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INVARIANT SUBSPACES OF $\mathscr{H}^p$ FOR MULTIPLY CONNECTED REGIONS

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To David Lowdenslager, in memoriam

A closed linear subspace of $\mathscr{H}^p(\mathrm{G})$ is said to be *invariant* if $zf(z)$ is in $\mathscr{M}$ for all $f(z) \in \mathscr{M}$. It is said to be fully invariant if $r(z)f(z)$ is in $\mathscr{M}$ for all $f \in \mathscr{M}$ and all rational functions $r(z)$ with poles in the complement of G. This paper investigates those invariant subspaces of $\mathscr{H}^p(\mathrm{G})$, for a multiply connected G, which are invariant but not fully invariant. We show that an invariant subspace $\mathscr{M}$ fails to be fully invariant if and only if there is one bounded component G_i of the complement of $\overline{\mathrm{G}}$ such that the ratio of any two functions in $\mathscr{M}$ has a pseudo-continuation to a meromorphic function in the Nevanlinna class of G_i. This allows us to give a complete characterization of those invariant subspaces of $\mathscr{H}^p(\mathrm{G})$ which contain the constants.

0. Introduction. Let G be a finitely connected bounded domain in C with smooth boundary contours. It is the purpose of this paper to study the closed linear subspaces of $\mathscr{H}^p(\mathbf{G})$ which are invariant under multiplication by z, that is, those subspaces $\mathscr{M}$ such that $zf(z)$ is in $\mathscr{M}$ whenever $f(z)$ is. The study of such spaces was initiated by Beurling [**1**] who gave a complete characterization of the invariant subspaces of $\mathscr{H}^2(\Delta)$, where Δ is the unit disk. Shortly thereafter Helson and Lowdenslager also investigated further problems of invariant subspaces using Beurling's methods.

Beurling's characterization is not difficult to extend to general simply connected domains, but the problem of characterizing the invariant subspaces for multiply connected domains is more complicated. A subspace $\mathscr{M}$ of $\mathscr{H}^p(\mathbf{G})$ is said to be fully invariant if it is invariant under multiplication by rational functions whose poles are in the complement of $\overline{\mathbf{G}}$. For simply connected domains all invariant subspaces are fully invariant, but this is no longer true for multiply connected G.

It is possible to give a characterization similar to Beurling's for the *fully* invariant subspaces of $\mathscr{H}^p(\mathbf{G})$. This was carried out for the annulus by Sarason [**12**] and for more general domains by Hasumi [**5**] and Voichick [**13**], [**14**].

Our results here depend on the notion of analytic (or meromorphic) pseudo-continuation. Two functions f_1 and f_2 of the Nevanlinna class N in abutting domains are said to be pseudo-continuations of each other across a smooth arc C in the boundary of both domains if the non-tangential limits of f_1 and f_2 are the same almost everywhere. In particular we show (Theorem 3) that if a closed invariant subspace $\mathscr{M}$ of $\mathscr{H}^p(\mathbf{G})$ is not fully invariant then there is some bounded component $\mathbf{G}_j$ of the complement of $\overline{\mathbf{G}}$ such that for any two functions f, g in $\mathscr{M}$ the function f/g has an analytic pseudo-continuation to $\mathbf{G}_j$.

This allows us to give a characterization of those closed invariant subspaces of $\mathscr{H}^p(\mathbf{G})$ which contain the constants. We also show that each closed invariant subspace of $\mathscr{H}^p(\mathbf{G})$ satisfying a restrictive hypothesis **H** is of the form

$$\mathscr{M} = \varphi \mathscr{M}_\chi$$

where φ is an inner function on **G**, χ a measurable function on the inner boundary contours of **G** whose modulus is constant on each of them, and $\mathscr{M}_\chi = \{f \in \mathscr{H}^p(\mathbf{G}) \colon f\chi$ extends to a function of class $\mathscr{H}^p$ in each bounded component of the complement of $\overline{\mathbf{G}}\}$. Simple examples show that not all invariant subspaces are of this form, and that the zeros (and possibly infinities) on the inner boundaries of **G** play a role in the characterization of invariant subspaces. It seems reasonable to suppose, however, that all invariant subspaces are of the form

$$h\mathscr{M}_\psi,$$

where $h \in \mathscr{H}^p(\mathbf{G})$ is a function whose outer part is continuous and non-vanishing on the outer boundary of **G**.

This paper is a revision (with simplified proofs of Theorems 2 and 4) of an unpublished preprint of mine from fifteen years ago. I had wanted to settle the conjecture above before publication, but I have been unable to do so. It appears to be difficult. Recently, Daniel Hitt [**6**] has established the truth of the conjecture for $\mathscr{H}^2(\mathbf{A})$, where **A** is the annulus. This seems to me to lend credence to the conjecture. He shows also that in this case the function χ can be taken to be the boundary values of an inner function in $\mathscr{H}^\infty(\Delta)$, Δ being the disk inside the annulus **A**.

The next two sections contain some general results on $\mathscr{H}^p(\mathbf{G})$, and in §3 we give a brief proof of the Hasumi-Sarason-Voichick characterization of fully invariant subspaces. In §5 we characterize the closed subspaces of $\mathscr{H}^p(\mathbf{G})$ which are invariant under the backward shift. This characterization is related to work of Douglas, Shapiro and Shields [2] on the backward shift on the disk. They also make use of analytic pseudo-continuation.

1. Inner functions and the Nevanlinna class. Let **G** be a finitely connected bounded domain in **C** with smooth (i.e., C^2) boundary Γ. In this section we reformulate the concepts of Blaschke product and inner functions in a manner suitable for function theory on a multiply connected domain **G** with smooth boundaries so that the classical factorization and divisibility theorems of Beurling remain. In the treatment here only single-valued holomorphic functions are used. In order to accomplish this, we shall often need to insert a harmonic measure into our formulae. This will mean that inner functions, etc., are only required to have moduli which are constant almost everywhere on each boundary contour of **G**, rather than having those which are one almost everywhere on the boundary of **G**. Sarason [12], Hasumi [5], and Voichick [14] take a different approach, and allow their functions to have multivalued arguments, but restrict inner functions, etc., to those whose boundary values are one almost everywhere.

By a harmonic measure on **G** we mean a harmonic function h whose boundary values are constant on each component of Γ. A bounded analytic function U on **G** is called a *unit* if $\log|U|$ is a harmonic measure. Equivalently, a bounded analytic function U is a unit if $|U|$ is constant on each boundary contour. The units of **G** form a finitely generated group under multiplication.

A bounded analytic function Φ in **G** is called a (generalized) *Blaschke product* if

$$\log|\Phi(z)| = \sum_{\nu} g(z, a_\nu) + h(z) \tag{1}$$

where $g(z, \zeta)$ is the Green's function for **G** and h is a harmonic measure.

We list below some standard properties of bounded analytic functions and Blaschke products. These are easily established using classical techniques (cf. [3], [4], and [7]).

LEMMA 1. *A bounded analytic function in* **G** *has non-tangential boundary values almost everywhere in* Γ. *If these boundary values vanish on a set of positive measure, so does the function.*

LEMMA 2. *If $\{a_\nu\}$ is the sequence of the zeros of a bounded analytic function f in* $\mathbf{G}$, *repeated according to multiplicity, then*

$$\sum d(a_\nu, \Gamma) < \infty.$$

If $\{a_\nu\}$ satisfies this condition, there is a Blaschke product Φ whose zeros are $\{a_\nu\}$, and it is unique apart from multiplication by a unit.

