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TOPOLOGICAL ENTROPY AND RECURRENCE OF COUNTABLE CHAINS

IBRAHIM A. SALAMA

We consider a symbolic dynamical system (X, σ) on a countable state space. We introduce a kind of topological entropy for such systems, denoted h^*, which coincides with usual topological entropy when X is compact. We use a pictorial approach, to classify a graph Γ (or a chain) as transient, null recurrent, or positive recurrent. We show that given $0 \leq \alpha \leq \beta \leq \infty$, there is a chain whose h^* entropy is β and where Gurevic entropy is α. We compute the topological entropies of some classes of chains, including larger chains built up from smaller ones by a new operation which we call the Cartesian sum.

Introduction. The importance of subshifts of finite type in ergodic theory and dynamical systems is well known. One needs also to study chains on a countably infinite set in order to analyze problems in various fields such as differentiable dynamics, coding for magnetic recording, nonuniqueness of equilibrium states in statistical mechanics, formal languages and automata, or even to analyze arbitrary subshifts. (See **[3]**, **[9]**, **[6]**, **[2]**, **[1]**, **[10]**, respectively.)

Let Γ be a strongly connected directed graph on a countable set of vertices $S = \{s_1, s_2, \dots\}$, and let

$$X(\Gamma) = \{x \in S^z \mid \text{ for all } i, \text{ there is an edge in } \Gamma \text{ from } x_i \text{ to } x_{i+1}\}.$$

If S has the discrete topology and S^z the product topology, then in the induced topology $X(\Gamma)$ (or simply X), together with the shift transformation σ defined by $(\sigma x)_i = x_{i+1}$ for all i, is a (non-compact) dynamical system, called the *chain* determined by the directed graph Γ. The topological entropy of X may be determined using Bowen's definition, to obtain $h_B(X)$ (see **[8]** for a definition). This definition depends on the metric we put on X. We consider the following two metric spaces.

1. For $x, y \in X$ define

$$d_1(x, y) = \sum_i \frac{1 - \delta(x_i, y_i)}{2^{|i|}},$$

where $\delta(s, t) = 1$ if $s = t$ and 0 if $s \neq t$.

2. Let $S = \{1, 2, \ldots, \}$ and define

$$d_2(x, y) = \sum_i \frac{1}{2^{|i|}} \frac{1 - \delta(x_i, y_i)}{\min(x_i, y_i)}$$

(we may use $S = \{1/n, n \geq 1\}$ and define $d_2(x, y) = \sum_i |x_i - y_i|/2^{|i|}$). On the other hand and without reference to any metric, we may use Gurevic's definition (Gurevic, [**4**]), $h_G(X) = h_G(\Gamma) = \sup_{\Gamma' < \Gamma} h(\Gamma')$, where the sup is taken over all (connected) finite subgraphs Γ' of Γ (and $h(\Gamma')$ is the usual topological entropy of the subshift of finite type determined by Γ', the logarithm of the maximal eigenvalue of the transition matrix).

Another possible approach, suggested by Gurevic [**4**], is to let $\overline{S}$ be the one-point compactification of S, and $\overline{X}$ the closure of X in $\overline{S}^z$. Then $\overline{X}$ is compact, and we may define $h_c(X)$ to be the ordinary topological entropy of $(\overline{X}, \sigma)$.

Fix a vertex S in Γ and define

$B_s^{(n)} =$ number of paths of length n in Γ from s to s;

$f_s^{(n)} =$ number of paths of length n in Γ from s to s with no other occurrences of s in between;

$F(\Gamma, s, z) = \sum f_s^{(n)} z^n$;

$B(\Gamma, s, z) = \sum B_s^{(n)} z^n$;

$F'(\Gamma, s, z) = \sum n f_s^{(n)} z^{n-1}$.

Let L_Γ and R_Γ be the radii of convergence of $F(\Gamma, s, z)$ and $B(\Gamma, s, z)$, respectively (they are independent of s). We will abbreviate L_Γ by L and R_Γ by R (generally, L_{Γ_i} by L_i and R_{Γ_i} by R_i).

