PACIFIC JOURNAL OF MATHEMATICS
Vol. 134, No. 1, 1988

INVARIANT SUBSPACES OF $\mathscr{H}^2$ OF AN ANNULUS

D. HITT

Fully invariant subspaces of the Hardy class $\mathscr{H}^2$(G) on a multiply connected domain G $\subset$ C, are those $\mathscr{M}$ such that

$$f(x) \in \mathscr{M} \Rightarrow Q(z)f(z) \in \mathscr{M},$$

for all rational functions Q whose poles are in the complement of $\overline{\mathbf{G}}$. Simply invariant subspaces are those $\mathscr{M}$ such that

$$f(z) \in \mathscr{M} \Rightarrow zf(z) \in \mathscr{M}.$$

Although the structure of the fully invariant subspaces is well known as a result of the work of Sarason, Hasumi, and Voichick, little work has been done on subspaces simply invariant but not fully invariant. In this paper we consider the special case G = A, where A denotes the annulus $\{z \in \mathbf{C}\colon 1 < |z| < R\}$. We classify the simply invariant (closed) subspaces $\mathscr{M}$ of $\mathscr{H}^2$(A).

0. Introduction and statement of results. The fully invariant subspaces of the Hardy class $\mathscr{H}^2(\mathbf{G})$ on a multiply connected domain $\mathbf{G} \subset \mathbf{C}$, as well as some of the simply invariant ones, have been classified (cf., [12], [23], [25], and [27]). Fully invariant subspaces are those $\mathscr{M}$ such that

$$f(z) \in \mathscr{M} \Rightarrow Q(z)f(z) \in \mathscr{M},$$

for all rational functions Q whose poles are in the complement of $\overline{\mathbf{G}}$. Simply invariant subspaces are those $\mathscr{M}$ such that

$$f(z) \in \mathscr{M} \Rightarrow zf(z) \in \mathscr{M}.$$

In this paper we consider the special case $\mathbf{G} = \mathbf{A}$, where $\mathbf{A}$ denotes the annulus $\{z \in \mathbf{C}\colon 1 < |z| < R\}$. We extend the results of Royden [23] by classifying the simply invariant subspaces $\mathscr{M}$ of $\mathscr{H}^2(\mathbf{A})$. Here and throughout this paper "subspace" means "closed subspace". If we also have $z^{-1}f(z) \in \mathscr{M}$ for all $f \in \mathscr{M}$, we say that $\mathscr{M}$ is doubly invariant or fully invariant. Note that this use of "fully invariant" is consistent with the use above.

Sarason [25], Hasumi [12], and Voichick [27, 28] were the original investiagators of fully invariant subspaces of $\mathscr{H}^2(\mathbf{A})$. They characterized them, as well as the subspaces of $\mathscr{L}^2(\partial\mathbf{A})$ which are invariant under multiplication by both z and $1/z$. They showed that the fully invariant subspaces of $\mathscr{H}^2(\mathbf{A})$ have the form $\Phi\mathscr{H}^2(\mathbf{A})$ (cf., Theorem 1 of [**23**]). Here Φ is an *inner function* on $\mathbf{A}$. We say that a bounded analytic function Φ on $\mathbf{A}$ is inner provided its boundary values on the two components of the boundary have constant absolute value almost everywhere, although not necessarily the same constant on both circles.

In the course of our proofs we need to classify a certain kind of space which we call *weakly invariant under the backwards shift*. This is done in Propositions 2 and 3. In his classification of kernels of Toeplitz operators on $\mathscr{H}^2$ of the unit desk, Hayashi [**16**] has independently developed ideas similar to those used here to prove these propositions. This suggests that these methods may be useful for further kinds of problems.

We recall that if $f \in \mathscr{H}^2(\mathbf{A})$, then $f = \Phi F$, with Φ inner, F outer, and the factors are unique up to units. Units are powers of z. A function $F \in \mathscr{H}^2(\mathbf{A})$ is said to be outer provided

$$\log|F(z)| = \frac{1}{2\pi}\int_{\partial\mathbf{A}} \log|F(\zeta)|\frac{\partial g(\zeta, z)}{\partial n_\zeta}\, ds_\zeta,$$

for every $z \in \mathbf{A}$. A consequence is that $F \in \mathscr{H}^2(\mathbf{A})$ is outer provided the integral definition above holds for a single $z \in \mathbf{A}$, and in this case, it holds for every $z \in \mathbf{A}$.

It is fundamental to the theory that any set on inner functions has a greatest common divisor. That is, if S is a set of inner functions on the annulus $\mathbf{A}$, then there is an inner function Φ defined on $\mathbf{A}$, which is a divisor of S in the sense that Ψ/Φ bounded for all $\Psi \in S$. Further, if Φ' is any other divisor of S, then Φ'/Φ is bounded.

We define the greatest common divisor of a set of functions to be the greatest common divisor of their inner parts. Hence there is no loss of generality in searching only for the simply invariant subspaces of $\mathscr{H}^2(\mathbf{A})$ which have greatest common divisor 1. A simply invariant subspace of $\mathscr{H}^2(\mathbf{A})$ whose greatest common divisor Φ is not 1 can be reduced to the other case by dividing by Φ.

Before stating our results we should mention the concept of analytic pseudocontinuation [**6**], [**23**]. Let $\Delta = \{z \in \mathbf{C}: |z| < 1\}$ and $\mathbf{E} = \{z \in \hat{\mathbf{C}}: |z| > 1\}$. If $f \in \mathscr{H}^2(\Delta)$ and $g \in \mathscr{H}^2(\mathbf{E})$, then f and g have boundary values almost everywhere on $\{z \in \mathbf{C}: |z| = 1\}$. If their boundary values are equal almost everywhere, then they are said to be pseudocontinuations of each other. (By the way, in this case, they are both constants.) If Φ is an inner function on $\mathbf{E}$, and $f \in \mathscr{H}^2(\Delta)$, then the statement $\Phi f \in \mathscr{H}^2(\mathbf{E})$ (respectively, $\Phi f \in \mathscr{H}^2(\mathbf{A})$) is to mean that Φf is a pseudocontinuation of a function $g \in \mathscr{H}^2(\mathbf{E})$ (respectively, a pseudocontinuation of a function $g \in \mathscr{H}^2(\mathbf{A})$).

We are now in a position to state

THEOREM A. *Let $\mathbf{A} = \{z \in \mathbf{C}: 1 < |z| < R\}$ and let $\mathscr{M}$ be a proper closed subspace of $\mathscr{H}^2(\mathbf{A})$ which is invariant under multiplication by z and which has greatest common divisor* 1. *Then*

$$\mathscr{M} = z^{m_0} f_\infty \mathscr{M}_{\psi_0},$$

where

$$\mathscr{M}_{\psi_0} = \{f \in \mathscr{H}^2(\mathbf{A}) \colon \psi_0(e^{i\vartheta}) f(e^{i\vartheta}) \in \mathscr{H}^2(\Delta)\},$$

ψ_0 is an inner function on Δ, m_0 is an integer, and f_∞ is an outer function in $\mathscr{H}^2(\mathbf{E})$.

(As above, "$\psi_0(e^{i\vartheta}) f(e^{i\vartheta}) \in \mathscr{H}^2(\Delta)$" means "$\psi_0(e^{i\vartheta}) f(e^{i\vartheta})$ form the boundary values a.e. of a function in $\mathscr{H}^2(\Delta)$".)