LEMMA 3. *A bounded analytic function f (not $\equiv 0$) may be factored into $f = \Phi g$ where Φ is a Blaschke product with the same zeros as f and g is a function without zeros having the same bound as f. The factorization is unique apart from units.*

LEMMA 4. *The product of two Blaschke products is a Blaschke product, and so is their quotient if it is analytic (i.e., has no poles).*

A bounded analytic function Φ in $\mathbf{G}$ is called an *inner function* if the nontangential boundary values of $|\Phi|$ on each boundary contour of $\mathbf{G}$ are almost everywhere equal to a constant. Note that this definition makes the constant 0 an inner function. An inner function ($\not\equiv 0$) is said to be non-trivial if it is not a unit.

An inner function with no zeros is called a *singular* function. Singular functions are those functions in $\mathscr{H}^\infty(\mathbf{G})$ for which

$$\log|\Phi(z)| = -\int_\Gamma \frac{\partial g(\zeta, z)}{\partial n}\, d\mu(\zeta) + h(z) \tag{2}$$

where μ is a positive measure on Γ which is singular with respect to the measure given by arc length, and h is a harmonic measure. The measure μ in this representation is unique.

LEMMA 5. *Every Blaschke product is an inner function. The product of two inner functions is an inner function, and so is their quotient if it is bounded.*

If Φ_1 and Φ_2 are inner functions, we say that Φ_1 divides Φ_2 if there is an inner function Φ_3 such that $\Phi_2 = \Phi_1\Phi_3$. If Φ_1 is not identically zero, then Φ_1 divides Φ_2 iff Φ_2/Φ_1 is bounded. Using the representations given in (1) and (2) one can establish the following lemma:

LEMMA 6. *Let $\{\Phi_\alpha\}$ be any collection of inner functions. Then there is an inner function Φ_0 which divides each Φ_α and has the property that if Φ is an inner function dividing each Φ_α, then Φ divides Φ_0.*

The function Φ_0 given by the lemma is called the greatest common divisor of the class $\{\Phi_\alpha\}$. If the class consists of two functions Φ_1 and Φ_2, we write $\Phi_0 = (\Phi_1, \Phi_2)$. The greatest common divisor of the class $\{0\}$ is 0.

A meromorphic function f in **G** is said to have bounded characteristic, or to belong to the Nevanlinna class N, if $f = f_1/f_2$ where f_1 and f_2 are bounded analytic functions in **G**.

LEMMA 7. *The functions of class N form a field. An analytic function f belongs to N if and only if* $\log|f|$ *has a positive harmonic majorant. Functions of class N have non-tangential boundary values almost everywhere, and, if two functions of class N have the same boundary values on a set of positive measure on* Γ, *they are identical.*

The existence of boundary values follows from Lemma 1. To see the unicity, we suppose $f = g$ on a set of positive measure on Γ, and that $f = f_1/f_2$ and $g = g_1/g_2$ with f_i, g_i bounded. Then $f_1 g_2 = f_2 g_1$ on a set of positive measure on Γ, and, since they are bounded analytic functions, we have $f_1 g_2 \equiv f_2 g_1$ in **G**. Thus $f = g$.

An analytic function f in N is called an *outer* function if

$$\log|f(\zeta)| = \frac{1}{2\pi}\int_\Gamma \log|f(z)|\frac{\partial g(z,\zeta)}{\partial n}\,ds, \tag{3}$$

where g is the Green's function of **G**.

LEMMA 8. *An outer function has no zeros. A function f is outer if* (3) *holds for a single point* $\zeta \in$ **G**. *The outer functions of N form a group under multiplication. An analytic function in N is both inner and outer if and only if it is a unit.*

LEMMA 9. *Every function f of class N may be factored into*

$$f = \frac{\Phi_1}{\Phi_2}F \tag{4}$$

where Φ_1 *and* Φ_2 *are inner functions with* $(\Phi_1, \Phi_2) = 1$ *and F an outer function. If f is not identically zero, then this factorization is unique except for units.*

An important subclass of the class N is the class N^+ consisting of those functions $f \in N$ which have a factorization of the form $f = \Phi_1 F$ where Φ_1 is inner and F is outer.

It is sometimes useful to extend these notions to a set $\mathbf{O}$ which is the union of a finite number of smoothly bounded domains $\{\mathbf{G}_j\}$ whose closures are disjoint. A function f is holomorphic in $\mathbf{O}$ if it is holomorphic in each $\mathbf{G}_j$. The function f is bounded (or of class N) iff each $f|_{\mathbf{G}_j}$ is. A function Ψ is an inner function on $\mathbf{O}$ iff $\Psi_j = \Psi|_{\mathbf{G}_j}$ is inner for each j. Note that some Ψ_j may be identically zero and others not.

The concept of inner and outer functions is due to Beurling **[1]** who first established the factorization into inner and outer functions for functions of class $\mathscr{H}^2$ on the disk.

2. The Hardy classes $\mathscr{H}^p$. An analytic function f in $\mathbf{G}$ is said to belong to the Hardy class $\mathscr{H}^p(\mathbf{G})$ if $|f|^p$ has a harmonic majorant in $\mathbf{G}$. For $p \geq 1$ these are Banach spaces, if we define

$$||f||_p = (u(\zeta_0))^{1/p}, \tag{5}$$

where ζ_0 is a fixed point of $\mathbf{G}$ and u is the least harmonic majorant of $|f|^p$. Different choices of ζ_0 give different norms, but these are equivalent. For a discussion of these spaces in this form see Parreau **[10]** or Rudin **[11]**.

We use $\mathscr{H}^\infty$ to denote the space of all bounded analytic functions on $\mathbf{G}$, and $\mathscr{H}$ to denote the union of all $\mathscr{H}^p$ with $p > 0$. If $p > q$, then $\mathscr{H}^p \subset \mathscr{H}^q$.

A harmonic function u in $\mathbf{G}$ is said to belong to the class $h^p(\mathbf{G})$ if $|u|^p$ has a harmonic majorant. The class $h^1(\mathbf{G})$ consists precisely of those harmonic functions which can be expressed as the difference of two non-negative harmonic functions. We list some standard properties of the $\mathscr{H}^p$ spaces as further lemmas. The following result is due to M. Riesz (cf. **[3]** and **[4]**).

LEMMA 10. *If $1 < p < \infty$, then a holomorphic function f in $\mathbf{G}$ belongs to $\mathscr{H}^p$ iff* $\operatorname{Re} f$ *belongs to h^p. If* $\operatorname{Re} f \in h^1$, *then $f \in \mathscr{H}^p$ for all $p < 1$.*

LEMMA 11. *Each f in $\mathscr{H}^p$ belongs to the Nevanlinna class N^+ and has a factorization*

$$f = \Phi_1 F, \tag{6}$$

where Φ_1 is inner and F is an outer function in $\mathscr{H}^p$ with the same norm as f.

LEMMA 12. *A function $f \in N$ with canonical factorization $\Phi_1\Phi_2^{-1}F$ belongs to $\mathscr{H}^p$ if and only if Φ_2 is a unit and the boundary values of*

$|F|$ belong to $\mathscr{L}^p$ on Γ. The $\mathscr{L}^p$ norm of the boundary values of f is an equivalent norm for $\mathscr{H}^p$.

Thus a function f in N^+ is in $\mathscr{H}^p$ if and only if its boundary values are in $\mathscr{L}^p$. We say that an inner function Φ divides an $\mathscr{H}^p$ function f if $f = \Phi g$ for some $g \in \mathscr{H}^p$.

COROLLARY. *An inner function Φ divides a function f in $\mathscr{H}^p$ if and only if Φ divides the inner part of f. If $\Phi \not\equiv 0$, we have $\|f\Phi^{-1}\| \leq m\|f\|$, where m^{-1} is the essential infimum of $|\Phi|$ on the boundary.*

The following proposition guarantees that certain functions which we construct are of class $\mathscr{H}$ and hence of class N.

