) *If $|\sigma| \neq 0$, then M is a Seifert manifold with fibers of multiplicities p, q, $|\sigma|$.*

(ii) *If $|\sigma| = 1$, then M is a lens space $L(|\alpha|, \beta q^2)$.*

(iii) *If* $|\sigma| = 0$, *then* M *is the connected sum of two lens spaces* $L(p, q)\#L(q, p)$.

Our results are the following. Theorem 1.3 is a generalization of Theorem 1.1.

Let $k(p, q)$ be the torus knot of type (p, q) in S^3, and B be a 3-ball contained in $S^3 - k(p, q)$. Define $S(k(p, q))$ and $\tilde{S}(k(p, q))$ by

$$(S^4, S(k(p,q))) = ((S^3, k(p,q)) - \operatorname{Int} B) \times S^1 \cup_{\mathrm{id}} S^2 \times D^2,$$
$$(S^4, \tilde{S}(k(p,q))) = ((S^3, k(p,q)) - \operatorname{Int} B) \times S^1 \cup_{\tau} S^2 \times D^2$$

where id is the natural identification of $\partial B \times S^1$ with $S^2 \times \partial D^2$ and τ is the map $(u, v) \mapsto (uv, v)$ where we identify S^2 with the Riemann sphere and D^2 with the unit disk in $\mathbf{C}$.

PROPOSITION 1.2. (*Lemma* 2.6 *and Proposition* 2.9.) *Any torus* T^2-*knot is equivalent to one and only one of the following*:

(i) $S(k(p, q))$, $1 < p < q$, $\gcd(p, q) = 1$;

(ii) $\tilde{S}(k(p, q))$, $1 < p < q$, $\gcd(p, q) = 1$;

(iii) *unknotted* T^2-*knot.*

THEOREM 1.3. (*See Proposition* 3.6, *Remark* 3.7, *Proposition* 3.9, *Corollary* 3.10, *Proposition* 3.11.) *Assume that a Dehn-surgery of type* (α, β, γ) (*see Definition* 3.2) *is performed along* $S(k(p, q))$ *or* $\tilde{S}(k(p, q))$. *Put* $\sigma = |pq\beta - \alpha|$. *The manifold obtained is denoted by* M.

(i) *If* $\sigma \neq 0$, *then* M *is the total space of a good torus fibration over* S^2 *with one twin singular fiber of multiplicity* p *and two multiple tori of multiplicity* q *and* σ.

(ii) *If* $\sigma = 1$, *then* M *is* $L_{|\alpha|}$ *or* $L'_{|\alpha|}$.

(iii) *If* $\sigma = 0$, *then* M *is an irrational connected sum along circles* [5] *of either* L_m *or* L'_m *and* $L(n, r) \times S^1$ *for some* m, n, r.

(iv) *If* $\gamma = 0$, *then* $M = (M_0 - \operatorname{Int} B^3) \times S^1 \cup_h S^2 \times D^2$ *where* M_0 *is the manifold obtained by a Dehn-surgery of type* (α, β) *along the torus knot of type* (p, q) *and* $h = \mathrm{id}$ (*if* $K = S(k(p, q))$), $h = \tau$ ($K = \tilde{S}(k(p, q))$). *Especially if* $(\alpha, \beta, \gamma) = (pq, 1, 0)$, $M = L_p\#L_q$.

(L_m *and* L'_m *are the manifold defined in* [9]. *See also* [3].)

We use standard notations. $N(X)$ means the tubular neighborhood of X. All the homology groups are with coefficients in $\mathbf{Z}$ unless otherwise indicated.

2. Torus T^2-knots.

DEFINITION 2.1. Let M^n (resp. N^{n-2}) be an n- (resp. $(n-2)$-) dimensional manifold. A submanifold K in M^n is called an N^{n-2}-*knot in* M^n if K is diffeomorphic to N^{n-2}. Let K, K' be two N^{n-2}-knots in M^n. K and K' are *equivalent* if there exists a diffeomorphism $h: (M^n, K) \to (M^n, K')$.

We will be mainly concerned with T^2-knots in S^4.

Recall that a T^1-knot (i.e. a classical knot) K in S^3 is called unknotted if K bounds a disk D^2 in S^3.