Vere-Jones ([**11**], [**12**]) studied the classification of the graph Γ as transient, null recurrent, or positive recurrent according to the following table:

	transient	null recurrent	positive recurrent
$F(\Gamma, s, R)$	< 1	$= 1$	$= 1$
$F'(\Gamma, s, R)$	$= \infty$	$= \infty$	$< \infty$
$B(\Gamma, s, R)$	$< \infty$	$= \infty$	$= \infty$
$\lim_{n\to\infty} B_s^{(n)} R^n$	$= 0$	$= 0$	> 0

Gurevic ([**4**], [**5**]) showed that if X is a connected chain with the d_2-metric, then $h_G(X) = h_c(X)$; and if Γ is a connected graph with $h_G(\Gamma) < \infty$, then $h_G(\Gamma) = -\log R$.

The paper is organized as follows: In §1 we introduce a new definition for the topological entropy of a symbolic dynamical system

(X, σ), which we call h^*. For a locally finite chain X, we show that $h_B(X) = h^*(X)$ when the d_1-metric is used; and that $h_B(X) = h_G(X)$ when the d_2-metric is used. In §2 we consider a geometric or pictorial approach to classify chains as transient, null recurrent, or positive recurrent. In §3 we show that given $1 < \alpha \leq \beta$, there exists a recurrent uniformly locally finite graph Γ such that $h_G(\Gamma) = \log \alpha$ and $h^*(\Gamma) = \log \beta$. If $\alpha = \beta$, then the Γ that we construct is positive recurrent. In §4 we consider some computation examples.

1. The h^* entropy. Let (X, σ) be a symbolic dynamical system with countable state space $S = \{s_1, s_2, \dots\}$. If S is finite, then X is compact and the topological entropy of X is given by $h(X) = \lim_n \frac{1}{n} \log B^{(n)}$, where $B^{(n)}$ is the number of blocks of length n in X. Taking this formula and applying it "formally" to symbolic dynamical systems with infinite state space S, we will end up assigning the same value, namely ∞, to these systems. However, it is reasonable to consider the rate of growth of the number of blocks in X starting with a fixed symbol or generally starting with a fixed block. Based on this notion we introduce a new definition for the topological entropy of (X, σ), which we shall call h^*, formulated as follows.

DEFINITION Let $T_s^{(n)}$ be the number of blocks of length n in X starting with $s \in S$. The entropy of X relative to s is defined by

$$h^*(X \mid s) = \overline{\lim_n} \tfrac{1}{n} \log T_s^{(n)},$$

and the entropy of X is defined by

$$h^*(X) = \sup_{s \in S} h^*(X \mid s).$$

We observe that if X is a finite connected chain, then $h^*(X) = h(X) = h_G(X) = h_B(X)$. Also, it is easy to see that if X is transitive, then $h^*(X) = h^*(X \mid s)$ for all $s \in S$, and if X is a countable connected chain, then $h_G(X) \leq h^*(X)$.

A chain X is *locally finite* if the corresponding graph Γ satisfies: If s is a state of Γ then the number of arrows coming in and going out of s is finite. The chain is *uniformly locally finite* if the number of arrows coming in and going out of any state is less than some fixed number, say m.

THEOREM (1.1). *If X is a locally finite connected chain with the metric d_1, then $h^*(X) = h_B(X)$.*

Proof. Let K be a compact subset of X. Then for every i, the number of different symbols appearing in the ith place in K is finite. For $\varepsilon = 2^{-i}$, $r_n(\varepsilon, K)$ is the number of different blocks in K of the form $s_{-i} \cdots s_{n+i}$. Let N_{-i} be the set of symbols appearing in K in the $-i$th place. Then $r_n(\varepsilon, K) = \sum_{s \in N_{-i}} r_n^s(\varepsilon, K)$, where $r_n^s(\varepsilon, K)$ is the number of blocks in K of the form $ss_{-(i-1)} \cdots s_{n+i}$ and $s \in N_{-i}$. Since $r_n(\varepsilon, K)$ is the sum of $\#\{N_{-i}\}$ nonnegative sequences, there is $s_0 \in N_{-i}$ such that

$$r(\varepsilon, K) = \overline{\lim}\, \tfrac{1}{n} \log r_n(\varepsilon, K) = \overline{\lim}\, \tfrac{1}{n} \log r_n^{s_0}(\varepsilon, K) \leq h^*(X).$$

Hence $h_B(X) \leq h^*(X)$.