PROPOSITION 1. *Let $\dot{z}$ denote the unit tangent vector at a point z of Γ, and let μ be a complex Borel measure on Γ. Then for each $p < 1$ the function f defined by*

$$f(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{\dot{z}\, d\mu(z)}{z - \zeta} \tag{7}$$

is of class $\mathscr{H}^p$ in $\mathbf{G}$ and in each component of the complement of $\overline{\mathbf{G}}$. Let f_j^+ denote the boundary values of the function from the outside and f_j^- from the inside on a component of Γ_j of Γ. Then

$$f_j^+(z) - f_j^-(z) = \mu'(z) \tag{8}$$

where μ' is the Radon-Nikodym derivative of μ with respect to arc length on Γ.

Proof. Since each complex measure is a linear combination of positive ones and $\mathscr{H}^p$ is a linear space, it suffices to prove the proposition for positive measure μ. Since Γ is C^2, there is a $\delta > 0$ such that at each $z \in \Gamma$ there are two circles of radius δ tangent to Γ at z, with one circle in $\mathbf{G}$ and one in $\tilde{\mathbf{G}}$. Then

$$\operatorname{Im}\frac{\dot{z}}{z-\zeta} > -\frac{1}{2}\delta^{-1}, \qquad \zeta \in \mathbf{G},$$

and

$$\operatorname{Im}\frac{\dot{z}}{z-\zeta} < \frac{1}{2}\delta^{-1}, \qquad \zeta \in \tilde{\mathbf{G}}.$$

Thus in $\mathbf{G}$ the real part of f differs from a positive harmonic function by a bounded function and so belongs to h^1. By Lemma 10, the function $f|_\mathbf{G}$ is in $\mathscr{H}^p(\mathbf{G})$ for each $p < 1$. Similarly for $g|_{\tilde{\mathbf{G}}}$. The fact that (8) holds at each point where the cumulative distribution function of $d\mu$ has a derivative follows from the argument in Nevanlinna [9], section 163, pp 193 f. □

LEMMA 14. *If h is a function of class $\mathscr{L}^p$ on Γ with $1 \le p < \infty$, and $g(z, \zeta)$ is the Green's function for* **G**, *then the function*

$$u(\zeta) = \frac{1}{2\pi} \int_\Gamma h(z) \frac{\partial g}{\partial n}\, ds,$$

is a harmonic function of class h^p.

Proof. Observe that $\partial g / \partial n$ is non-negative and

$$\frac{1}{2\pi} \int_\Gamma \frac{\partial g}{\partial n}\, ds = 1.$$

Hence by the Hölder inequality

$$|u(\zeta)|^p \le \frac{1}{2\pi} \int_\Gamma |h(z)|^p \frac{\partial g}{\partial n}\, ds,$$

and the integral on the right is a harmonic majorant for $|u|^p$. □

PROPOSITION 2. *Let h be a function of class $\mathscr{L}^p$ on Γ with $1 < p < \infty$. Then the function f defined by*

$$f(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{h(z)}{z - \zeta}\, dz \tag{9}$$

is of class $\mathscr{H}^p$ in **G** *and in each component of the complement of* $\overline{\mathbf{G}}$. *Moreover, $f(\infty) = 0$. If f_j^+ and f_j^- denote the limits from outside and from inside on a component Γ_j of Γ, then*

$$f_j^+(z) - f_j^-(z) = h(z) \tag{10}$$

almost everywhere.

Proof. It suffices to prove the proposition when h is real. If δ is the constant in the proof of Proposition 1, then in **G** we have

$$\left| \operatorname{Re} \frac{2i\dot{z}}{z - \zeta} - \frac{\partial g(z, \zeta)}{\partial n} \right| < \delta^{-1}.$$

Consequently, $\operatorname{Re} f$ differs in **G** by a bounded harmonic function from

$$\frac{1}{2\pi} \int_\Gamma h(z) \frac{\partial g(z, \zeta)}{\partial n}\, ds.$$

Since the latter is in $\mathscr{H}^p$ by Lemma 14, we have $\operatorname{Re} f \in h^p$. By the theorem of Riesz (Lemma 10) we have $f \in \mathscr{H}^p$. A similar argument applies to each component of the complement of $\overline{\mathbf{G}}$, and the fact that (10) holds follows from Proposition 1. □

LEMMA 15. *Let* $f \in \mathscr{H}^p(\mathbf{G})$ *for* $1 \leq p \leq \infty$. *Then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\zeta}\,dz = \begin{cases} f(\zeta), & \textit{if } \zeta \in \mathbf{G}; \\ 0, & \textit{if } \zeta \in \tilde{\mathbf{G}}. \end{cases} \tag{11}$$

A function analytic in the complement $\tilde{\mathbf{G}}$ of $\overline{\mathbf{G}}$ is said to be of class $\mathscr{H}^r$ in $\tilde{\mathbf{G}}$ if it is of class $\mathscr{H}^r$ in each component of the complement. If $f \in \mathscr{H}^1(\tilde{\mathbf{G}})$, and $f(\infty) = 0$, then

$$-\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\zeta}\,dz = \begin{cases} f(\zeta), & \text{if } \zeta \in \tilde{\mathbf{G}}; \\ 0, & \text{if } \zeta \in \mathbf{G}. \end{cases} \tag{12}$$

This enables us to characterize the topological dual of $\mathscr{H}^p(\mathbf{G})$ as $\mathscr{H}_0^q(\tilde{\mathbf{G}})$ for $1 < p < \infty$, where $q^{-1} = p^{-1} = 1$, and where $\mathscr{H}_0^q(\tilde{\mathbf{G}})$ is the subset of $\mathscr{H}^q(\tilde{\mathbf{G}})$ consisting of those functions f with $f(\infty) = 0$:

PROPOSITION 3. *Let* $\mathbf{G}$ *be a finitely connected, bounded domain whose boundary* Γ *is* C^2. *Then for each continuous linear functional* L *on* $\mathscr{H}^p(\mathbf{G})$, $1 < p < \infty$, *there is a unique function* $g \in \mathscr{H}_0^q(\tilde{\mathbf{G}})$ *with* $q^{-1} = p^{-1} = 1$ *such that*

$$L(f) = \frac{1}{2\pi i}\int_\Gamma f(z)g(z)\,dz. \tag{13}$$

Proof. By the Hahn-Banach theorem we may extend L to a bounded linear functional on $\mathscr{L}^p(\Gamma)$. By the Riesz representation theorem there is a function $k \in \mathscr{L}^q(\Gamma)$ such that

$$L(f) = \frac{1}{2\pi i}\int_\Gamma f(z)k(z)\,dz.$$

Define g in $\tilde{\mathbf{G}}$ and h in $\mathbf{G}$ by

$$g(\zeta) = -\frac{1}{2\pi i}\int_\Gamma \frac{k(z)}{z-\zeta}\,dz, \qquad \zeta \in \tilde{\mathbf{G}},$$

and

$$h(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{k(z)}{z-\zeta}\,dz, \qquad \zeta \in \mathbf{G}.$$