DEFINITION 2.2. A T^2-knot K in S^4 is called *unknotted* if K bounds a solid torus $S^1 \times D^2$ in S^4.

Any two unknotted T^2-knots are equivalent.

REMARK 2.3. There exist three isotopy classes of embeddings $T^2 \to S^4$ such that their images are unknotted (see Theorem 5.3 in [7]). But we are considering a T^2-knot itself, not its embedding map.

Recall that a T^1-knot K in S^3 is called a torus knot if K is essentially embedded in $\partial N(U)$, where U is an unknotted T^1-knot in S^3.

DEFINITION 2.4. A T^2-knot K in S^4 is called a *torus knot* if K is incompressibly embedded in $\partial N(U)$, where U is an unknotted T^2-knot in S^4.

LEMMA 2.5. *Let K, K' be incompressible 2-tori in T^3 such that $[K] = [K']$ in $H_2(T^3)$. Then, there is an ambient isotopy which carries K to K'.*

Proof. We may assume that $(T^3, K) = (S^1 \times S^1 \times S^1, S^1 \times S^1 \times \{*\})$ without loss of generality.

By Theorem VI.34 and VI.17 in [4], there exists a diffeomorphism $f: (T^3, K') \to (T^3, K)$. Since $f_*([K]) = [K]$, there exists a diffeomorphism $g: (T^3, K) \to (T^3, K)$ with $f_* = g_*$. Since $f^{-1} \cdot g: (T^3, K) \to (T^3, K')$ satisfies $(f^{-1} \cdot g)_* = \mathrm{id}$, it is isotopic to the identity map. □

Let U be an unknotted T^2-knot in S^4. $\overline{S^4 - N(U)}$ is a *twin* (see [7]). We denote the twin by the symbol Tw. A twin consists of two $S^2 \times D^2$'s plumbed at two points with opposite signs. Let R, S be the cores of two $S^2 \times D^2$'s. They generate $H_2(Tw)$. Let $D(r)$, $D(s)$ be 2-disks properly embedded in Tw such that $R \cdot D(r) = S \cdot D(s) = 1$ and $R \cdot D(s) = S \cdot D(r) = 0$. $\partial D(r)$ and $\partial D(s)$ are circles in $\partial(Tw)$. We call them r and s respectively. Their homology classes in $H_1(\partial(Tw))$ are well-defined. Choose a circle l in $\partial(Tw)$ such that $\langle l, r, s\rangle$ is an oriented basis of $H_1(\partial(Tw))$. Two l's are mapped to each other by some diffeomorphism between Tw's which fixes r and s (see Remark 2.5 in [3]). Next, we consider the manifold $D^2 \times T^2 \cong D^2 \times S^1 \times S^1$. Let $\bar{l}$, $\bar{r}$, $\bar{s}$ be the circles $\partial D^2 \times \{*\} \times \{*\}$, $\{*\} \times S^1 \times \{*\}$, $\{*\} \times \{*\} \times S^1$ in $\partial(D^2 \times T^2)$ respectively. $S^4 = Tw \cup_f T^2 \times D^2$ where $f_*[\bar{l}\ \bar{r}\ \bar{s}] = [l\ r\ s]$. Put $T_0 = T^2 \times D^2 \cap Tw \subset S^4$. Assume that K is a torus T^2-knot contained in T_0. Denote K by $K(p, q, q')$ if $[K] = p(r \times s) + q(s \times l) + q'(l \times r)$ in $H_2(\partial(Tw))$ where $p, q, q' \in \mathbf{Z}$. Note that by Lemma 2.5, p, q, q' determines the knot type of K.

Let $h: T_0 \to T_0$ be a diffeomorphism with $h_*[l\ r\ s] = [l\ r\ s]A^h$, where $A^h \in \mathrm{GL}_3\mathbf{Z}$. There is a diffeomorphism $\bar{h}: S^4 \to S^4$ such that $\bar{h}|T_0 = h$ if and only if $A^h \in H$, where

$$H = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \in \mathrm{GL}_3\mathbf{Z} \,\middle|\, a + b + c + d \equiv 0 \pmod 2 \right\}$$