Now, since X is locally finite, for $s \in S$, $K_s = \{x \in X \mid x_0 = s\}$ is compact. If $\varepsilon = 2^{-i}$, then $r_n(\varepsilon, T^{-i}K_s)$ is the number of blocks in K_s of the form $ss_{-(i-1)} \cdots s_{n+1}$, and $h^*(X \mid s) = \overline{\lim}\, \frac{1}{n} \log r_n(\varepsilon, T^{-i}K_s) \leq h_B(X)$. Thus, $h^*(X) \leq h_B(X)$. □

The notion of the h^*-entropy may be extended to a general topological space (not necessarily compact) and a homeomorphism $T: X \to X$ as follows. For an open cover α of X and a compact $K \subset X$ let $N_K^n(\alpha)$ denote the minimum number of sets in $\bigvee_{i=0}^{n-1} T_\alpha^{-i}$ needed to cover K, and let $H_K^n(\alpha) = \log N_K^n(\alpha)$. The entropy of T relative to α given compact $K \subset X$ is defined as $h^*(T, \alpha; K) = \overline{\lim}_n \frac{1}{n} H_K^n(\alpha)$. The entropy of T relative to α is defined as $h^*(T, \alpha) = \sup_{K \subset X} h^*(T, \alpha; K)$, where K is compact. Finally, the entropy of T is defined as $h^*(T) = \sup_\alpha h^*(T, \alpha)$. The following fact follows from a straightforward argument.

PROPOSITION (1.2). *If X is compact, then $h^*(T) = h(T)$.*

PROPOSITION (1.3). *If X is a connected chain with the d_2-metric, then $h_B(X) = h_G(X)$.*

Proof. Let K be a compact subset of X, $r_n(\varepsilon, K)$ = the smallest cardinality of an (n, ε)-spanning set of K, $r(\varepsilon, K) = \overline{\lim}_n \frac{1}{n} \log r_n(\varepsilon, K)$, and $h_B(K) = \lim_{\varepsilon \to 0} r(\varepsilon, K)$, so that $h_B(X) = \sup_{K \subset X} h_B(K)$.

If X' is a connected finite subchain of X, then $h_G(X') \leq h_B(X)$; hence $h_G(X) \leq h_B(X)$. On the other hand, let $\overline{X}$ be the completion of X in the d_2-metric. Since (X, d_2) is totally bounded, $(\overline{X}, d_2)$ is compact. If X'' is a compact subset of X, then X'' is compact as a subset of $\overline{X}$, and $h_B(X'') \leq h_B(\overline{X}) = h(\overline{X}) = h_G(X)$. Hence $h_B(X) \leq h_G(X)$. □

2. Classification of chains. In this section we present a criterion enabling us to decide if a connected graph Γ is transient, null recurrent, or positive recurrent. Our approach is geometric or pictorial in nature, avoiding the computational and combinatorial problems usually encountered. The idea is to relate the value of $h_G(\Gamma)$ to the values of $h_G(\Gamma')$, where Γ' is either a subgraph of Γ or a graph containing Γ. In general terms, our results on this question may be summarized as follows: Γ is transient if and only if we can "expand" or "contract" Γ without changing h_G, Γ is positive-recurrent if and only if any "expansion" or "contraction" of Γ will change h_G in the right direction; and Γ is null-recurrent if and only if we can "contract" Γ without changing h_G but any "expansion" of Γ will increase h_G. We prove the following.

THEOREM (2.1). (i) *If* $\Gamma_0 < \Gamma_2$ *and* $h_G(\Gamma_0) = h_G(\Gamma_2)$, *then* Γ_0 *is transient.*

(ii) *If* Γ_0 *is transient, then there exists* $\Gamma_2 > \Gamma_0$ *such that* $h_G(\Gamma_0) = h_G(\Gamma_2)$ *and* Γ_2 *is transient.*

THEOREM (2.2). *Let* Γ_0 *be such that* $R_0 = L_0$. *Then there exists* $\Gamma < \Gamma_0$ *such that* $h_G(\Gamma) = h_G(\Gamma_0)$.