Then g and h are in $\mathscr{H}^q$ by Proposition 2, and $k = g + h$. Since $hf \in \mathscr{H}^1(\mathbf{G})$ for $f \in \mathscr{H}^p(\mathbf{G})$, we have

$$\frac{1}{2\pi i}\int_\Gamma h(z)f(z)\,dz = 0.$$

Hence

$$L(f) = \frac{1}{2\pi i} \int_\Gamma g(z) f(z)\, dz.$$

Since $g(\infty) = 0$, it remains only to show the uniqueness of g. Suppose that for some $g \in \mathscr{H}^q(\tilde{\mathbf{G}})$ we had

$$\frac{1}{2\pi i} \int_\Gamma g(z) f(z)\, dz = 0$$

for each $f \in \mathscr{H}^p(\mathbf{G})$. Since $(z-\zeta)^{-1} \in \mathscr{H}^p(\mathbf{G})$ whenever $\zeta \in \tilde{\mathbf{G}}$, we must have

$$g(\zeta) = -\frac{1}{2\pi i} \int_\Gamma \frac{g(z)}{z-\zeta}\, dz = 0.$$

□

3. Fully invariant subspaces of $\mathscr{H}^p$. Let $\mathbf{G}$ be a bounded domain in $\mathbf{C}$ with smooth boundary Γ, and let $\mathbf{G}_0, \mathbf{G}_1, \dots, \mathbf{G}_{n-1}$ be the components of the complement of $\overline{\mathbf{G}}$. We say that a closed subspace $\mathscr{M}$ of $\mathscr{H}^p(\mathbf{G})$ is *fully invariant* if there are points $\alpha_j \in \mathbf{G}_j$ such that $(z-\alpha_j)^{-1} f(z)$ is in $\mathscr{M}$ whenever f is. We may take $\alpha_0 = \infty$ for the unbounded component $\mathbf{G}_0$, in which case we require that $zf(z)$ is in $\mathscr{M}$ whenever f is, i.e. that $\mathscr{M}$ is invariant in our previous sense. Since any function holomorphic in an open set containing $\overline{\mathbf{G}}$ can be uniformly approximated on $\overline{\mathbf{G}}$ by rational functions with poles only at the points α_j, we see that the fully invariant subspaces of $\mathscr{H}^p(\mathbf{G})$ are just those closed subspaces which are invariant under multiplication by any function holomorphic in a neighborhood of $\overline{\mathbf{G}}$. This definition is not changed by conformal mapping, and we may use it to define the notion of *fully invariant* subspaces for domains in $\hat{\mathbf{C}}$ which contain ∞ and also for open sets in $\hat{\mathbf{C}}$ which are a finite union of components.

The characterization of the fully invariant subspaces of $\mathscr{H}^p(\mathbf{G})$ for multiply connected regions $\mathbf{G}$ closely resembles Beurling's characterization for $\mathscr{H}^p(\Delta)$. This characterization was carried out for the annulus by Sarason **[12]**, and for the more general case by Hasumi **[5]**, and Voichick **[13]**, **[14]**. Their results are summarized in the following Theorem. We give a brief proof utilizing methods we shall use later.

If φ is an inner function in $\mathbf{G}$, we denote by $\varphi\mathscr{H}^p$ the space of all $f \in \mathscr{H}^p$ which are multiples of φ, i.e., those f for which there exists $g \in \mathscr{H}^p$ such that $f = \varphi g$. These are clearly closed subspaces of $\mathscr{H}^p$ invariant under multiplication by rational functions with poles outside $\mathbf{G}$. The following theorem states that these are the only closed fully invariant subspaces.

THEOREM 1. *Let $\mathscr{M}$ be a closed subspace of $\mathscr{H}^p(\mathbf{G})$ for $1 \leq p \leq \infty$ (weak* closed if $p = \infty$) which is invariant under multiplication by rational functions with poles outside* $\mathbf{G}$. *Then there is an inner function φ on* $\mathbf{G}$ *such that $\mathscr{M} = \varphi\mathscr{H}^p$.*

Proof. Let φ be the greatest common divisor of the inner parts of functions of $\mathscr{M}$. Then $\mathscr{M} \subset \varphi\mathscr{H}^p$. Let $\mathscr{X}^0$ denote the annihilator of $\mathscr{X}$ in $\mathscr{H}_0^p(\tilde{\mathbf{G}})$. Since $\mathscr{M}$ is closed, $\mathscr{M} = \mathscr{M}^{00}$. Thus in order to show that $\varphi\mathscr{H}^p \subseteq \mathscr{M}$, it suffices to show that each linear functional L in $\mathscr{M}^0$ satisfies $L(\phi g) = 0$ for each $g \in \mathscr{H}^p$. In the case of $\mathscr{H}^\infty$, we have assumed $\mathscr{M}$ is weak* closed in $\mathscr{L}^\infty$, and so it suffices to consider weak* continuous linear functionals, i.e., those which are represented by integrals of $\mathscr{L}^1$ functions.

If every continuous linear functional which vanishes on $\mathscr{M}$ is zero, then $\mathscr{M} = \mathscr{H}^p$, and the theorem holds with $\varphi = 1$. Otherwise, let L be a non-zero continuous linear functional in $\mathscr{M}^0$. Then there is a $k \in \mathscr{L}^q(\Gamma)$, $p^{-1} + q^{-1} = 1$, such that

$$L(f) = \frac{1}{2\pi i}\int_\Gamma f(z)k(z)\,dz.$$

Define Tf in the complement of Γ by

$$Tf(\zeta) = L((z-\zeta)^{-1}f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)k(z)\,dz}{z-\zeta}. \tag{14}$$

Then for $r < 1$ we have $Tf \in \mathscr{H}^r$ in $\mathbf{G}$ and each component of the complement of $\overline{\mathbf{G}}$. If $f \in \mathscr{M}$, and $\mathscr{M}$ is fully invariant, then $Tf \equiv 0$ outside $\mathbf{G}$. By Proposition 1 the function fk is the set of boundary values of a function in $\mathscr{H}^r$ of $\mathbf{G}$. The quotient of this function by f is a meromorphic function in $\mathbf{G}$ of class N whose non-tangential boundary values are almost everywhere equal to k. Denote this function by k, and let $k = \Phi_1\Phi_2^{-1}K$ be its canonical factorization. If $f \in \mathscr{M}$ has the canonical factorization $f = \Phi F$, then $fk = \Phi\Phi_1\Phi_2^{-1}FK$. Since $fk \in \mathscr{H}^r$, Φ_2 must divide Φ. Because this holds for each $f \in \mathscr{M}$, Φ_2 must divide φ.

Let $g \in \mathscr{H}^p$ be any multiple of φ. Then

$$gk = h\varphi k = h\varphi\Phi_1\Phi_2^{-1}K$$

is a function of class N whose canonical factorization contains no inner function in the denominator and whose boundary values are in $\mathscr{L}^1$. Thus $gk \in \mathscr{H}^1$ by Lemma 11, and

$$L(g) = \frac{1}{2\pi i}\int_\Gamma g(z)k(z)\,dz = 0.$$

Consequently L annihilates $\phi\mathscr{H}^p(\mathbf{G})$, and so $\phi\mathscr{H}^p(\mathbf{G}) \subset \mathscr{M}$. Thus $\mathscr{M} = \phi\mathscr{H}^p(\mathbf{G})$. □

A slight modification of this proof gives the standard characterization of the closed ideals in the algebra $\mathscr{A}(\mathbf{G})$ of continuous functions on $\overline{\mathbf{G}}$ which are analytic in $\mathbf{G}$.

If $\mathbf{O}$ is an open set which is the finite union of regions $\{\mathbf{G}_j\}$ with disjoint closures and smooth boundaries, we may still define fully invariant as before: A closed subspace of $\mathscr{H}^p(\mathbf{O})$ is fully invariant if $rf \in \mathscr{M}$ whenever $f \in \mathscr{M}$ and r is a rational function with poles in the complement of $\overline{\mathbf{O}}$. It is not difficult to see that a closed subspace $\mathscr{M}$ of $\mathscr{H}^p(\mathbf{O})$ is fully invariant if and only if the subspace $\mathscr{M}_j = \{g\colon g = f|_{\mathbf{G}_j},\ f \in \mathscr{M}\}$ is a fully invariant subspace of $\mathscr{H}^p(\mathbf{G}_j)$. Then Theorem 1 still holds with $\mathbf{G}$ replaced by $\mathbf{O}$. Note that in this case the inner function ϕ is given by giving inner functions ϕ_j on each of the components $\mathbf{G}_j$ of $\mathbf{O}$. Some of the ϕ_j may be identically zero without ϕ being identically zero.