(see Theorem 5.3 in [7] and Lemma 2.6 in [3]). If

$$h_*[l\ r\ s] = [l\ r\ s] \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix},$$

then

$$h_*[r \times s\ \ s \times l\ \ l \times r] = [r \times s\ \ s \times l\ \ l \times r] \begin{bmatrix} ad - bc & 0 & 0 \\ 0 & \varepsilon d & -\varepsilon c \\ 0 & -\varepsilon b & \varepsilon a \end{bmatrix}.$$

Therefore if

$$\begin{bmatrix} p \\ q \\ q' \end{bmatrix} = \begin{bmatrix} ad - bc & 0 & 0 \\ 0 & \varepsilon d & -\varepsilon c \\ 0 & -\varepsilon b & \varepsilon a \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \\ q_1' \end{bmatrix},$$

$K(p, q, q')$ and $K(p_1, q_1, q_1')$ have the same knot type. Since

$$\mathrm{GL}_2\mathbf{Z} \Big/ \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a + b + c + d \equiv 0 \pmod 2 \right\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\},$$

every torus knot is equivalent to $K(p, q, 0)$ or $K(p, q, q)$ for some non-negative integers p, q with $\gcd(p, q) = 1$.

It is easy to check the following:

LEMMA 2.6. *$K(p,q,0) = S(k(p,q))$ and $K(p,q,q) = \tilde{S}(k(p,q))$.*

Therefore, every torus T^2-knot is equivalent to $S(k(p,q))$ or $\tilde{S}(k(p,q))$ for some p, q. Since $k(p,q) = k(q,p)$, we have $K(p,q,0) = K(q,p,0)$ and $K(p,q,q) = K(q,p,p)$. It is clear that $\pi_1(S^3 - k(p,q)) \cong \pi_1(S^4 - K(p,q,0)) \cong \pi_1(S^4 - K(p,q,q))$. The following theorem is known. For example, see **[10]**, p. 54.

THEOREM 2.7. (*O. Schreier.*) *If $1 < p < q$ and $1 < p' < q'$, then $\pi_1(S^3 - k(p,q)) \cong \pi_1(S^3 - k(p',q'))$ if and only if $p = p'$, $q = q'$.* □

The exteriors of $K(p,q,0)$ and $K(p,q,q)$ have the same homotopy type. But,

LEMMA 2.8. *If $p > 1$ and $q > 1$, then the exteriors of $K(p,q,0)$ and $K(p,q,q)$ have different diffeomorphism types.*

Proof. Put $K = K(p,q,0)$ or $K(p,q,q)$, $k = k(p,q)$. Recall that $S^4 - \operatorname{Int} N(K) = (B^3 - \operatorname{Int} N(k)) \times S^1 \cup S^2 \times D^2$. Let i be the inclusion map $\partial N(K) \to S^4 - \operatorname{Int} N(K)$ and $*$ be a point in $\partial N(k)$. Then, $[\{*\} \times S^1] \in \pi_1(\partial N(K))$ generates $\operatorname{Ker}(i_*: \pi_1(\partial N(K)) \to \pi_1(S^4 - \operatorname{Int} N(K)))$.

Fact. Let X be a spin 4-manifold with $\partial X = T^3$. Let C_1, C_2 be two loops in ∂X with a diffeomorphism $\partial X \to S^1 \times S^1 \times S^1$ which maps C_1, C_2 to $\{*\} \times \{*\} \times S^1$ and $\{*'\} \times \{*\} \times S^1$ $(* \neq *')$. Assume that $[C_1] = [C_2] = 0$ in $H_1(X; \mathbf{Z}/2)$. Let D_i be a 2-chain in X such that (a) $\partial D_i \pmod 2 = C_i$ $(i = 1, 2)$, (b) $[D_1] = [D_2]$ in $H_2(X, \partial X; \mathbf{Z}/2)$, (c) D_1 and D_2 meet transversely. Then, $D_1 \cdot D_2 \pmod 2$ is determined by $[C_1]$ in $H_1(\partial X; \mathbf{Z}/2)$ (see the proof of Lemma 2.10 in [3]).

Put $Y([C_1]) = D_1 \cdot D_2 \in \mathbf{Z}/2$.