REMARKS. A couple of observations are in order here: (1) This is an existence theorem. Given an arbitrary inner function ψ on Δ, if ψ is not a finite Blaschke product, then $\mathscr{M}_\psi$ will contain functions unbounded on $\partial\Delta$. Hence for $f \in \mathscr{H}^2(\mathbf{E})$, $f\mathscr{M}_\psi$ will not in general even be a subset of $\mathscr{H}^2(\mathbf{A})$. Theorem A asserts that there exists an f_∞ and a ψ_0 such that $f_\infty \mathscr{M}_{\psi_0}$ is a subspace of $\mathscr{H}^2(\mathbf{A})$ having the desired properties. (2) If the greatest common divisor of an $\mathscr{M}$ is Ψ_0, and $\mathscr{M}$ otherwise meets the conditions of Theorem A, then $\mathscr{M} = \Psi_0 z^{m_0} f_\infty \mathscr{M}_{\psi_0}$. (3) We can also make a uniqueness statement (Theorem B below).

DEFINITION. Let $\mathscr{M}$ be as in Theorem A. A normalized decomposition of $\mathscr{M}$ with respect to a point $p \in \Delta$ is a factoring

$$\mathscr{M} = z^{m_0} f_\infty \mathscr{M}_{\psi_0},$$

where f_∞ is outer in $\mathscr{H}^2(\mathbf{E})$ with normalization

$$\frac{1}{2\pi} \int_0^{2\pi} |f_\infty(e^{i\vartheta})|^2 \, d\vartheta = 1,$$

and ψ_0 is an inner function on Δ such that $\psi_0(p) > 0$.

THEOREM B. *Let $\mathscr{M}$ be as in Theorem A. We can choose m_0, f_∞, and ψ_0 so that*

$$\mathscr{M} = z^{m_0} f_\infty \mathscr{M}_{\psi_0}$$

is a normalized decomposition of $\mathscr{M}$ with respect to a point $p \in \Delta$ and if

$$\mathscr{M} = z^{m_0} g_\infty \mathscr{M}_{\varphi_0}$$

is another normalized decomposition of $\mathscr{M}$ with respect to p, then

$$\Re f_\infty(\infty) \geq \Re g_\infty(\infty),$$

with equality holding iff $f_\infty \equiv g_\infty$ and $\psi_0 \equiv \varphi_0$.

The normalization that for some p, $\psi_0(p) > 0$, $\varphi_0(p) > 0$, is intended to rule out the possibility that $\psi_0 = e^{i\alpha}\varphi_0$ for some $e^{i\alpha}$ not equal to 1.

1. Outline of Proof. We will first define and characterize the subspaces which are weakly invariant under the backwards shift operator on $\mathscr{H}^2(\Delta)$ (see Propositions 2 and 3). Then we will show that the problem of classifying the simply invariant subspaces $\mathscr{M}$ of $\mathscr{H}^2(\mathbf{A})$ reduces to classifying the spaces weakly invariant under the backwards shift operator. For future reference, we will adopt the standard notation $\Delta_R = \{z \in \mathbf{C}\colon |z| < R\}$.

A backwards shift invariant subspace $\mathscr{F}$ of $\mathscr{H}^2(\Delta)$ is a subspace such that

$$f \in \mathscr{F} \Rightarrow \frac{f(s) - f(0)}{s} \in \mathscr{F}.$$

Note that a backwards shift invariant space is the orthogonal complement of an invariant (under multiplication by z) space.

A subspace $\mathscr{F}$ of $\mathscr{H}^2(\Delta)$ which is weakly invariant under the backwards shift operator is a subspace such that

$$f \in \mathscr{F}, \quad f(0) = 0 \Rightarrow \frac{f(s)}{s} \in \mathscr{F}. \tag{1}$$

If $\mathscr{M}$ is a simply invariant subspace of $\mathscr{H}^2(\mathbf{A})$, then we can form

$$\mathscr{E} = \mathscr{M} \cap \mathscr{H}^2(\mathbf{E}).$$

It may very well be that $\mathscr{E} = \{0\}$. This will be the case if $\mathscr{M} = z\mathscr{H}^2(\Delta_R)$. In any event, $\mathscr{E}$ will have the property that

$$g \in \mathscr{E}, \quad g(\infty) = 0 \Rightarrow zg(z) \in \mathscr{E}. \tag{2}$$

By switching dummy variables to

$$s = \frac{1}{z}, \tag{3}$$

we see that any $\mathscr{E} = \mathscr{M} \cap \mathscr{H}^2(\mathbf{E})$ corresponds to a subspace $\mathscr{F} \subseteq \mathscr{H}^2(\Delta)$ which is weakly invariant under the backwards shift operator. Knowing the possibilities for $\mathscr{F}$ and appropriately modifying $\mathscr{E}$ will allow us to push through to a solution for what $\mathscr{M}$ can be.

2. Spaces weakly invariant under the backwards shift operator, Part I. Let $\mathscr{F}$ be a subspace of $\mathscr{H}^2(\Delta)$ such that

$$f \in \mathscr{F}, \quad f(0) = 0 \Rightarrow \frac{f(s)}{s} \in \mathscr{F}. \tag{1}$$

We will call spaces satisfying (1) *weakly invariant under the backwards shift operator*. This name is chosen because their distinguishing property is weaker than the property of being invariant under the backwards shift B. By B we refer to the adjoint of the shift. For $f \in \mathscr{H}^2(\Delta)$,

$$(B(f))(s) = \frac{f(s) - f(0)}{s}.$$

Let us now suppose that in addition to being weakly invariant under the backwards shift operator, the greatest common divisor of $\mathscr{F}$ is 1. This involves no important loss of generality. The Szegő kernel function for $\mathscr{F}$ is defined by

$$k_{\mathscr{F}}(s, \bar{\sigma}) = \sum_{j=1}^{\infty} \varphi_j(s)\overline{\varphi_j(\sigma)}, \tag{4}$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is any orthonormal basis for $\mathscr{F}$. It is well-defined and has the reproducing property:

$$\langle f(s), k_{\mathscr{F}}(s, \bar{\sigma})\rangle = f(\sigma), \tag{5}$$

for all $f \in \mathscr{F}$. Thus

$$\|k_{\mathscr{F}}(s, \bar{\sigma})\|_2^2 = k_{\mathscr{F}}(\sigma, \bar{\sigma}). \tag{6}$$