THEOREM (2.3). *The following conditions for* Γ_0 *are equivalent*:

(i) Γ_0 *is positive recurrent.*

(ii) $R_0 < L_0$.

(iii) *For all* $\Gamma_1 < \Gamma_0$, $h_G(\Gamma_1) < h_G(\Gamma_0)$.

(iv) $F(\Gamma_0, s, L_0) > 1$.

COROLLARY (2.4). *If* Γ_0 *is transient, then there exists* $\Gamma_1 < \Gamma_0$ *such that* $h_G(\Gamma_1) = h_G(\Gamma_0)$ *and* Γ_1 *is transient.*

COROLLARY (2.5). Γ_0 *is recurrent if and only if for all* $\Gamma_2 > \Gamma_0$, $h_G(\Gamma_2) > h_G(\Gamma_0)$.

COROLLARY (2.6). Γ_0 *is null-recurrent if and only if*: (i) *there exists* $\Gamma_1 < \Gamma_0$ *such that* $h_G(\Gamma_1) = h_G(\Gamma_0)$, *and* (ii) *for all* $\Gamma_2 > \Gamma_0$ *we have* $h_G(\Gamma_2) > h_G(\Gamma_0)$.

Proof of Theorem (2.1). (i) Let s be a state in Γ_0 and Γ_2. Then $F(\Gamma_2, s, R_2) \leq 1$. Since $\Gamma_0 < \Gamma_2$ and $R_0 = R_2$, we have $F(\Gamma_0, s, R_0) < 1$.

(ii) Since Γ_0 is transient, $F(\Gamma_0, s, R_0) = \sum_n f_s^{(n)} R_0^n < 1$. Hence, for some m we can find a positive integer k such that $k > f_s^{(m)}$

and $\sum_{n \neq m} f_s^{(n)} R_0^n + kR_0^m < 1$. Let Γ_2 be the graph obtained by adding a loop of length $(k - f_s^{(m)})$ based at s to Γ_0. Thus $\Gamma_0 < \Gamma_2$ and $F(\Gamma_2, s, R_0) < 1$. Since $R_2 \leq R_0$, then $F(\Gamma_2, s, R_2) \leq F(\Gamma_2, s, R_0) < 1$, and Γ_2 is transient. To show $R_0 = R_2$, note that $L_0 = L_2$ and $R_2 \leq R_0 \leq L_0$. If $R_2 < R_0$, then $R_2 < L_2$ and $F'(\Gamma_2, s, R_2) < \infty$, contradicting that Γ_2 is transient.

Proof of Theorem (2.2). For a state s in Γ_0, let $\mathrm{Fol}(s) = \{s' \mid$ there is an arrow from s to s' in $\Gamma_0\}$. Let s_0 be a state in Γ_0 such that $\#\mathrm{Fol}(s_0) \geq 2$. Let $s^* \in \mathrm{Fol}(s_0)$, let Γ_1 be the subgraph obtained by removing the edge $s_0 s^*$ from Γ_0, and let Γ_2 be the subgraph obtained by the removal from Γ_0 the edges $s_0 s'$, $s' \in \mathrm{Fol}(s_0) - s^*$. Let i be such that $L_i = \min(L_1, L_2)$. Then $R_0 \leq R_i \leq L_i = L_0$. Since $L_0 = R_0$, we have $R_0 = R_i$ and $h_G(\Gamma_0) = h_G(\Gamma_i)$. □