4. Analytic pseudo-continuation. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be disjoint plane domains whose boundaries have a smooth arc Γ in common, and let f_1 and f_2 be meromorphic functions of class N on $\mathbf{D}_1$ and $\mathbf{D}_2$. We say that f_2 is an *analytic pseudo-continuation* of f_1 across Γ if the boundary values of f_1 and f_2 on Γ are equal almost everywhere. Note that an analytic pseudo-continuation of f_1 to $\mathbf{D}_2$ across Γ is unique if it exists.[1]

If f_1 and f_2 are in $\mathscr{H}^1$ (or any $\mathscr{H}^p$, $1 \le p \le \infty$) and f_2 is an analytic pseudo-continuation of f_1, then Lemma 15 may be used to show that there is a holomorphic function f on $\mathbf{D}_1 \cup \mathbf{D}_2 \cup \Gamma$ with $f|_{\mathbf{D}_i} = f_i$. In this case f_2 is an ordinary analytic continuation of f.

If φ is an inner function (or a quotient of inner functions) in the unit disk Δ, then the function $\tilde{\varphi}$ defined in $\tilde{\Delta}$ by

$$\tilde{\varphi}(z) = \overline{\varphi(1/\bar{z})}$$

is an inner function (or quotient of inner functions) in $\tilde{\Delta}$ and $\tilde{\varphi}^{-1}$ is the analytic pseudo-continuation of φ to $\tilde{\Delta}$.

For each inner function ψ on the complement $\tilde{\mathbf{G}}$ of $\overline{\mathbf{G}}$ we define the subspace $\mathscr{M}_\psi$ of $\mathscr{H}^p(\mathbf{G})$ to be the space of those functions $f \in \mathscr{H}^p(\mathbf{G})$ such that the function $f\psi|_\Gamma$ is almost everywhere equal to the nontangential boundary values of a function in $\mathscr{H}_0^p(\tilde{\mathbf{G}})$. The following proposition characterizes $\mathscr{M}_\psi$ in terms of its annihilators.

[1] This is no longer true if we do not require a pseudo-continuation to be of class N. See Kahane and Katznelson [8].

PROPOSITION 4. *Let ψ be an inner function in $\mathscr{H}^\infty(\tilde{\mathbf{G}})$. Then the annihilator in $\mathscr{H}^p(\mathbf{G})$ of $\psi\mathscr{H}_0^q(\tilde{\mathbf{G}})$ is just the subspace*

$$\mathscr{M}_\psi = \{f \in \mathscr{H}^p(\mathbf{G})\colon f\psi \text{ extends from } \Gamma \text{ to a function in } \mathscr{H}_0^p(\tilde{\mathbf{G}})\}.$$

Proof. Let $f \in \mathscr{M}_\psi$ and $g = \psi h$ be an arbitrary element of $\psi\mathscr{H}_0^q(\tilde{\mathbf{G}})$. Then $f\psi$ extends to a function in $\mathscr{H}_0^p(\tilde{\mathbf{G}})$ and so $fg = f\psi h$ extends to a function in $\mathscr{H}_0^1(\tilde{\mathbf{G}})$ with a double zero at ∞. Hence

$$\int_\Gamma f(z)g(z)\,dz = 0,$$

and $f \in [\psi\mathscr{H}_0^q(\tilde{\mathbf{G}})]^0$. This shows that $\mathscr{M}_\psi \subset [\psi\mathscr{H}_0^q(\tilde{\mathbf{G}})]^0$.

Let f be any element in $[\psi\mathscr{H}_0^q(\tilde{\mathbf{G}})]^0$. For any $\zeta \in \mathbf{G}$, the function $(z-\zeta)^{-1}$ is in $\mathscr{H}_0^q(\tilde{\mathbf{G}})$ and so

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)\psi(z)}{z-\zeta}\,dz \equiv 0, \qquad \zeta \in \mathbf{G}.$$

If we set

$$h(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)\psi(z)}{z-\zeta}\,dz$$

for $\zeta \in \tilde{\mathbf{G}}$, then Proposition 2 implies that $h \in \mathscr{H}_0^p(\tilde{\mathbf{G}})$, and that h has the boundary limits $f(z)\psi(z)$ almost everywhere on Γ. Thus $f \in \mathscr{M}_\psi$. This shows that

$$\mathscr{M}_\psi \subset [\psi\mathscr{H}_0^q(\tilde{\mathbf{G}})]^0. \qquad \square$$

In the remaining sections we use the concept of analytic pseudo-continuation to characterize some of the invariant subspaces of $\mathscr{H}^p(\mathbf{G})$ which are not fully invariant. We begin by looking at invariant subspaces for the backward shift.

5. Subspaces of $\mathscr{H}^p$ invariant under the backward shift operator. In this section, we assume $\mathbf{G}$ is a bounded domain containing the origin. The backward shift operator S^* is defined on the space of analytic functions on $\mathbf{G}$ by

$$(S^*f)(z) = \frac{f(z)-f(0)}{z}. \tag{15}$$

It is a left inverse of the shift operator (the operator S which sends f into zf) and maps $\mathscr{H}^p(\mathbf{G})$ into itself. On $\mathscr{H}^2(\Delta)$, where Δ is the unit disk, S^* is the Hilbert space adjoint of S.

The methods of the preceding section can be used to characterize the closed subspaces of $\mathscr{H}^2(\mathbf{G})$ which are invariant under S^*. Douglas, Shapiro, and Shields [2] have considered this problem from a slightly different viewpoint.

For each inner function ψ on $\tilde{\mathbf{G}}$ we denote by $\mathscr{M}_\psi$ the space of functions $f \in \mathscr{H}^p(\mathbf{G})$ such that $f\psi$ can be extended to a function in $\mathscr{H}^p(\tilde{\mathbf{G}})$ which vanishes at ∞. Clearly, $\mathscr{M}_\psi$ is a closed linear subspace of $\mathscr{H}^p(\mathbf{G})$. If $f \in \mathscr{M}_\psi$, so is $S^*f = (f - f(0))/z$: The first term times ψ has a continuation in $\mathscr{H}^p(\mathbf{G})$, namely $(1/z)$ times the continuation of $f\psi$. Likewise for the second term, and both vanish at ∞. Thus $\mathscr{M}_\psi$ is a closed invariant subspace for S^*. We shall show that these are the only closed subspaces of $\mathscr{H}^p(\mathbf{G})$ which are invariant under S^*.

THEOREM 2. *Let $\mathscr{M}$ be a closed subspace of $\mathscr{H}^p(\mathbf{G})$ which is invariant under S^*. If $1 < p < \infty$, there is an inner function ψ on* **G** *such that $\mathscr{M} = \mathscr{M}_\psi$.*

Proof. Let L_k be the linear functional on $\mathscr{H}^p(\mathbf{G})$ given by

$$L_k[f] = \frac{1}{2\pi i}\int_\Gamma f(z)k(z)\,dz,$$

with $k \in \mathscr{H}_0^q(\tilde{\mathbf{G}})$. Then

$$L_k[S^*f] = \frac{1}{2\pi i}\int_\Gamma \frac{f(z) - f(0)}{z}k(z)\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)k(z)\,dz}{z},$$

since

$$\int_\Gamma \frac{k(z)\,dz}{z} = 0.$$

Thus

$$L_k[S^*f] = L_{Sk}[f], \tag{16}$$

where S is the linear operator on $\mathscr{H}_0^q(\tilde{\mathbf{G}})$ which takes $k(z)$ into $z^{-1}k(z)$.