Let

$$f_0(s) = \frac{k_{\mathscr{F}}(s, 0)}{\sqrt{k_{\mathscr{F}}(0, 0)}}, \tag{7}$$

so that f_0 has unit norm. An arbitrary $f \in \mathscr{F}$ may be written in the form

$$f(s) = \alpha_0 f_0(s) + f_1(s) \tag{8}$$

where $f_1 \perp f_0$. Hence

$$\|f\|_2^2 = |\alpha_1|^2 + \|f_1\|_2^2. \tag{9}$$

Since f_1 is perpendicular to f_0, it is perpendicular to $k_{\mathscr{F}}(s, 0)$. Thus their inner product is zero and the reproducing property (5) implies

$$f_1(0) = 0. \tag{10}$$

We note that $f_1(s)/s$ is in $\mathscr{F}$, since this space is weakly invariant under the backwards shift operator. Thus we may repreat this process for $f_1(s)/s$, getting

$$\frac{f_1(s)}{s} = \alpha_1 f_0(s) + f_2(s)$$

with

$$\|f_1\|^2 = |\alpha_1|^2 + \|f_2\|^2.$$

Continuing recursively, we get

$$\frac{f_j(s)}{s} = \alpha_j f_0(s) + f_{j+1}(s) \tag{11}$$

with

$$\|f_j\|^2 = |\alpha_j|^2 + \|f_{j+1}\|^2. \tag{12}$$

Linking these equations together gives

$$f(s) = (\alpha_0 + \alpha_1 s + \cdots + \alpha_j s^j) f_0(s) + s^j f_{j+1}(s) \tag{13}$$

and

$$\|f\|^2 = |\alpha_0|^2 + |\alpha_1|^2 + \cdots + \|f_{j+1}\|^2. \tag{14}$$

Since Taylor series are unique,

$$\frac{f(s)}{f_0(s)} = \sum_{j=0}^{\infty} \alpha_j s^j. \tag{15}$$

Also

$$\left\| \frac{f}{f_0} \right\|^2 = \sum_{j=0}^{\infty} |\alpha_j|^2 \leq \|f\|^2, \tag{16}$$

with equality holding iff $\|f_j\|^2 \to 0$.

This shows that f_0 is outer, because

$$\frac{f}{f_0} \in \mathscr{H}^2(\Delta) \quad \forall f \in \mathscr{F},$$

and by assumption the greatest common divisor of $\mathscr{F}$ is 1.

Let

$$J : \mathscr{F} \to \mathscr{H}^2(\Delta)$$

be the map defined by

$$J(f) = \frac{f}{f_0}.$$

Then $\|Jf\| \le \|f\|$. Further

$$\mathscr{J} = \{f \in \mathscr{F} : \|Jf\| = \|f\|\}$$

is a subspace of $\mathscr{F}$. Indeed, if

$$f(s) = (\alpha_0 + \alpha_1 s + \cdots + \alpha_j s^j) f_0(s) + s^j f_{j+1}(s)$$

and

$$g(s) = (\beta_0 + \beta_1 s + \cdots + \beta_j s^j) f_0(s) + s^j g_{j+1}(s)$$

with $\|f_j\| \to 0$, $\|g_j\| \to 0$, then similar statements can be made about $f+g$. Consequently, $\mathscr{J}$ is a linear manifold. Because J is an isometry when restricted to $\mathscr{J}$, $\mathscr{J}$ is complete and hence closed.

PROPOSITION 1. *Let $\mathscr{F}$ be a closed, non-trivial, subspace of $\mathscr{H}^2(\Delta)$ weakly invariant under the backwards shift operator, with greatest common divisor* 1. *Then f_0, the function of unit norm in $\mathscr{F}$ maximizing $\Re f(0)$, is outer. If J is defined by*

$$J(f) = \frac{f}{f_0},$$

then J is a bounded linear operator

$$J : \mathscr{F} \to \mathscr{H}^2(\Delta)$$

with $\|J\|_{op} = 1$, that is, $\|Jf\| \le \|f\|$. The set

$$\mathscr{J} = \{f \in \mathscr{F} : \|Jf\| = \|f\|\},$$

is a closed linear subspace of $\mathscr{F}$.

3. Spaces weakly invariant under the backwards shift operator, Part II. Proposition 1 leads to the question of whether $\mathscr{J}$ is ever all of $\mathscr{F}$, and we shall prove that in fact it always is. In any event, it is immediate that $J(\mathscr{J})$ is a backwards shift invariant closed subspace of $\mathscr{H}^2(\Delta)$. That is, $J(\mathscr{J})$ is invariant under the backwards shift map

$$(B(f))(s) = \frac{f(s) - f(0)}{s}.$$

We also have $1 \in J(\mathscr{J})$. It follows from Theorem 2 of **[23]** that

$$J(\mathscr{J}) = \mathscr{N}_\psi, \tag{17a}$$

where ψ is either an inner function on Δ or $\psi \equiv 0$, and

$$\mathscr{N}_\psi = \{f \in \mathscr{H}^2(\Delta) : f(e^{i\vartheta})\overline{\psi(e^{i\vartheta})} \in \mathscr{H}^2(\mathbf{E})\}. \tag{17b}$$

Here $\psi \equiv 0$ represents the case that $J(\mathscr{J}) = \mathscr{H}^2(\Delta)$.

Our tool in this section is the Szegő kernel function, so we compute it for $\mathscr{N}_\psi$.

Since $\mathscr{N}_\psi$ is backwards shift invariant, it is the orthogonal complement of an invariant space. By Beurling's theorem, an invariant subspace of $\mathscr{H}^2(\Delta)$ must have the form $\xi\mathscr{H}^2(\Delta)$, where ξ is an inner function on Δ. Since $1 \in \mathscr{N}_\psi$, we have

$$1 \perp \xi.$$

Thus $\xi(0) = 0$. That is, we must have $\xi(s) = s\varphi(s)$, where φ is also an inner function on Δ. For any polynomial $P(s)$,

$$\int_{|s|=1} s\varphi(s)\overline{\psi(s)}P(s)\,|ds| = 0$$

because ψ is orthogonal to $s\varphi(s)\mathscr{H}^2(\Delta)$. Hence $\varphi\bar{\psi}$ is the boundary value of an analytic function, i.e., φ is a multiple of ψ. On the other hand, $\varphi \in \mathscr{N}_\psi$, so ψ is a multiple of φ. Hence,

$$\mathscr{N}_\psi = \left(s\psi(s)\mathscr{H}^2(\Delta)\right)^\perp. \tag{18}$$

If $\{\varphi_j\}_{j=1}^\infty$ is any orthonormal basis for $\mathscr{H}^2(\Delta)$, then $\{s\psi(s)\varphi_j(s)\}_{j=1}^\infty$ is an orthonormal basis for $s\psi(s)\mathscr{H}^2(\Delta)$. Thus

$$\begin{aligned} k_{s\psi(s)\mathscr{H}^2(\Delta)}(s,\bar{\sigma}) &= \sum_{j=1}^\infty s\psi(s)\varphi_j(s)\overline{\sigma\psi(\sigma)\varphi_j(\sigma)} \\ &= s\bar{\sigma}\,\psi(s)\overline{\psi(\sigma)}\sum_{j=1}^\infty \varphi_j(s)\overline{\varphi_j(\sigma)} \\ &= s\bar{\sigma}\,\psi(s)\overline{\psi(\sigma)}k_{\mathscr{H}^2(\Delta)}(s,\bar{\sigma}). \end{aligned} \tag{19}$$

Taking $\{s^n\}_{n=0}^\infty$ to be our orthonormal basis for $\mathscr{H}^2(\Delta)$ gives

$$k_{s\psi(s)\mathscr{H}^2(\Delta)}(s,\bar{\sigma}) = \frac{s\psi(s)\overline{\sigma\psi(\sigma)}}{1-s\bar{\sigma}}. \tag{20}$$

The union of orthonormal bases for $\mathscr{N}_\psi$ and $\mathscr{N}_\psi^\perp = s\psi(s)\mathscr{H}^2(\Delta)$ is an orthonormal basis for $\mathscr{H}^2(\Delta)$. Hence

$$k_{\mathscr{N}_\psi}(s,\bar{\sigma}) + k_{s\psi(s)\mathscr{H}^2(\Delta)}(s,\bar{\sigma}) = k_{\mathscr{H}^2(\Delta)}(s,\bar{\sigma}), \tag{21}$$

and so

$$k_{\mathscr{N}_\psi}(s,\bar{\sigma}) = \frac{1-s\psi(s)\overline{\sigma\psi(\sigma)}}{1-s\bar{\sigma}}. \tag{22}$$