For $K(p,q,0)$ (resp. $K(p,q,q)$), $Y([\{*\} \times S^1]) = 0$ (resp. 1). This completes the proof. □

We have proved :

PROPOSITION 2.9. *Any torus T^2-knot is equivalent to one and only one of the following*:

(i) $K(p,q,0)$, $1 < p < q, \gcd(p,q) = 1$;
(ii) $K(p,q,q)$, $1 < p < q, \gcd(p,q) = 1$;
(iii) *unknotted T^2-knot.* □

For the definition of *good torus fibrations* (*GTF*), see **[6]** or **[3]**, §3.

LEMMA 2.10. *The exteriors of $K(p,q,0)$ and $K(p,q,q)$ have structures of good torus fibrations with one twin singular fiber of multiplicity p and one multiple torus of multiplicity q.*

Proof. Recall that $S^4 = Tw \cup_f T^2 \times D^2$ where $f_*[\bar{l}\ \bar{r}\ \bar{s}] = [l\ r\ s]$ and that $T_0 = Tw \cap T^2 \times D^2 \subset S^4$. Let pr: $T^3 = T^1 \times T^2 \to T^1$ be the projection map to the first coordinate and let $h\colon T_0 \to T^3$ be a diffeomorphism such that $(\mathrm{pr}\cdot h)_*$ maps l, r, s to $q\gamma$, $p\gamma$, 0 (resp. $q\gamma$, $p\gamma$, $p\gamma$) where γ is a generator of $H_1(T^1)$. Since $-pl + qr \mapsto 0$ and $s \mapsto 0$ (resp. $-pl+qr \mapsto 0$ and $s-r \mapsto 0$), $\mathrm{pr}\cdot h$ has fiber $(-pl+qr)\times s = q(r\times s)+p(s\times l)$ (resp. $(-pl+qr)\times(s-r) = q(r\times s)+p(s\times l)+p(l\times r)$). By Proposition 3.12 and Definition 3.16 in [3], $\mathrm{pr}\cdot h$ extends to a GTF $f\colon S^4 \to S^2$ with one twin singular fiber of multiplicity p and one multiple torus of multiplicity q. Since a general fiber in T_0 is $K(p,q,0)$ (resp. $K(p,q,q)$), the lemma is proved. □

3. Dehn-surgery along a torus T^2-knot. In this section, K denotes $K(p,q,0)$ or $K(p,q,q)$ and k denotes $k(p,q)$ (the classical torus knot of type (p,q) in S^3). We assume that K is embedded in S^4 as is in the proof of Lemma 2.10.

We choose a basis $\langle m, l_1, l_2\rangle$ of $H_1(\partial N(K))$ as follows. Let m denote the meridian curve of k, which can be regarded as the meridian curve of K. Let l_0 be the preferred longitude of k. Put $l_1 = l_0 \times \{*\} \subset (B^3 - \mathrm{Int}\, N(k)) \times S^1 \cup S^2 \times D^2 = S^4 - \mathrm{Int}\, N(K)$. We orient m and l_1 so that $pqm + l_1 = -pl + qr$ in T_0. Put $l_2 = s$ for $K(p,q,0)$ and $l_2 = s - r$ for $K(p,q,q)$. Note that l_2 generates the kernel of $\pi_1(\partial N(K)) \to \pi_1(S^4 - \mathrm{Int}\, N(K))$.

REMARK 3.1. Note that for $K = K(p,q,0)$ or $K(p,q,q)$ a general fiber contained in $\partial N(K)$ represents $(pqm + l_1) \times l_2$ in $H_2(\partial N(K))$.

DEFINITION 3.2. Let K be a torus T^2-knot in S^4. $N(K)$ is diffeomorphic to $T^2 \times D^2$. Let $i\colon \partial N(K) \to \partial(S^4 - \mathrm{Int}\, N(K))$ be the natural identification and $h\colon \partial N(K) \to \partial N(K)$ be a diffeomorphism such that $i\cdot h(m) = \alpha m + \beta l_1 + \gamma l_2$. $M = \overline{S^4 - N(K)} \cup_{i\cdot h} N(K)$ is called the manifold obtained by *Dehn-surgery of type* (α, β, γ) *along* K.