Let $\mathscr{M}$ be a closed subspace of $\mathscr{H}^p(\mathbf{G})$ invariant under S^*. Then (16) implies that $\mathscr{M}^0$ is invariant under S. But this means that $\mathscr{M}^0$ is a closed fully invariant subspace of $\mathscr{H}_0^q(\tilde{\mathbf{G}})$, since the complement of $\tilde{\mathbf{G}}$ has only the one component $\overline{\mathbf{G}}$. By Theorem 1 applied to $\mathscr{H}_0^q(\tilde{\mathbf{G}})$ we have

$$\mathscr{M}^0 = \psi\mathscr{H}_0^q(\tilde{\mathbf{G}})$$

for some inner function ψ on $\tilde{\mathbf{G}}$. Hence

$$\mathscr{M} = [\psi\mathscr{H}_0^q(\tilde{\mathbf{G}})]^0 = \mathscr{M}_\psi. \qquad \square$$

6. Invariant subspaces of $\mathscr{H}^p$ for multiply connected regions. Let **G** be a finitely connected bounded region with smooth boundary. As usual we denote the unbounded component of the complement of $\overline{\mathbf{G}}$ by $\mathbf{G}_0$, and the bounded components by $\mathbf{G}_1, \ldots, \mathbf{G}_{n-1}$. We let $-\Gamma_j$ denote the boundary of $\mathbf{G}_j$, $0 \leq j \leq n-1$. In this section we characterize some of the closed subspaces $\mathscr{M}$ of $\mathscr{H}^p(\mathbf{G})$ which are invariant under multiplication by z. We begin with a theorem which gives us information about the invariance of such a subspace under multiplication by a rational function.

THEOREM 3. *Let $\mathscr{M}$ be a closed subspace of $\mathscr{H}^p(\mathbf{G})$, $1 \leq p \leq \infty$ (weak* closed if $p = \infty$), which is invariant under multiplication by z, and let $\mathbf{G}_j$ be a bounded component of the complement of $\overline{\mathbf{G}}$. Then either $\mathscr{M}$ is invariant under multiplication by rational functions with poles in $\mathbf{G}_j$ or else each pair of elements f, g in $\mathscr{M}$ has the property that f/g has a pseudo-continuation of class N in $\mathbf{G}_j$. The inner part of the denominator of this pseudo-continuation is a multiple of an inner function Ψ in $\mathbf{G}_j$ which depends only on g.*

Proof. Suppose there is a function $g \in \mathscr{M}$ such that $(z-a)^{-1}g \notin \mathscr{M}$ for some $a \in \mathbf{G}_j$. Then it suffices to show that for each $f \in \mathscr{M}$ the quotient f/g has a pseudo-continuation to $\mathbf{G}_j$. Let L be a continuous linear functional on $\mathscr{H}^p(\mathbf{G})$ such that $L[\mathscr{M}] = 0$ and $L[(z-a)^{-1}g] \neq 0$. Then there is a function $k \in \mathscr{L}^q(\Gamma)$, $p^{-1} + q^{-1}$, such that for each $f \in \mathscr{H}^p$ we have

$$L(f) = \frac{1}{2\pi i} \int_\Gamma f(z)k(z)\,dz.$$

Let Γ_0 be the outer boundary of $\mathbf{G}$ and $-\Gamma_j$ the boundary of $\mathbf{G}_j$. Let k_0 and k_j be the restrictions of k to Γ_0 and Γ_j. By adding a constant to k, if necessary, we may assume $k_0 \not\equiv 0$ and $k_j \not\equiv 0$.

The functions Tf defined by

$$(Tf)(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)k(z)\,dz}{z-\zeta}$$

are of class $\mathscr{H}^r$ for each $r < 1$ in the complement of Γ. If $f \in \mathscr{M}$, then for each $\zeta \in \mathbf{G}_0$, the function $(z-\zeta)^{-1}f \in \mathscr{M}$. Hence for $f \in \mathscr{M}$ the function Tf is identically zero in $\mathbf{G}_0$. By Proposition 2 the function Tf has the boundary values fk_0 almost everywhere on Γ_0 as we approach Γ_0 non-tangentially from $\mathbf{G}$.

Let $g \in \mathscr{M}$ be the function chosen at the beginning of the proof. Then $(Tg)(a) \neq 0$, and so $Tg \not\equiv 0$ in $\mathbf{G}_j$. Since the function Tg in $\mathbf{G}$ has boundary values gk_0 almost everywhere on Γ_0, we see that $Tf/Tg = f/g$ in $\mathbf{G}$. Let $(Tf)^-$ and $(Tg)^-$ denote the boundary values on Γ_i of Tf and Tg in $\mathbf{G}_j$, and $(Tf)^+$ and $(Tg)^+$ the boundary values on Γ_j approached from $\mathbf{G}$. Then $(Tf)^+ - (Tf)^- = k_j f$ and $(Tg)^+ - (Tg)^- = k_j g$. Thus $(Tf)^-/(Tg)^-$ has the boundary values f/g on Γ_j. Since $Tg \not\equiv 0$ in $\mathbf{G}_j$, the function $(Tf)/(Tg)$ is of class N in $\mathbf{G}_j$. Thus f/g has a pseudo-continuation to $\mathbf{G}_j$ of class N. The denominator of the canonical factorization of this function is a multiple of the inner part of Tg. □

The next lemma characterizes those closed invariant subspaces of $\mathscr{H}^p$, $1 < p < \infty$, which contain the constant functions. We let $\mathbf{G}_0$ denote the unbounded component of the complement of $\overline{\mathbf{G}}$ and $\{\mathbf{G}_i\}$ the bounded components, and let Γ_0 be the outer boundary of $\mathbf{G}$ and $-\Gamma_i$ the boundary of $\mathbf{G}_i$. For a function $k \in \mathscr{H}^p(\tilde{\mathbf{G}})$ we use the notation $k = \langle k_0, k_1, \ldots, k_{n-1}\rangle$ to indicate that $k|_{\mathbf{G}_j} = k_j$.

LEMMA 16. *Let $\mathscr{M}$ be a closed invariant subspace of $\mathscr{H}^p(\mathbf{G})$, $1 < p < \infty$. Then $1 \in \mathscr{M}$ if and only if every annihilator $k = \langle k_0, k_1, \ldots, k_{n-1}\rangle$ of $\mathscr{M}$ in $\mathscr{H}_0^q(\tilde{\mathbf{G}})$ has $k_0 \equiv 0$.*

Proof. Suppose $k_0 \equiv 0$ for each $k \in \mathscr{M}$. Then 1 is annihilated by k for each $k \in \mathscr{M}^0$. This implies $1 \in \mathscr{M}$, since $\mathscr{M}$ is closed.