Recall that f_0 is the normalized maximal function for $\mathscr{F}$. Proposition 1 says that dividing elements of $\mathscr{J}$ by f_0 is an isometry. We can use this fact to perform a calculation of $k_{\mathscr{J}}(0,0)/k_{\mathscr{F}}(0,\bar{\sigma})$. Our method is similar to the one used above, with f_0 playing a role analogous to that of an inner function. Take any orthonormal basis for $\mathscr{J}$. Divide each element of this basis by f_0 to get an orthonormal basis for $\mathscr{N}_\psi$. Comparing the Szegő kernels computed with these bases gives

$$\frac{k_{\mathscr{J}}(0,0)}{k_{\mathscr{J}}(0,\bar{\sigma})}\frac{k_{\mathscr{J}}(s,\bar{\sigma})}{k_{\mathscr{J}}(s,0)} = k_{\mathscr{N}_\psi}(s,\bar{\sigma}).$$

Then it follows from (22) that

$$\frac{k_{\mathscr{J}}(0,0)}{k_{\mathscr{J}}(0,\bar{\sigma})}\frac{k_{\mathscr{J}}(s,\bar{\sigma})}{k_{\mathscr{J}}(s,0)} = \frac{1-s\psi(s)\overline{\sigma\psi(\sigma)}}{1-s\bar{\sigma}}. \tag{23}$$

Rearranging this equation gives

$$\begin{aligned}(24)\qquad & k_{\mathscr{J}}(0,\bar{\sigma})k_{\mathscr{J}}(s,0)\psi(s)\overline{\psi(\sigma)}\\ &= \frac{k_{\mathscr{J}}(0,\bar{\sigma})k_{\mathscr{J}}(s,0)-k_{\mathscr{J}}(0,0)k_{\mathscr{J}}(s,\bar{\sigma})(1-s\bar{\sigma})}{s\bar{\sigma}}.\end{aligned}$$

Knowing the kernel function tells us everything, and so we examine the right-hand side of (24) with $\mathscr{J}\subseteq\mathscr{F}$ replaced by the space $\mathscr{F}$.

Let

$$f(s,\bar{\sigma}) = \frac{k_{\mathscr{F}}(0,\bar{\sigma})k_{\mathscr{F}}(s,0)-k_{\mathscr{F}}(0,0)k_{\mathscr{F}}(s,\bar{\sigma})}{s\bar{\sigma}} + k_{\mathscr{F}}(0,0)k_{\mathscr{F}}(s,\bar{\sigma}). \tag{25}$$

Since $\mathscr{F}$ is a space weakly invariant under the backwards shift operator, $f(s,\bar{\sigma})\in\mathscr{F}$. If we had formed $f(s,\bar{\sigma})$ for $\mathscr{J}$ instead of $\mathscr{F}$, then changing σ would just change it by a multiplicative complex constant. We shall see that the same is true for $f(s,\bar{\sigma})$. That is, we shall see that changing σ will just scale $f(s,\bar{\sigma})$ by a complex constant. This fact will essentially finish our work with spaces weakly invariant under the backwards shift operator. We begin by computing the inner product of $f(s,\bar{\sigma})$ with $f(s,\bar{\tau})$, for two arbitrary non-zero points σ and τ in Δ.

The inner product $\langle f(s,\bar{\sigma}), f(s,\bar{\tau})\rangle$ consists of four terms, since each factor consists of two terms. Remembering that division by s is an isometry on $\partial\Delta$, we see the first term of the inner product is

$$\begin{aligned}\frac{1}{\tau\bar{\sigma}}\Big(&k_{\mathscr{F}}(0,\bar{\sigma}),k_{\mathscr{F}}(\tau,0)k_{\mathscr{F}}(0,0)-k_{\mathscr{F}}(0,0)k_{\mathscr{F}}(\tau,0)k_{\mathscr{F}}(0,\bar{\sigma})\\ &-k_{\mathscr{F}}(0,0)k_{\mathscr{F}}(0,\bar{\sigma})k_{\mathscr{F}}(\tau,0)+(k_{\mathscr{F}}(0,0))^2k_{\mathscr{F}}(\tau,\bar{\sigma})\Big).\end{aligned}$$

The next two terms are each equal to

$$k_{\mathscr{F}}(0,0)\frac{k_{\mathscr{F}}(0,\bar{\sigma})k_{\mathscr{F}}(\tau,0)-k_{\mathscr{F}}(0,0)k_{\mathscr{F}}(\tau,\bar{\sigma})}{\tau\bar{\sigma}},$$

and the final term is

$$(k_{\mathscr{F}}(0,0))^2 k_{\mathscr{F}}(\tau,\bar{\sigma}).$$

Of the nine summands in the inner product, six cancel, and the three remaining combine to give

$$\langle f(s,\bar{\sigma}), f(s,\bar{\tau})\rangle = k_{\mathscr{F}}(0,0) f(\tau,\bar{\sigma}). \tag{26}$$

Equation (26) represents the reproducing property, so $f(s,\bar{\tau})/k_{\mathscr{F}}(0,0)$ is the Szegő kernel function for the closure $\mathscr{L}$ of the linear space spanned by $\{f(s,\bar{\sigma})\}$ as σ ranges over $\mathbf{\Delta}$.

Thus, for $g \in \mathscr{L}$,

$$\begin{aligned}
\frac{|g(\sigma)|^2}{\|g\|^2} &\le \frac{f(\sigma,\bar{\sigma})}{k_{\mathscr{F}}(0,0)} \\
&= \frac{|k_{\mathscr{F}}(0,\bar{\sigma})|^2}{|\sigma|^2 k_{\mathscr{F}}(0,0)} + k(\sigma,\bar{\sigma})\left(1-\frac{1}{|\sigma|^2}\right) \\
&\le \frac{|k_{\mathscr{F}}(0,\bar{\sigma})|^2}{|\sigma|^2 k_{\mathscr{F}}(0,0)}.
\end{aligned} \tag{27}$$

Hence,

$$\left|\frac{g(\sigma)}{k_{\mathscr{F}}(\sigma,0)}\right| \le \frac{\|g\|}{|\sigma|\sqrt{k_{\mathscr{F}}(0,0)}}, \tag{28}$$

for all $\sigma \in \mathbf{\Delta}$. The maximum principle then gives

$$\left|\frac{g(\sigma)}{k_{\mathscr{F}}(\sigma,0)}\right| \le \frac{\|g\|}{\sqrt{k_{\mathscr{F}}(0,0)}}, \tag{29}$$

and so

$$|g(\sigma)| \le \frac{|k_{\mathscr{F}}(\sigma,0)|}{\sqrt{k_{\mathscr{F}}(0,0)}}\|g\|. \tag{30}$$

Equation (30) holds for all σ in the unit disk. Thus we let $|\sigma| \to 1$, and see that (30) holds for all σ on the boundary $\partial\mathbf{\Delta} = \{\sigma \in \mathbf{C}\colon |\sigma| = 1\}$. If strict inequality held on a set of positive measure on the boundary, we could integrate and obtain $\|g\|^2 < \|g\|^2$. Hence

$$|g(\sigma)| = \frac{|k_{\mathscr{F}}(\sigma,0)|}{\sqrt{k_{\mathscr{F}}(0,0)}}\|g\| \quad \text{a.e. on } \partial\mathbf{\Delta}.$$

We may conclude that

$$g(\sigma) = \varphi(\sigma) k_{\mathscr{F}}(\sigma,0)\cdot c, \tag{31}$$

where φ is an inner function and c is a constant.