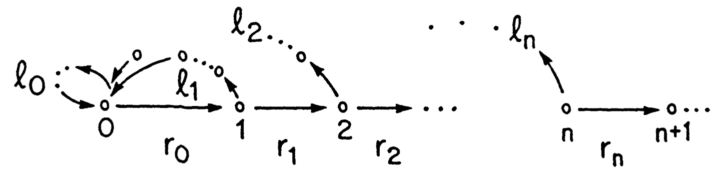
Proof of Theorem (2.3). (i) If $R_0 < L_0$, then $F'(\Gamma_0, s, R_0) < \infty$ and Γ_0 is positive recurrent. If $R_0 = L_0$, then by Theorem (2.2) there is $\Gamma_1 < \Gamma_0$ such that $R_1 = R_0$. Hence by (i) of Theorem (2.1) Γ_1 is transient. Since $F'(\Gamma_1, s, R_0) = \infty$ and $F'(\Gamma_1, s, R_0) \leq F'(\Gamma_0, s, R_0), \Gamma_0$ cannot be positive recurrent.

(ii) We show that: Γ_0 is not positive-recurrent if and only if there is $\Gamma_1 < \Gamma_0$, $h_G(\Gamma_1) = h_G(\Gamma_0)$. So, assume that Γ_0 is not positive-recurrent, then by (i) above we have $R_0 = L_0$; hence, by Theorem (2.2) there is $\Gamma_1 < \Gamma_0$ such that $h_G(\Gamma_1) = h_G(\Gamma_0)$. On the other hand, if there exists $\Gamma_1 < \Gamma_0$ with $h_G(\Gamma_1) = h_G(\Gamma_0)$, then Γ_0 is not positive-recurrent as shown in the proof of (i) above.

(iii) Assume $F(\Gamma_0, s, L_0) > 1$. Since $F(\Gamma_0, s, R_0) \leq 1$, we have $R_0 < L_0$ and by (i) above Γ_0 is positive-recurrent. If Γ_0 is positive-recurrent, then $R_0 < L_0$. Since $F(\Gamma_0, s, R_0) = 1$, we have $F(\Gamma_0, s, L_0) > 1$. □

Finally, Corollary (2.4) follows from part (ii) of the proof of Theorem (2.3), Corollary (2.5) follows from Theorem (2.1), and Corollary (2.6) follows from part (ii) of the proof of Theorem (2.3) and Corollary (2.5).

3. Relation between h_G and h^*. In this section we study the relation between h_G and h^* using the following class of countable graphs:



Here: r_i = the number of paths from state i to state $i+1$, and l_i = the number of arcs between state i and state 0 (as shown above). A graph of this form is indexed by two nonnegative integer sequences $\{r_n\}$, $\{l_n\}$ and is denoted by $\Gamma(\{r_n\}, \{l_n\})$.

Let Γ_0 be any connected countable graph (not necessarily of the form above) and let s be a state of Γ_0. Define $C_s^{(n)}$ = the number of sequences $ss_{i_1}\cdots s_{i_{n-1}}$ in Γ_0 such that $s_j \neq s$ for $j = i_1, \ldots, i_{n-1}$, and recall that $T_s^{(n)}$ = the number of sequences of length n in Γ_0 that start with s. Let $C(\Gamma_0, s, z) = \sum_n C_s^{(n)} z^n$, $T(\Gamma_0, s, z) = \sum_n T_s^{(n)} z^n$, and $Q_0(A_0)$ be the radius of convergence of $C(\Gamma_0, s, z)(T(\Gamma_0, s, z))$.

Towards proving the main result of this section, we use the following Lemmas:

LEMMA (3.1). *For a connected graph Γ_0, we have*

$$h^*(\Gamma_0) = \max\{\log Q_0^{-1}, h_G(\Gamma_0)\}.$$

Proof. Let s be a state in Γ_0. Then $T_s^{(n)} = \sum_{i=1}^n B_s^{(n-i)} C_s^{(i)}$, and $A_0 = \min(R_0, Q_0)$. Since $h^*(\Gamma_0) = \log A_0^{-1}$, $h_G(\Gamma_0) = \log R_0^{-1}$, the result follows. □

LEMMA (3.2). *Let $\Gamma_0(\{r_n\}, \{l_n\})$ be a graph with associated sequence $\{C_0^{(n)}\}$. Then*

(i) *If* $\lim_n \frac{1}{n} \log \prod_{i=1}^n r_i = \log \beta$, *then* $\lim_n \frac{1}{n} \log C_0^{(n)} = \log \beta$.