Montesinos showed that any homotopy 4-sphere obtained by Dehn-surgery along an unknotted T^2-knot in S^4 is diffeomorphic to S^4 (see [7], p. 187). Pao studied the 4-manifolds with effective T^2-action in [9]. All the 4-manifolds obtained by Dehn-surgeries along an unknotted T^2-knot are contained in his list. See also [3].

Note that the diffeomorphism type of M is determined by K and $i \cdot h(m)$ (see Remark 2.7. in [3]).

Dehn-surgery along any T^2-knot can be defined in the same way, but the description of the type of the surgery might be a little complicated.

PROPOSITION 3.3. *Assume that a Dehn-surgery of type* (α, β, γ) *is performed along* $K(p, q, 0)$ (*resp.* $K(p, q, q)$). *Denote the manifold by* M.

(i) *If* $\gcd(\alpha, \beta) = 1$, *then* $\pi_1(M)$ *is isomorphic to the fundamental group of the manifold obtained by a Dehn-surgery of type* (α, β) *along a torus* T^1*-knot of type* (p, q).

(ii) $H_1(M) \cong \mathbf{Z}/\alpha$.

(iii) M *is spin if and only if* $\beta\gamma \equiv 0$ (*resp.* $(1-\beta)\gamma \equiv 0$) $\pmod 2$ *or* $\alpha \equiv 1 \pmod 2$.

Proof. The proofs of (i) and (ii) are almost clear. For (iii), we need a lemma.

LEMMA 3.4. *Put* $X = S^4 - \operatorname{Int} N$ *where* N *is a tubular neighborhood of a* T^2*-knot* (*not necessarily a torus* T^2*-knot*) K. *Let* Y *be the function defined in the proof of Lemma* 2.8. *Assume that* $\operatorname{Ker}(i_*\colon H_1(\partial X; \mathbf{Z}/2) \to H_1(X; \mathbf{Z}/2)) = \{0, e_1, e_2, e_3\}$ *where* $i\colon \partial X \to X$ *is the inclusion map. Then, one of* $Y(e_i)$*'s is* 1, *the others are* 0.

Proof. Note that if $\{i, j, k\} = \{1, 2, 3\}$, then $e_k = e_i + e_j$ holds.

Let V be a 2-sided 3-dimensional submanifold of S^4 such that $\partial V = K$ and that $(N, N \cap V)$ is diffeomorphic to $(D^2 \times T^2, r \times T^2)$ where r is a radius of D^2. Put $V_0 = \partial N \cap V$. Note that e_1, e_2, e_3 are represented by curves in V_0. Assume that $i_*\colon H_1(V_0; \mathbf{Z}/2) \to H_1(V; \mathbf{Z}/2)$ is injective (i is the inclusion map). Then, the Mayer-Vietoris exact sequence shows that $H_2(V; \mathbf{Z}/2) \to H_2(V, V_0; \mathbf{Z}/2)$ is surjective. Since $H_3(V, V_0; \mathbf{Z}/2) \to H_2(V_0; \mathbf{Z}/2)$ is bijective, therefore $H_2(V; \mathbf{Z}/2) \to H_2(V, V_0; \mathbf{Z}/2)$ is injective. By Poincaré duality, $H_2(V, V_0; \mathbf{Z}/2)$ is isomorphic to $H^1(V; \mathbf{Z}/2)$, which is isomorphic to $H_1(V; \mathbf{Z}/2)$. Therefore $\chi(V) = 1$. Put $\bar{V} = V \cup_\partial S^1 \times D^2$. Then,

$$\chi(\bar{V}) = \chi(V) + \chi(S^1 \times D^2) - \chi(\partial V) = 1.$$

Since $\bar{V}$ is a closed 3-manifold, this is a contradiction. Therefore $\operatorname{Ker}(i_*\colon H_1(V_0; \mathbf{Z}/2) \to H_1(V; \mathbf{Z}/2))$ contains e_i, one element of $\{e_1, e_2, e_3\}$. Since V has trivial normal bundle, we can move V slightly into V' so that $V \cap V' = \varnothing$. Therefore $Y(e_i) = 0$.