This representation holds for all $g \in \mathscr{L}$. If φ_1 and φ_2 can occur in (31), then because $\mathscr{L}$ is closed under addition, some non-trivial linear combination of φ_1 and φ_2 is also inner. Thus φ_1 is a multiple of φ_2, and hence $\mathscr{L}$ is one dimensional. Further,

$$\mathscr{L} \subseteq \mathscr{J},$$

because division by f_0 doesn't affect the norm of anything in $\mathscr{L}$. Hence φ divides ψ.

Putting (25) into (31) gives

$$\frac{k_{\mathscr{F}}(0,0)}{k_{\mathscr{F}}(0,\bar{\sigma})} \frac{k_{\mathscr{F}}(s,\bar{\sigma})}{k_{\mathscr{F}}(s,0)} = \frac{1 - s\varphi(s)\overline{\sigma c(\sigma)}}{1 - s\bar{\sigma}}. \tag{32}$$

Because of the symmetry properties of the Szegő kernel, we conclude

$$c(\sigma) = c_1 \varphi(\sigma),$$

where c_1 is a constant. To establish $c_1 = 1$, we have to be just a little bit fussy. We already have

$$\frac{1 - s\varphi(s)\overline{\sigma\varphi(\sigma)}}{1 - s\bar{\sigma}} \in J(\mathscr{J}),$$

because φ divides ψ. Thus, if $c_1 \neq 1$, we take a linear combination of

$$\frac{1 - s\varphi(s)\overline{\sigma\varphi(\sigma)}}{1 - s\bar{\sigma}}$$

and

$$\frac{1 - s\varphi(s)\overline{c_1\sigma\varphi(\sigma)}}{1 - s\bar{\sigma}}$$

to get $1/(1 - s\bar{\sigma})$ in $J(\mathscr{M})$. The uniform closure of $\{1/(1 - s\bar{\sigma})\}_{|\sigma|<1}$ includes the polynomials. The kernel function $k_{\mathscr{F}}(s, 0)$ is outer. Beurling's theorem then would give $\mathscr{F} = \mathscr{H}^2(\Delta)$, and we have the trivial case.

So we write $c_1 = 1$, and this means that the maximal function for $\mathscr{F}$ is in the linear span of the maximal function for $\mathscr{J}$, since φ divides ψ. In either case

$$\mathscr{F} = \mathscr{J}.$$

We summarize this as a proposition:

Proposition 2. *Let $\mathscr{F}$ be a closed subspace of $\mathscr{H}^2(\Delta)$ which is weakly invariant under the backwards shift operator, and has greatest common divisor* 1. *Then*

$$\mathscr{F} = f_0 \mathscr{N}_\psi,$$

where

$$\mathcal{N}_\psi = \{f \in \mathcal{H}^2(\Delta) : f(e^{i\vartheta})\overline{\psi(e^{i\vartheta})} \in \mathcal{H}^2(\mathbf{E})\},$$

$\psi(s)$ is inner on Δ, and f_0 is an outer function in $\mathcal{H}^2(\Delta)$. We may take f_0 to be the function of norm 1 in $\mathcal{F}$ maximizing $\Re f(0)$; in this case, the map

$$F_0 : \mathcal{N}_\psi \to f_0 \mathcal{N}_\psi$$

given by $F_0(g) = f_0 g$ is an isometry.

If we relax the requirement that the greatest common divisor of $\mathcal{F}$ be 1, we then have the following proposition.

PROPOSITION 3. *If $\mathcal{F}$ is a closed subspace of $\mathcal{H}^2(\Delta)$ which is weakly invariant under the backwards shift operator, then*

$$\mathcal{F} = \varphi f_0 \mathcal{N}_\psi,$$

where $\psi(s)$ is inner on Δ, and φ is the greatest common divisor of $\mathcal{F}$. We may take φf_0 to be the function of norm 1 in $\mathcal{F}$ maximizing $\Re f(0)$; in this case, the map

$$F_0 : \mathcal{N}_\psi \to f_0 \mathcal{N} \psi$$

given by $F_0(g) = f_0 g$ is an isometry.

4. Simply invariant subspaces of $\mathcal{H}^2(\mathbf{A})$. We are now in a position to classify the simply invariant subspaces $\mathcal{M}$ of $\mathcal{H}^2(\mathbf{A})$, having classified the subspaces of $\mathcal{H}^2(\Delta)$ which are weakly invariant under the backwards shift operator.

We do this in two steps. First we consider $\mathcal{E} = \mathcal{M} \cap \mathcal{H}^2(\mathbf{E})$. As we've mentioned, possibly $\mathcal{E} = \{0\}$ (e.g., if $\mathcal{M} = z\mathcal{H}^2(\Delta_R)$), but $\mathcal{E}$ has the property that

$$g \in \mathcal{E}, \quad g(\infty) = 0 \Rightarrow zg(z) \in \mathcal{E}. \tag{2}$$

We see from preceding work (by letting $s = 1/z$), that $\mathcal{E} = f_\infty \mathcal{E}_\psi$, where f_∞ is the function of norm 1 in $\mathcal{E}$ maximizing $\Re f(\infty)$, $\psi(s)$ is inner on Δ, and

$$\mathcal{E}_\psi = \{f \in \mathcal{H}^2(\mathbf{E}) : f(e^{i\vartheta})\psi(e^{i\vartheta}) \in \mathcal{H}^2(\Delta)\}.$$

Let $\mathcal{E} = f_\infty \mathcal{E}_\psi$, and let $\hat{\mathcal{E}}$ be the smallest closed simply invariant subspace of $\mathcal{H}^2(\mathbf{A})$ containing $\mathcal{E}$.

We will show that $\hat{\mathcal{E}}$ is $f_\infty \mathcal{M}_\psi$, where

$$\mathcal{M}_\psi = \{f \in \mathcal{H}^2(\mathbf{A}) : \psi(e^{i\vartheta})f(e^{i\vartheta}) \in \mathcal{H}^2(\Delta)\}.$$

(Also, of course f_∞ is the maximal function of $\mathscr{E}$, normalized to have norm 1.) Thus, if we knew that $\mathscr{M} = \hat{\mathscr{E}}$ for some $\mathscr{E}$, we would be done. This however is not true in general, but our second step will be to show that $z^{-m}\mathscr{M}$ is $\hat{\mathscr{E}}$ for some integer m and some $\mathscr{E}$ satisfying

$$(2) \qquad g \in \mathscr{E}, \quad g(\infty) = 0 \Rightarrow zg(z) \in \mathscr{E}.$$

5. Related subspaces of $\mathscr{H}^2(\mathbf{E})$. Let $\mathscr{E}$ be a given subspace of $\mathscr{H}^2(\mathbf{E})$ satisfying (2), and let f_∞ be as above. We note first that if $f \in \mathscr{M}_\psi$, then $f = g + h$, where $g \in \mathscr{H}^2(\mathbf{\Delta}_R)$ and $h \in \mathscr{E}_\psi$, say normalized so that $g(0) = 0$. Thus $f_\infty \mathscr{M}_\psi$ is certainly in $\hat{\mathscr{E}}$. We must prove $f_\infty \mathscr{M}_\psi$ is closed to get $f_\infty \mathscr{M}_\psi = \hat{\mathscr{E}}$.