(ii) $h^*(\Gamma_0) = \max\{\log \beta, h_G(\Gamma_0)\}$.

Proof. Let $n_i = l_i + (i+1)$, $i = 0, 1, 2, \ldots$, and $W_k = \prod_{i=1}^k r_i$, $i = 1, 2, \ldots$ ($W_0 = 1$). To count the number of sequences of length k in Γ_0 that start with 0, let $S_k = \{i = 0, 1, \ldots, k-1 \mid n_i \geq k\}$, then $C_0^{(k)} = \sum_{i \in S_k} W_i$ and $T_0^{(n)} = \sum_{i=1}^n B_0^{(n-i)} C_0^{(i)}$. Noting that $W_k \leq C_0^{(k)} \leq \sum_{i=1}^k W_i$, and $\lim_n \frac{1}{n} \log \prod_{i=1}^n r_i = \lim_n \frac{1}{n} \log W_n = \log \beta$, then $\lim_n \frac{1}{n} \log C_0^{(n)} = \log \beta$. That shows (i) and (ii) follows by Lemma (3.1). □

LEMMA (3.3). *Let $m < x < (m+1)$. Then there is a sequence $\{r_n\}$ such that*

(i) $r_n = m$ *or* $m+1$,

(ii) *for all* n:

$$\prod_{i=1}^n r_i \leq x^n < \left(\frac{m+1}{m}\right) \prod_{i=1}^n r_i,$$

and

(iii) $\lim_n \frac{1}{n} \log \prod_{i=1}^n r_i = \log x$.

For a graph $\Gamma_0(\{r_n\}, \{l_n\})$, let $W_n = \prod_{i=1}^n r_i$, and let W_0 be the radius of convergence of the series $\sum_n W_n z^n$. Then by Lemma (3.2) we have

LEMMA (3.4). *For $\Gamma_0(\{r_n\}, \{l_n\})$, if $L_0 = W_0$, then $h_G(\Gamma_0) = h^*(\Gamma_0)$.*

LEMMA (3.5). *For $\Gamma_0(\{r_n\}, \{l_n\})$, if $\lim_n \frac{1}{n} \log \prod_{i=1}^n r_i = \log W_0^{-1}$ and $\{l_n\}$ is bounded (say by k), then $L_0 = W_0$ and $h_G(\Gamma_0) = h^*(\Gamma_0)$.*

Proof. Note that $f_0^{(n)} \leq \sum_{i=0}^n W_i$, hence $W_0 \leq L_0$. To show that $L_0 \leq W_0$, consider the sequence $\{n_j \mid l_{n_j} \neq 0\}$. Then for every j we have $f_0^{(n_j+l_{n_j})} \geq W_{n_j}$, hence

$$\overline{\lim_j} \frac{1}{n_j} \log f_0^{(n_j+l_{n_j})} \geq \underline{\lim_j} \frac{1}{n_j} \log W_{n_j} = \log W_0^{-1},$$

and $L_0 \leq W_0$. By Lemma (3.4) we have $h_G(\Gamma_0) = h^*(\Gamma_0)$. □

LEMMA (3.6). *Let $\Gamma_0(\{r_n\}, \{l_n\})$ be such that $\{r_n\}$ is a special sequence satisfying the conditions of Lemma* (3.3) *and $\{l_n\}$ is bounded (say by k). Then $h_G(\Gamma_0) = h^*(\Gamma_0)$ and Γ_0 is positive recurrent.*

Proof. As before consider the sequence $\{n_j \mid l_{n_j} \neq 0\}$. Then for every j we have

$$W_{n_j} = \prod_{i=1}^{n_j} r_i \leq f_0^{(n_j+l_{n_j})} \quad \text{and} \quad \frac{W_{n_j}}{x^{n_j+l_{n_j}}} \leq \frac{f_0^{(n_j+l_{n_j})}}{x^{n_j+l_{n_j}}}.$$