Let m be the meridian curve of K and we consider the manifold $Q = X \cup_a Tw$ with the attaching map $a\colon \partial(Tw) \to \partial X$ satisfying $a_*[l\ r\ s] = [\tilde{m}\ \tilde{e}_j\ \tilde{e}_k]$ where $\langle \tilde{m}, \tilde{e}_j, \tilde{e}_k \rangle$ is a basis of $H_1(\partial X; \mathbf{Z})$ whose mod 2 reduction is $\langle m, e_j, e_k \rangle$. The Mayer-Vietoris sequence with coefficients in $\mathbf{Z}/2$

$$H_2(X) \oplus H_2(Tw) \xrightarrow{j_*} H_2(Q) \to H_1(\partial X) \xrightarrow{i_*} H_1(X) \oplus H_1(Tw)$$

shows that

$$H_2(Q; \mathbf{Z}/2) = \operatorname{Im}(j_*) \oplus \langle [D(e_j) + D(r)], [D(e_k) + D(s)] \rangle,$$

where $D(c)$ is a mod 2 2-chain satisfying $\partial D(c) = c$ and $D(e_j), D(e_k) \subset X$, $D(r), D(s) \subset Tw$. The self-intersection number on $\operatorname{Im}(j_*)$ is zero since X and Tw are subsets of S^4. Since $[D(e_j) + D(r)]^2 = Y(e_j) + Y(r) = Y(e_j)$ and $[D(e_k) + D(s)]^2 = Y(e_k) + Y(s) = Y(e_k)$, we have

$$\begin{aligned}[D(e_i) + D(r+s)]^2 &= [D(e_j) + D(e_k) + D(r) + D(s)]^2 \\ &= [D(e_j) + D(r)]^2 + [D(e_k) + D(s)]^2 \\ &= Y(e_j) + Y(e_k).\end{aligned}$$

On the other hand, $[D(e_i) + D(r+s)]^2 = Y(e_i) + Y(r+s) = 0 + 1 = 1$. Therefore $Y(e_j) = 0$, $Y(e_k) = 1$ or $Y(e_j) = 1$, $Y(e_k) = 0$. □

We now continue with the proof of Proposition 3.3.

If α is odd, $H_2(M; \mathbf{Z}/2)$ is zero. Therefore M is spin. Assume that α is even. The Mayer-Vietoris sequence with coefficients in $\mathbf{Z}/2$

$$\begin{aligned}H_2(D^2 \times T^2) \oplus H_2(X) \xrightarrow{j_*} H_2(M) &\to H_1(\partial(D^2 \times T^2)) \\ &\xrightarrow{i_*} H_1(D^2 \times T^2) \oplus H_1(X)\end{aligned}$$

shows that $H_2(M; \mathbf{Z}/2) = \operatorname{Im}(j_*) \oplus \langle [D_M + D(\alpha m + \beta l_1 + \gamma l_2)] \rangle$ where X is the knot exterior and D_M is the meridian disk of $D^2 \times T^2$ and $D(\alpha m + \beta l_1 + \gamma l_2)$ is a mod 2 2-chain in X with $\partial D(\alpha m + \beta l_1 + \gamma l_2) = \alpha m + \beta l_1 + \gamma l_2$. For $K = K(p, q, 0)$ (resp. $K(p, q, q)$), $Y(l_2) = 0$ (resp. 1) and $Y(l_1) = 0$. By Lemma 3.4, $Y(l_1 + l_2) = 1$ (resp. 0). Therefore $[D_M + D(\alpha m + \beta l_1 + \gamma l_2)]^2 = \beta\gamma \mod 2$ (resp. $(1 - \beta)\gamma \mod 2$). Since the self-intersection number on $\operatorname{Im}(j_*)$ is zero, this completes the proof. □

Proposition 3.5. *If a closed 4-manifold M is obtained by Gluck-surgery along an S^2-knot in S^4, then M is also obtained by Dehn-surgery along a T^2-knot in S^4.*