If

$$f_\infty(g_n + h_n) \to f \quad \text{in } \mathscr{H}^2(\mathbf{A}),$$

where $g_n \in \mathscr{H}^2(\mathbf{\Delta}_R)$, $g_n(0) = 0$, and $h_n \in \mathscr{E}_\psi$, then

$$f_\infty(z)(g_n(z) + h_n(z)) \to f(z) \quad \text{in } \mathscr{L}^2(\partial \mathbf{\Delta}_R),$$

so that

$$g_n(z) + h_n(z) \to \frac{f(z)}{f_\infty(z)} \quad \text{in } \mathscr{L}^2(\partial \mathbf{\Delta}_R),$$

since f_∞ is nice on $\partial \mathbf{\Delta}_R$.

The functions g_n and h_n lie in orthogonal subspaces of $\mathscr{L}^2(\partial \mathbf{\Delta}_R)$ since g_n has Fourier coefficients equal to zero for the nonpositive integers and h_n has Fourier coefficients equal to zero for the positive integers. Thus, for some $g \in \mathscr{H}^2(\mathbf{\Delta}_R)$,

$$g_n \to g \quad \text{in } \mathscr{H}^2(\mathbf{\Delta}_R).$$

This means

$$g_n \to g \quad \text{uniformly on } \partial \mathbf{\Delta}.$$

Hence $f_\infty h_n$ converges in $\mathscr{L}^2(\partial \mathbf{\Delta})$. Since multiplication by f_∞ is an isometry on $\mathscr{E}_\psi$,

$$h_n \to h \quad \text{in } \mathscr{H}^2(\mathbf{E}).$$

This implies $h \in \mathscr{E}_\psi$, and so

$$f_\infty(g_n + h_n) \to f_\infty(g + h) \in f_\infty \mathscr{M}_\psi.$$

We summarize this more formally:

Proposition 3. *Let $\mathscr{E}$ be a closed subspace of $\mathscr{H}^2(\mathbf{E})$, such that*

$$g \in \mathscr{E}, \quad g(\infty) = 0 \Rightarrow zg(z) \in \mathscr{E}.$$

Then

$$\mathscr{E} = f_\infty \mathscr{E}_\psi,$$

where ψ is inner on Δ, f_∞ is the function of norm 1 in $\mathscr{E}$ maximizing $\Re f_\infty(\infty)$. Further, if $\hat{\mathscr{E}}$ is the smallest closed subspace of $\mathscr{H}^2(\mathbf{A})$ which contains $\mathscr{E}$ and for which

$$z\hat{\mathscr{E}} \subseteq \hat{\mathscr{E}},$$

then

$$\hat{\mathscr{E}} = f_\infty \mathscr{M}_\psi.$$

Now we want to show that given a simply invariant subspace $\mathscr{M}$ of $\mathscr{H}^2(\mathbf{A})$, then $z^{-m}\mathscr{M} = \hat{\mathscr{E}}$ for some $\mathscr{E}$. Our principal tool will be the Szegő kernel function $k_{\mathscr{M}}(z, \zeta)$.

6. Boundary smoothness of extremals. First, we obtain an expression for $k_{\mathscr{H}^2(\mathbf{A})}(z, \bar{\zeta})$. For $\mathbf{A}$ it is convenient to use the rotationally invariant norm

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\vartheta})|^2\, d\vartheta + \frac{1}{2\pi}\int_0^{2\pi} |f(Re^{i\vartheta})|^2\, d\vartheta. \tag{33}$$

This is equivalent, but not isometric, to norming via least harmonic majorants. The norm (33) leads to the usual class $\mathscr{H}^2(\mathbf{A})$, viz., sums of functions form $\mathscr{H}^2(\mathbf{E})$ and $\mathscr{H}^2(\Delta_R)$. With the norm given by (33), $\{z^n\}_{n=-\infty}^{n=\infty}$ is orthogonal and linearly spans $\mathscr{H}^2(\mathbf{A})$. We have

$$\|z^m\|^2 = 1 + R^{2m}, \tag{34}$$

so that

$$k_{\mathscr{H}^2(\mathbf{A})}(z, \bar{\zeta}) = \sum_{m=-\infty}^{m=\infty} \frac{z^m \bar{\zeta}^m}{1 + R^{2m}}. \tag{35}$$

Equation (35) shows $k_{\mathscr{H}^2(\mathbf{A})}(z, \bar{\zeta})$ to be analytic in z on $\partial\Delta_R$.

Let $\mathscr{M}$ be a simply invariant subspace of $\mathscr{H}^2(\mathbf{A})$ with greatest common divisor 1. We will use a technique of Royden [**23**, Theorem 1] to prove that $k_{\mathscr{M}}(z, \bar{\zeta})$ is analytic on a neighborhood of $\partial\Delta_R$.

We norm $\mathscr{M}$ by (33). We let $\mathscr{M}^\perp$ be the orthogonal complement of $\mathscr{M}$ in $\mathscr{H}^2(\mathbf{A})$. Letting $k_{\mathscr{M}^\perp}$ be the kernel function for $\mathscr{M}^\perp$, we have

$$k_{\mathscr{H}^2(\mathbf{A})}(z, \bar{\zeta}) = k_{\mathscr{M}}(z, \bar{\zeta}) + k_{\mathscr{M}^\perp}(z, \bar{\zeta}), \tag{36}$$

because we can take as a complete orthonormal basis for $\mathscr{H}^2(\mathbf{A})$ the union of orthonormal bases for $\mathscr{M}$ and $\mathscr{M}^\perp$.

We fix a $\zeta_0 \in \mathbf{A}$ and for each $f \in \mathscr{M}$ we form

$$(Tf)(\omega) = \left\langle \frac{f(z)}{z-\omega}, k_{\mathscr{M}^\perp}(z, \overline{\zeta_0}) \right\rangle. \tag{37}$$

Here, the inner product is taken on z. Simple invariance gives

$$\frac{f(z)}{z-\omega} \in \mathscr{M}, \quad \text{if } |\omega| > R, \tag{38}$$

so by orthogonality

$$(Tf)(\omega) \equiv 0, \quad \forall \omega \in \mathbf{E}_R = \{z \in \mathbf{C} \cup \{\infty\} : |z| > R\}. \tag{39}$$

Now,

$$\begin{aligned} (Tf)(\omega) &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z-\omega} \bar{k}_{\mathscr{M}^\perp}(z, \overline{\zeta_0}) \frac{dz}{z} \\ &\quad + \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z-\omega} \bar{k}_{\mathscr{M}^\perp}(z, \overline{\zeta_0}) \frac{dz}{z} \\ &= F_1(\omega) + F_2(\omega), \end{aligned} \tag{40}$$

where

$$F_1(\omega) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z-\omega} \bar{k}_{\mathscr{M}^\perp}(z, \overline{\zeta_0}) \frac{dz}{z}, \tag{41}$$

$$F_2(\omega) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z-\omega} \bar{k}_{\mathscr{M}^\perp}(z, \overline{\zeta_0}) \frac{dz}{z}. \tag{42}$$