We now suppose $1 \in \mathscr{M}$, and take any $k \in \mathscr{M}^0$. Since 1 belongs to $\mathscr{M}$, so does the function $(z-\zeta)^{-1}$ for each $\zeta \in \mathbf{G}_0$. Hence

$$\frac{1}{2\pi i}\int_\Gamma \frac{k(z)\,dz}{z-\zeta} \equiv 0$$

for $\zeta \in \mathbf{G}_0$. But

$$\frac{1}{2\pi i}\int_\Gamma \frac{k(z)\,dz}{z-\zeta} = \frac{1}{2\pi i}\int_{\Gamma_0} \frac{k_0(z)\,dz}{z-\zeta} + \frac{1}{2\pi i}\sum_{j=1}^{n-1}\int_{\Gamma_j} \frac{k_j(z)\,dz}{z-\zeta} = k_0(\zeta)$$

for $\zeta \in \mathbf{G}_0$. Thus $k_0(\zeta) \equiv 0$. □

Let

$$\mathbf{G}_I = \bigcup_{j=1}^{n-1} \mathbf{G}_j$$

be the union of the bounded components of the complement of $\overline{\mathbf{G}}$, and let

$$\Gamma_I = \bigcup_{j=1}^{n-1} \Gamma_j$$

be the negative of the boundary of $\mathbf{G}_I$. Then Lemma 16 says that the annihilator $\mathscr{M}^0$ of an invariant subspace $\mathscr{M}$ which contains 1 is in effect a subspace of $\mathscr{H}^q(\mathbf{G}_I)$. This enables us to establish the following theorem characterizing those invariant subspaces.

THEOREM 4. *Let $\mathscr{M}$ be a closed invariant subspace of $\mathscr{H}^p(\mathbf{G})$, $1 < p < \infty$, and suppose $1 \in \mathscr{M}$. Then there is an inner function ψ on $\mathbf{G}_I$ such that $\mathscr{M}$ is the space $\mathscr{M}_\psi$ consisting of those $f \in \mathscr{H}^p(\mathbf{G})$ such that $f\psi$ is the boundary values of a function in $\mathscr{H}^p(\mathbf{G}_I)$.*

Proof. By Lemma 16 each $L \in \mathscr{M}^0$ is of the form

$$L[f] = L_k[f] = \frac{1}{2\pi i}\int_{\Gamma_I} f(z)k(z)\,dz$$

for some $k \in \mathscr{H}^q(\mathbf{G}_I)$. If S is the operator on $\mathscr{H}^p(\mathbf{G})$ and $\mathscr{H}^q(\mathbf{G}_I)$ which takes g into zg, then

$$L_k[Sf] = \frac{1}{2\pi i}\int_{\Gamma_I} zf(z)k(z)\,dz = L_{Sk}[f].$$

This shows that the annihilator $\mathscr{M}^0$ of $\mathscr{M}$ is an invariant subspace of $\mathscr{H}^p(\mathbf{G}_I)$. Since each component of $\tilde{\mathbf{G}}$ is a bounded simply connected domain, $\mathscr{M}^0$ must be a fully invariant subspace of $\mathscr{H}^q(\mathbf{G}_I)$. By Theorem 1 there is an inner function ψ on $\mathbf{G}_I$ such that $\mathscr{M}^0 = \psi\mathscr{H}^q(\mathbf{G}_I)$. If we extend ψ to $\tilde{\mathbf{G}}$ by setting $\psi \equiv 0$ in $\mathbf{G}_0$, then ψ is an inner function in $\mathscr{H}_0^\infty(\tilde{\mathbf{G}})$, and

$$\mathscr{M}^0 = \psi\mathscr{H}_0^q(\tilde{\mathbf{G}}).$$

Thus

$$\mathscr{M} = [\psi\mathscr{H}_0^q(\tilde{\mathbf{G}})]^0 = \mathscr{M}_\psi$$

by Proposition 3. □

This theorem gives us a one-to-one correspondence between invariant subspaces of $\mathscr{H}^p(\mathbf{G})$ containing 1 and the inner functions on $\mathbf{G}_I$. Observe that the spaces $\mathscr{M}_\psi$ are closed invariant subspaces for each ψ. Also $\mathscr{M}_\psi \subseteq \mathscr{M}_{\psi'}$ if and only if ψ divides ψ'.

A subspace $\mathscr{M}$ of $\mathscr{H}^p$ is said to be a *reduced* subspace if the greatest common divisor of the inner parts of the functions of $\mathscr{M}$ is 1. If $\mathscr{M}$ is a closed invariant subspace of $\mathscr{H}^p$ and φ is the greatest common divisor of the inner parts of functions of $\mathscr{M}$, then the subspace $\mathscr{M}' = \{f\colon f\varphi \in \mathscr{M}\}$ is a reduced closed invariant subspace of $\mathscr{H}^p$ and $\mathscr{M} = \varphi\mathscr{M}'$. Thus to classify the closed invariant subspaces of $\mathscr{H}^p$ it suffices to classify the reduced ones.

The problem of giving an effective description of reduced invariant subspaces of $\mathscr{H}^p(\mathbf{G})$ which do not contain 1 seems to be difficult. I have only been able to characterize those subspaces $\mathscr{M}$ that satisfy a fairly strong additional hypothesis:

Hypothesis H. *The subspace* $\mathscr{M}$ *contains a function* h *with the property that on the inner boundaries* Γ_I *of* $\mathbf{G}$ *the boundary values of* h *satisfy* $m \le |h| \le M$ *almost everywhere for suitable constants* M, $m > 0$.

Some of the significance of the hypothesis H is given by the following lemma:

LEMMA 17. *Let* $\mathscr{M}$ *be a reduced invariant subspace of* $\mathscr{H}^p(\mathbf{G})$ *satisfying hypothesis* H. *Then for each* $k = \langle k_0, k_1, \dots, k_{n-1}\rangle$ *in* $\mathscr{H}_0^q(\tilde{\mathbf{G}})$ *which annihilates* $\mathscr{M}$ *the function* k_0 *extends to a function in* $\mathscr{H}_0^q(\mathbf{G}_0 \cup \mathbf{G} \cup \Gamma_0)$.

Proof. Let $k \in \mathscr{H}_0^q(\tilde{\mathbf{G}})$ annihilate $\mathscr{M}$, and define Tf as usual by

$$(Tf)(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)k(z)\,dz}{z-\zeta}.$$

Then $Tf \equiv 0$ in $\mathbf{G}_0$ for $f \in \mathscr{M}$. Thus for $f \in \mathscr{M}$ the function Tf in $\mathbf{G}$ is a function of class $\mathscr{H}^r$, $r < 1$, with boundary values fk_0 on Γ_0. Hence k_0 has an extension to $\mathbf{G}$ of class N whose denominator divides the inner part of each $f \in \mathscr{M}$. Since $\mathscr{M}$ is reduced, this denominator is one and k_0 extends to a function of class N^+ in $\mathbf{G}$. Since its boundary values on Γ_0 are in $\mathscr{L}^q$, we see that k_0 is regular and of class N^+ in $\mathbf{G} \cup \mathbf{G}_0 \cup \Gamma_0$.

Let h be the function in $\mathscr{M}$ satisfying hypothesis H, and let Γ_0' be a curve in $\mathbf{G}$ homologous to Γ_0. Then for ζ in the region between Γ_0' and Γ_I, we have

$$\begin{aligned} h(\zeta)k_0(\zeta) = (Th)(\zeta) &= \frac{1}{2\pi i}\int_{\Gamma_0} \frac{h(z)k_0(z)\,dz}{z-\zeta} + \frac{1}{2\pi i}\int_{\Gamma_I} \frac{h(z)k_I(z)\,dz}{z-\zeta} \\ &= \frac{1}{2\pi i}\int_{\Gamma_0'} \frac{h(z)k_0(z)\,dz}{z-\zeta} + \frac{1}{2\pi i}\int_{\Gamma_I} \frac{h(z)k_I(z)\,dz}{z-\zeta}. \end{aligned}$$

Since $hk_0 \in \mathscr{L}^q(\Gamma_0')$ and $hk_I \in \mathscr{L}^q(\Gamma_I)$, we see that hk_0 belongs to $\mathscr{H}^q$ in the region between Γ_0' and Γ_I. Thus the boundary values hk_0 belong to $\mathscr{L}^q$ on Γ_I. Since h is bounded from below on Γ_I, we have the boundary values of k_0 on Γ_I in $\mathscr{L}^q$, and so k_0 is in $\mathscr{H}^q$ of the exterior of Γ_I. □