Proof. Here K denotes the S^2-knot. Identify $N(K) \cong S^2 \times D^2$ with $\hat{\mathbf{C}} \times D$ where $\hat{\mathbf{C}}$ is the Riemann sphere and D is the unit disk in $\mathbf{C}$. Recall that $M = (S^4 - \operatorname{Int} N(K)) \cup_\tau N(K)$ is the manifold obtained by Gluck-surgery along K where $\tau(u, v) = (uv, v)$. Observe that $\tau(\{|u| = c\} \times \partial D) = \{|u| = c\} \times \partial D$ for any $c \in \mathbf{R} \cup \{\infty\}$. Embed $(D^3 \times D^1, D^3 \times \partial D^1)$ in $(S^4 - \operatorname{Int} N(K), \partial N(K))$ with $D^3 \times \{-1\} \subset D_0 \times \partial D^2$ and $D^3 \times \{1\} \subset D_\infty \times \partial D^2$ where $D_0 = \{|z| \leq 1/9\} \subset \hat{\mathbf{C}}$ and $D_\infty = \{|z| \geq 9\} \subset \hat{\mathbf{C}}$. Let H denote its image. One can consider H as a 1-handle attached to $N(K)$. Verify that there exists an annulus A_0 (resp. A_∞) properly embedded in $D_0 \times D^2$ (resp. $D_\infty \times D^2$) such that $K' = (\hat{\mathbf{C}} - \operatorname{Int} D_0 - \operatorname{Int} D_\infty) \times \{0\} \cup A_0 \cup U \times D^1 \cup A_\infty$ is an embedded torus where U is an unknot in D^3. Especially, if K is unknotted, then so is K'.

K' has a tubular neighborhood $N(K')$ such that

$$N(K')|(\hat{\mathbf{C}} - \operatorname{Int}(D_0 \cup D_\infty)) = N(K)|(\hat{\mathbf{C}} - \operatorname{Int}(D_0 \cup D_\infty))$$

and $N(K') \subset N(K) \cup H$ and $N(K')|(U \times \{*\}) = N_0(U) \times \{*\}$ where $N_0(U)$ is a tubular neighborhood of U in D^3.

Let $f\colon \partial N(K') \to \partial N(K')$ be a diffeomorphism such that

$$f|(\hat{\mathbf{C}} - \operatorname{Int}(D_0 \cup D_\infty)) \times \partial D = \tau|(\hat{\mathbf{C}} - \operatorname{Int}(D_0 \cup D_\infty)) \times \partial D$$

and

$$f(\partial N(K') \cap D^3 \times \{*\}) = \partial N(K') \cap D^3 \times \{*\}.$$

Put $M' = (S^4 - \operatorname{Int} N(K')) \cup_f N(K')$.

Construct a diffeomorphism $F\colon M' \to M$ as follows. Put

$$F|(S^4 - (D_0 \cup D_\infty) \times D - H) = \mathrm{id}.$$

$M'_* = (D^3 - \operatorname{Int} N_0(U)) \times \{*\} \cup_f N_0(U) \times \{*\}$ is the manifold obtained by Dehn-surgery of type $(1, 1)$ or $(1, -1)$ along U in D^3. Therefore there exists a diffeomorphism $F_*\colon M'_* \to D^3 \times \{*\}$ with $F_*|\partial = \mathrm{id}$. Put $F|M'_* = F_*$. Finally, extend $F|\partial(D_0 \times D)$ and $F|\partial(D_\infty \times D)$ to $D_0 \times D$ and $D_\infty \times D$.

This completes the proof. □

PROPOSITION 3.6. *If a Dehn-surgery of type (α, β, γ) is performed along $K(p, q, 0)$ or $K(p, q, q)$ and $\sigma = |pq\beta - \alpha| \neq 0$, then the manifold obtained is the total space of a good torus fibration over S^2 with one twin singular fiber of multiplicity p and two multiple tori of multiplicity q and σ.*

Proof. Put $K = K(p, q, 0)$ or $K(p, q, q)$. By Lemma 2.10, the exterior of K has the structure of GTF. The intersection number of

$i \cdot h(m) = \alpha m + \beta l_1 + \gamma l_2$ and the fiber $(pqm + l_1) \times l_2$ in $\partial N(K)$ is $\pm(\alpha - pq\beta)$. Therefore, after surgery, a fiber is homologous to $\pm(\alpha - pq\beta)C$ in $N(K)$, where C is the core of $N(K)$. Now the proposition is proved. (See Definition 3.16 in [3].) □

REMARK 3.7. If $\sigma = 1$, then the manifold has only one twin singular fiber and one multiple torus. Therefore it is diffeomorphic to $L_{|\alpha|}$ or $L'_{|\alpha|}$ by the Main Theorem of [3] and Proposition 3.3.(ii). If $\alpha \equiv 0$ (mod 2), then it is $L_{|\alpha|}$ if and only if either $K = K(p, q, 0)$ and $\gamma \equiv 0$ (mod 2) or $K = K(p, q, q)$. See Proposition 3.3.(iii).