Because F_1 and F_2 are Cauchy integrals of $\mathscr{L}^1$ functions, our general theory (see Royden [23] or Duren [7]) tells us that

$$F_1 \in \mathscr{H}^2(\mathbf{E}), \qquad F_2 \in \mathscr{H}^r(\Delta_R), \tag{43}$$

for all $r \in (0,1)$. Moreover, F_2 has boundary values

$$F_2(\omega) = f(\omega)\bar{k}_{\mathscr{M}^\perp}(\omega, \overline{\zeta_0})\frac{1}{\omega} - F_1(\omega) \quad \text{on } \partial\Delta_R. \tag{44}$$

The function F_1 is analytic on $\partial\Delta_R$, and so

$$f(\omega)\bar{k}_{\mathscr{M}^\perp}(\omega, \overline{\zeta_0}) = \Psi_f(\omega) K_f(\omega) \quad \text{on } \partial\Delta_R, \tag{45}$$

where $\Psi_f(\omega)$ is inner on $\mathbf{A}$ and $K_f(\omega)$ is outer on $\mathbf{A}$. This is because (45) gives the boundary values of $F_1 + F_2$, which is a function in $\mathscr{H}^r(\mathbf{A})$, for all $r < 1$, and hence a function in $\mathscr{N}^+(\mathbf{A})$. Here $\mathscr{N}^+(\mathbf{A})$ is the subset of the Nevanlinna class of the annulus consisting of those functions which have only trivial inner part in their denominators in their canonical factorings.

We can express (45) in the form

$$\bar{k}_{\mathscr{M}^\perp}(\omega, \overline{\zeta_0}) = \frac{\Psi_f(\omega) K_f(\omega)}{f(\omega)} \quad \text{on } \partial\Delta_R. \tag{46}$$

Then we let f range over $\mathscr{M}$ (which has greatest common divisor 1), to see that $\bar{k}_{\mathscr{M}^\perp}(\omega, \overline{\zeta_0})$ has the boundary values on $\partial\Delta_R$ of an $\mathscr{N}^+(\mathbf{A})$ function. But then $k_{\mathscr{M}^\perp}(\omega, \bar{\zeta}_0)$ is the boundary values of a function in $\mathscr{N}^+(R < |\omega| < R^2)$. Being in $\mathscr{L}^2$, this makes $k_{\mathscr{M}^\perp}(\omega, \overline{\zeta_0})$ the boundary values of a function in $\mathscr{H}^2(R < |\omega| < R^2 - \varepsilon)$. Hence, $k_{\mathscr{M}^\perp}(z, \overline{\zeta_0})$ is analytic somewhat beyond $\partial\Delta_R$. Because

$$k_{\mathscr{H}^2(\mathbf{A})}(z, \bar{\zeta}) = k_{\mathscr{M}}(z, \bar{\zeta}) + k_{\mathscr{M}^\perp}(z, \bar{\zeta}) \tag{36}$$

and

$$k_{\mathscr{H}^2(\mathbf{A})}(z, \bar{\zeta}) = \sum_{m=-\infty}^{m=\infty} \frac{z^m \bar{\zeta}^m}{1 + R^{2m}}, \tag{35}$$

we see that $k_{\mathscr{M}}(z, \bar{\zeta})$ also is analytic a little beyond $\partial\Delta_R$. We then have the following proposition.

PROPOSITION 4. *If $\mathscr{M}$ is a closed simply invariant subspace of $\mathscr{H}^2(\mathbf{A})$ with greatest common divisor* 1, *then the maximal function $k_{\mathscr{M}}(z, \bar{\zeta})$ for $\mathscr{M}$, normed by*

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\vartheta})|^2 \, d\vartheta + \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\vartheta})|^2 \, d\vartheta,$$

can be analytically continued across $\{z \in \mathbf{C} \colon |z| = R\}$.

7. Outer and kernel functions in $\mathscr{M}$. We now make an observation about the closed linear span of $k_{\mathscr{M}}(z, \bar{\zeta})$ as ζ varies.

PROPOSITION 5. *Let $\mathscr{M}$ be a closed simply invariant subspace of $\mathscr{H}^2(\mathbf{A})$ with greatest common divisor* 1, *where $\mathscr{H}^2(\mathbf{A})$ is normed by*

$$\|f\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\vartheta})|^2 \, d\vartheta + \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\vartheta})|^2 \, d\vartheta.$$

Let $k_{\mathscr{M}}(z, \bar{\zeta})$ be the Szegő reproducing kernel. If S is any subset of $\mathbf{A}$ *that has a limit point interior to the annulus, then the closed linear span of*

$$\{k_{\mathscr{M}}(z, \bar{\zeta})\}$$

as ζ ranges over S is $\mathscr{M}$. Consequently

$$\{k_{\mathscr{M}}(z, \bar{\zeta})\}_{\zeta \in S}$$

has greatest common divisor 1.

Proof. Any function in $\mathscr{M}$ which is orthogonal to

$$\{k_{\mathscr{M}}(z, \bar{\zeta})\}_{\zeta \in S}$$

vanishes on S and hence vanishes identically. □

8. The relation of $\mathscr{M}$ to $\mathscr{H}^2(\mathbf{E})$. In order to carry out the analysis of this section, we first need to factor $k_{\mathscr{M}}(z, \bar{\zeta})$ ($\mathscr{M}$ as above). We let $I_{\mathscr{M}}(z, \bar{\zeta})$ be the inner function on $\mathbf{E}$ having the zeros of $k_{\mathscr{M}}(z, \bar{\zeta})$ which have modulus in $(1, R]$. We then write

$$k_{\mathscr{M}}(z, \bar{\zeta}) = I_{\mathscr{M}}(z, \bar{\zeta}) K_{\mathscr{M}}(z, \bar{\zeta}),$$

where $K_{\mathscr{M}}(z, \bar{\zeta})$ is a zero free function in $\mathscr{H}^2(\mathbf{A})$. Thus $\log K_{\mathscr{M}}(z, \bar{\zeta})$ can be expressed as the sum of a Laurent series and a multiple of $\log z$. Hence

$$k_{\mathscr{M}}(z, \bar{\zeta}) = z^{m_{\zeta}} I_{\mathscr{M}}(z, \bar{\zeta}) F_{\mathscr{M}}(z, \bar{\zeta}) G_{\mathscr{M}}(z, \bar{\zeta}),$$

where $F_{\mathscr{M}}(z, \bar{\zeta})$ is a zero-free function in $\mathscr{H}^2(\mathbf{E})$ and $G_{\mathscr{M}}(z, \bar{\zeta})$ is an outer function in $\mathscr{H}^2(\Delta_R)$.

There are only countably many $m_{\bar{\zeta}}$ but uncountably many ζ. Thus at least one of the countably many sets

$$S(m) = \{\zeta \in \mathbf{A} \colon m_{\bar{\zeta}} = m\}$$

is uncountable and hence has an interior limit point. Fix one such uncountable set $S = S(m)$.

We set

$$\mathscr{E} = (z^{-m}\mathscr{M}) \cap \mathscr{H}^2(\mathbf{E}).$$

By Beurling's theorem,

$$I_{\mathscr{M}}(z, \bar{\zeta}) F_{\mathscr{M}}(z, \bar{\zeta}) \in \mathscr{E},$$

for all $\zeta \in S$, so $\mathscr{E}$ has greatest common divisor 1, considered as a subset of $\mathscr{H}^2(\mathbf{E})$. We also have

$$f \in \mathscr{E}, \quad f(\infty) = 0 \Rightarrow zf(z) \in \mathscr{E}. \tag{2}$$

Thus our previous work concerning subspaces weakly invariant under the backwards shift operator (and related topics) applies.