LEMMA 18. *Let $\mathbf{G}_j$ be a simply connected region with smooth boundary Γ_j, and let u be a complex-valued measurable function on Γ_j with $\log|u|$ integrable. Then there is a $\chi \in \mathscr{L}^\infty(\Gamma_j)$, with $|\chi| \equiv 1$, such that for each $f \in \mathscr{L}^p(\Gamma_j)$ the function fu is the boundary value of a function of class N^+ in $\mathbf{G}_j$ if and only if $f\chi$ is the boundary value of a function of class $\mathscr{H}^p$ in $\mathbf{G}_j$.*

Proof. Let v be the outer function in $\mathbf{G}_j$ whose boundary values have modulus $|u|$ almost everywhere. Set $\chi = u/v$. Then $f\chi$ has an extension to $\mathbf{G}_j$ of class N^+ if and only if fu does. But an extension of $f\chi$ to $\mathbf{G}_j$ of class N^+ is of class $\mathscr{H}^p$ since $f\chi \in \mathscr{L}^p$. □

THEOREM 5. *Let $\mathscr{M}$ be a closed invariant subspace of $\mathscr{H}^p(\mathbf{G})$, $1 < p < \infty$, and suppose $\mathscr{M}$ satisfies hypothesis* H. *Then there is an inner function φ in $\mathbf{G}$ and a measurable function χ on Γ_I with $|\chi|$ constant almost everywhere on each component of Γ_I such that $\mathscr{M} = \phi\mathscr{M}_\chi$, where $\mathscr{M}_\chi = \{f \in \mathscr{H}^p\colon f\chi$ has an extension from Γ_I to $\mathscr{H}^p(\mathbf{G}_I)\}$. The function φ is the greatest common divisor of the inner parts of the functions in $\mathscr{M}$.*

Proof. Without loss of generality we may suppose that $\mathscr{M}$ is reduced. Let f_1 be a fixed function in $\mathscr{M}$, $f_1 \not\equiv 0$. It follows from Theorem 3 that there is an inner function Ψ in $\mathbf{G}_I$ such that for each $f \in \mathscr{M}$ the function $f\Psi/f_1$ has an extension to $\mathbf{G}_I$ of class N^+. Let ψ be the greatest common divisor of all such inner functions Ψ. Since the denominator of the extension of class N of f/f_1 divides each Ψ it divides their greatest common divisor ψ. Thus $f\psi/f_1$ has an extension of class N^+. Let χ be the function of constant modulus given by Lemma 18 for $u = \psi f_1^{-1}$. Then $f\chi$ extends to $\mathscr{H}^p(\mathbf{G}_I)$ for all $f \in \mathscr{M}$, and so $\mathscr{M} \subset \mathscr{M}_\chi$.

Since $\mathscr{M}$ is closed, we will have $\mathscr{M}_\chi \subset \mathscr{M}$ if every linear functional L on $\mathscr{H}^p(\mathbf{G})$ which vanishes on $\mathscr{M}$ also vanishes on $\mathscr{M}_\chi$. Such an L is represented by

$$L[f] = \frac{1}{2\pi i}\int_\Gamma f(z)k(z)\,dz$$

where $k = \langle k_0, k_I\rangle \in \mathscr{H}_0^q(\tilde{\mathbf{G}})$. By Lemma 17 the funxction k_0 is of class $\mathscr{H}_0^q$ on the exterior of Γ_I. Thus

$$L[f] = \frac{1}{2\pi i}\int_{\Gamma_I} f(z)[k_0(z) + k_I(z)]\,dz.$$

As usual we define Tf in $\tilde{\Gamma}$ by

$$T[f] = L[(z-\zeta)^{-1}f] = \frac{1}{2\pi i}\int_{\Gamma_I} \frac{f(k_0+k_I)\,dz}{z-\zeta}.$$

Then Tf is a function of class $\mathscr{H}^r$ in $\tilde{\Gamma}$ for each $r < 1$.

If f is in $\mathscr{M}$ so is $(z-\zeta)^{-1}f$ for $\zeta \in \mathbf{G}_0$. Thus for $f \in \mathscr{M}$ we have $Tf \equiv 0$ in $\mathbf{G}_0$ and hence also everywhere outside Γ_I. Consequently the function $(k_0 + k_I)f$ has an extension Tf to $\mathbf{G}_I$ of class $\mathscr{H}^r$ for each $r < 1$. Since $(k_0 + k_I)f$ is in $\mathscr{L}^1(\Gamma)$, its extension is in $\mathscr{H}^1(\mathbf{G}_I)$. Because $f\psi/f_1$ has an extension of class N^+, we see that the extension of $(k_I + k_0)f_1$ is divisible by ψ.

Let g be in $\mathscr{M}_\chi$. The fact that $g\chi$ has an extension to $\mathbf{G}_I$ of class $\mathscr{H}^p$ implies that $g\chi/f_1$ has an extension of class N^+. If $\psi_j \not\equiv 0$,

$$(k_j - k_0)g = \frac{(k_j - k_0)f_1}{\psi_j}\frac{g\psi_j}{f_1}$$

has an extension to $\mathbf{G}_i$ of class N^+. Since $g \in \mathscr{L}^p$ and the k's are in $\mathscr{L}^q$, this extension is of class $\mathscr{H}^1$. Thus

$$L(g) = \frac{1}{2\pi i}\int_{\Gamma_j} (k_j(z) - k_0(z))g(z)\,dz = 0$$

by the Cauchy theorem. This proves the theorem. □

I have only been able to characterize the invariant subspaces of $\mathscr{H}^p(\mathbf{G})$ under the restrictive hypothesis H. That not every invariant subspace of $\mathscr{H}^p(\mathbf{G})$ is of the form $\varphi\mathscr{M}_\chi$ with $|\chi| \equiv$ constant can be seen by the following example, due to Hitt:

Let $\mathbf{G}$ be the annulus $\{z\colon 1 < |z| < R\}$ and h the function $h(z) = z - 1$. Then the space $\mathscr{M} = h(z)\mathscr{H}^p(\Delta_R)$, where $\Delta_R = \{z\colon |z| < R\}$, is a reduced closed invariant subspace of $\mathscr{H}^p(\mathbf{G})$. Suppose some space $\mathscr{M}_\chi$, with $|\chi| = 1$ almost everywhere on $|z| = 1$, contains $h(z)$. Then χh extends to a function g in $\mathscr{H}^p(\Delta)$. Let g have the factorization

$$g = \psi g_0$$

where ψ is inner and g_0 outer on Δ. Then

$$\psi = \frac{\chi h}{g_0},$$

and since h/g_0 is outer $\psi = \chi$. Consequently

$$1 \in \mathscr{M}_\psi = \mathscr{M}_\chi \neq \mathscr{M},$$

since every function in $\mathscr{M}$ has a zero at 1.

This example shows that zeros (and presumably infinities) on Γ_I of the outer parts of functions in $\mathscr{M}$ play a role in determining $\mathscr{M}$. It seems to me likely, however, that each closed invariant subspace $\mathscr{M}$ of $\mathscr{H}^p(\mathbf{G})$ is of the form $\mathscr{M} = h\mathscr{M}_\chi$ where h is a function in $\mathscr{H}^p(\mathbf{G})$ whose outer part is continuous and non-zero on the outer boundary Γ_0 of $\mathbf{G}$. We may even be able to take χ to be the boundary values of an inner function on $\mathbf{G}_I$. Hitt **[6]** shows this is true for $\mathbf{G} = \{z\colon 1 < |z| < R\}$ and $p = 2$.

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