COROLLARY 3.8. *The manifold obtained by the Gluck-surgery along an untwisted spun* (S^2-) *knot of any torus* (S^1-) *knot is the* 4*-sphere.*

Proof. By the proof of Proposition 3.5, the manifold is diffeomorphic to the one obtained by a Dehn-surgery of type $(1, 0, \pm 1)$ along a torus T^2-knot. Since $L_1 = S^4$, Corollary is proved. □

The following Proposition is almost clear.

PROPOSITION 3.9. *If a Dehn-surgery of type* $(\alpha, \beta, 0)$ *is performed along* $K(p, q, 0)$ (*resp.* $K(p, q, q)$), *then the manifold obtained is* $(M_0 - \operatorname{Int} B^3) \times S^1 \cup_h S^2 \times D^2$ *where* M_0 *is the manifold obtained by a Dehn-surgery of type* (α, β) *along* $k(p, q)$ *and* $h = \mathrm{id}$ (*resp.* τ).

COROLLARY 3.10. *If a Dehn-surgery of type* $(pq, 1, 0)$ *is performed along* $K(p, q, 0)$ *or* $K(p, q, q)$, *then the manifold obtained is* $L_p \# L_q$.

Proof. The manifold obtained is $(L - \operatorname{Int} B^3) \times S^1 \cup S^2 \times D^2$ where $L = L(p, q) \# L(q, p)$. Corollary 4.10 in [3] completes the proof. □

PROPOSITION 3.11. *If a Dehn-surgery of type* (α, β, γ) *is performed along* $K(p, q, 0)$ *or* $K(p, q, q)$ *and* $\sigma = |pq\beta - \alpha| = 0$, *then the manifold obtained is an irrational connected sum along circles of either* L_m *or* L'_m *and* $L(n, r) \times S^1$ *for some* m, n, r.

Proof. In this case, the meridian of 4-dimensional solid torus sT^4 is attached in a fiber of a GTF of the knot exterior. Recall that the knot exterior X is made of Tw and $D^2 \times T^2$ pasted together along A, where A is diffeomorphic to $D^1 \times T^2$.

Put $\partial(D^2 \times T^2) - \operatorname{Int} A = B$ and $\partial(Tw) - \operatorname{Int} A = C$. B and C are diffeomorphic to $T^2 \times D^1$. Let $h \colon \partial(sT^4) \to B \cup C$ be the attaching map. Since $B \cap C$ is a disjoint union of two fibers, we may assume that $h^{-1}(B) = \partial D^2 \times S^1 \times [*, **] \subset \partial D^2 \times S^1 \times S^1 = \partial(D^2 \times T^2)$, $h^{-1}(C) = \partial D^2 \times S^1 \times [**, *] \subset \partial D^2 \times S^1 \times S^1 = \partial(D^2 \times T^2)$. Put $V = D^2 \times S^1 \times [*, **]$ and $V' = D^2 \times S^1 \times [**, *]$. $V \cup V' = sT^4$. The manifold obtained by the Dehn-surgery is

$$\begin{aligned} M &= D^2 \times T^2 \cup Tw \cup sT^4 \\ &= D^2 \times T^2 \cup Tw \cup (V \cup V') = (D^2 \times T^2 \cup V) \cup (Tw \cup V'). \end{aligned}$$

Note that $\partial(D^2 \times T^2 \cup V) = \partial(Tw \cup V') = S^2 \times S^1$. If we attach $D^3 \times S^1$ to $D^2 \times T^2 \cup V$ (resp. $Tw \cup V'$) in the natural way, $V \cup D^3 \times S^1$ (resp. $V' \cup D^3 \times S^1$) is diffeomorphic to $D^2 \times T^2$.

It is easy to show that $D^2 \times T^2 \cup D^2 \times T^2$ is $L(n, r) \times S^1$ for some n, r. The proof of Theorem 4.1 in [3] says that $Tw \cup D^2 \times T^2$ is L_m or L'_m. This completes the proof. □

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