We have

$$\mathscr{M}_1 = z^m \hat{\mathscr{E}} \subseteq \mathscr{M},$$

and

$$k_{\mathscr{M}}(z, \bar{\zeta}) \in \mathscr{M}_1$$

for all $\zeta \in S$. Now S is uncountable. By for example dividing the annulus into countably many disjoint sub-annuli, we see that one such sub-annulus contains uncountably many points of S. Hence S contains an interior limit point, and Proposition 5 gives

$$\mathscr{M} \subseteq \mathscr{M}_1,$$

and so

$$\mathscr{M} = \mathscr{M}_1.$$

Since we know $\hat{\mathscr{E}} = f_\infty \mathscr{M}_\psi$ (where f_∞ and ψ have their usual meanings), we have established Theorem A. An examination of our proofs establishes the uniqueness statement also.

9. Open questions and directions for further research. It is an open problem to determine the simply invariant subspaces of $\mathscr{H}^p(\mathbf{A})$ for $p \neq 2$. Beyond this, there are at least three directions for further investigation. One of these is to replace the annulus $\mathbf{A}$ with some other domain $\mathbf{G}$, possibly of a different connectivity. The question would be to determine the subspaces invariant under multiplication by z. On a related note, perhaps one would try to determine the subspaces invariant under multiplication by a subgroup of the group of units for that domain. Already with the annulus, if both boundary curves are slightly perturbed, it is not clear what the invariant subspaces under multiplication by z are.

If we keep the same domain $\mathbf{A}$ then it would be interesting to know exactly which functions f_∞ can occur in our factor decomposition. If we do not know exactly which f_∞ can occur, it would be good to determine sufficient conditions for a function f to be f_∞ in some decomposition.

Finally, we note that Beurling's theorem for $\mathscr{H}^2(\Delta)$ can be formulated as a statement about square-summable sequences, namely, a characterization of the subspaces of $l^2(\mathbf{N})$ invariant under the shift. This is because $\mathscr{H}^2(\Delta)$ is isometric to $l^2(\mathbf{N})$, where each $n \in \mathbf{N}$ is given unit mass. Under this isometry, multiplication by z in $\mathscr{H}^2(\Delta)$ corresponds to shifting sequences in $l^2(\mathbf{N})$.

Professor Sarnak has pointed out that our theorem is a characterization of the subspaces of $l^2(\mathbf{Z}, \mu)$ which are invariant under the shift, where μ is a measure of $\mathbf{Z}$ making $l^2(\mathbf{Z})$ isometric to $\mathscr{H}^2(\mathbf{A})$. He asks the interesting question of how to determine the shift invariant subspaces of $l^2(\mathbf{Z}, \lambda)$ for a different measure λ on $\mathbf{Z}$. That is, one would look for a domain $\mathbf{G}$ and a space of functions on $\mathbf{G}$—perhaps $\mathscr{H}^2(\mathbf{G})$.

One would hope to have a map from $\mathscr{H}^2(\mathbf{G})$ to itself, perhaps multiplication by some fixed function, that would be isometric to shifting $l^2(\mathbf{Z}, \lambda)$. Function theoretic means would then be used to determine the invariant subspaces. This appears to be difficult to do in general.

10. Acknowledgments. This paper contains the contents of Chapter 3 of the author's doctoral dissertation, written under the supervision of Professor Royden, to whom the author is grateful for the suggestion of the problem, as well as for encouragement and discussion during the course of the investigation.

The author also would like to acknowledge several long and fruitful discussions with Professor Katznelson, who suggested and encouraged many simplifications.

Finally the author would like to thank Professor Sarason for directing his attention to the work of Hayashi.

Added in proof. Since submission of this paper, Professor Sarason has written *Nearly Invariant Subspaces of the Backward Shift*, which contains an alternative approach to and an extension of some of the results in this paper.

REFERENCES

[1] L. Ahlfors, *Bounded analytic functions*, Duke Math. J., **14** (1947), 1–11.

[2] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math., **81** (1949), 239–255.

[3] L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice Hall, Englewood Cliffs, New Jersey (1968).

[4] L. de Branges and J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart, and Winston, New York (1966).

[5] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, volume I, Wiley, New York (1962).

[6] R. Douglas, H. Shapiro and A. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*, Annales de l'Institut Fourier, **20** (1970), 37–76.

[7] P. Duren, *The Theorem of $\mathscr{H}^p$ Spaces*, Academic Press, New York (1970).

[8] S. Fisher, *Function Theory on Planar Domains: a Second Course in Complex Analysis*, Wiley, New York (1983).

[9] P. Garabedian, *Schwarz's lemma and the Szegö kernel function*, Trans. Amer. Math. Soc., **67** (1949), 1–35.

[10] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York (1981).

[11] T. Gronwall, *On the maximum modulus of an analytic function*, Annals of Math., Second Series, **16** (1914), 77–81.

[12] M. Hasumi, *Invariant subspace theorems for finite Riemann surfaces*, Canad. J. Math., **18** (1966), 240–255.

[13] M. Hasumi, Lecture Notes in Mathematics, volume 1027: *Hardy Classes on Infinitely Connected Riemann Surfaces*, Springer-Verlag, Berlin (1983).

[14] M. Hasumi and T. Srinivasan, *Doubly invariant subspaces*, II, Pacific J. Math., **14** (1964), 525–535.
[15] ———, *Invariant subspaces of continuous functions*, Canad. J. Math., **17** (1965), 643–651.
[16] E. Hayashi, *The kernel of a Toeplitz operator*, to appear.
[17] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York (1964).
[18] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, New Jersey (1962).
[19] J.-P. Kahane and Y. Katznelson, *Sur le comportement radial des fonctions analytiques*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Série A, **272** (1971), 718–719.
[20] R. Nevanlinna, *Eindeutige Analytische Funktionen*, Springer-Verlag, Berlin (1936).
[21] M. Parreau, *Sur les Moyennes des Fonctions Harmoniques et Analytiques et la Classification des Surfaces de Riemann*, Annales de l'Institut Fourier, **3** (1951), 103–197.
[22] H. Royden, *The boundary values of analytic and harmonic functions*, Math. Z., **78** (1962), 1–24.
[23] ———, *Invariant subspaces of $\mathscr{H}^p$ for multiply connected regions*, to appear.
[24] W. Rudin, *Analytic functions of class $\mathscr{H}^p$*, Trans. Amer. Math. Soc., **78** (1955), 46–66.
[25] D. Sarason, *The $\mathscr{H}^p$ spaces of an annulus*, Amer. Math. Soc., Providence, Rhode Island (1965).
[26] M. Schiffer, *Appendix. Some Recent Developments in the Theory of Conformal Mapping* in *Dirichlet's Principle* by R. Courant, Interscience Publishers, Incorporated, New York (1950), 249–323.
[27] M. Voichick, *Ideals and invariant subspaces of analytic functions*, Trans. Amer. Math. Soc., **111** (1964), 493–512.
[28] ———, *Invariant subspaces of Riemann surfaces*, Canad. J. Math., **18** (1966), 399–403.

Received June 10, 1986. This research was supported in part by NSF grant MCS 83-01379.

UNIVERSITY OF CALIFORNIA
SANTA CRUZ, CA 95064