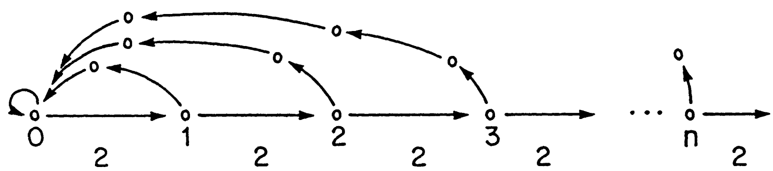
By (ii) of Lemma (3.3), we have $W_{n_j}/x^{n_j} > m/(m+1) > 0$. Since $\{l_n\}$ is bounded, we have

$$\lim_{j\to\infty} \frac{f_0^{(n_j+l_{n_j})}}{x^{n_j+l_{n_j}}} \geq \lim_{j\to\infty} \frac{W_{n_j}}{x^{n_j}} \left(\frac{1}{x^{l_{n_j}}}\right) > 0,$$

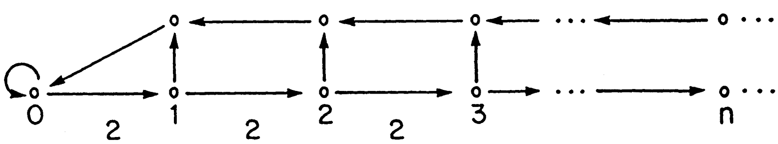
hence $F(\Gamma_0, 0, x^{-1}) = \infty$. By the proof of Lemma (3.5) we note that $L_0 = x^{-1}$, hence $F(\Gamma_0, 0, L_0) = \infty$ and by Theorem (2.3) Γ_0 is positive recurrent. □

We consider two examples. The first shows that positive recurrence and equality of h_G and h^* can happen without $\{l_n\}$ being bounded. The second shows we may have positive recurrence with $h_G < h^*$.

EXAMPLE (3.7). Let $\Gamma_0(\{r_n\}, \{l_n\})$ be given by



That is $r_n = 2$ for all $n \geq 1$ and $l_n = n + 1, n = 0, 1, 2, \ldots$. Γ_0 may be "represented" in a locally finite form by Γ_0' as follows:



(These two graphs have the same values of h_G and h^*.)

Note that $L_0 = 1/\sqrt{2}$ and $F(\Gamma_0, 0, L_0) = \infty$. Thus, by Theorem (2.3) Γ_0 is positive recurrent. Also $F(\Gamma_0, 0, \frac{1}{2}) = 1$, hence $h_G(\Gamma_0) = \log 2$. Finally, since $W_0 = \frac{1}{2}$, then by Lemma (3.2) $h^*(\Gamma_0) = \log 2$.

EXAMPLE (3.8). Let $\Gamma_1(\{r_n\}, \{l_n\})$ be the same as the Γ_0 in Example (3.7), except with the loop based at 0 removed. Again by the same argument as in Example (3.7), Γ_1 is positive recurrent. Since $\Gamma_1 < \Gamma_0$, then by Theorem (2.3) $h_G(\Gamma_1) < h_G(\Gamma_0) = \log 2$. By Lemma (3.2), $h^*(\Gamma_1) = \log 2$.

THEOREM (3.9). *Given $1 < \alpha \leq \beta$, there exists a recurrent uniformly locally finite graph Γ_0 such that $h_G(\Gamma_0) = \log \alpha$ and $h^*(\Gamma_0) = \log \beta$. If $\alpha = \beta$, then the Γ_0 that we construct is positive recurrent.*

Proof. Let $m < \beta \leq m + 1$, and choose a sequence $\{r_n\}$ satisfying the conditions of Lemma (3.3) with β replacing x. We construct the kth partial sum, S_k, for the loop series of Γ_0 as follows: Let $n_1 = 1$ and set $S_1(\alpha) \equiv 1/\alpha < 1$. Let n_2 be the smallest integer such that

$$S_2(\alpha) = S_1(\alpha) + \frac{r_1}{\alpha^{n_2}} < 1.$$

Note that $n_2 \geq 2$. Assuming that we have $n_1 < n_2 < \cdots < n_k$, let n_{k+1} be the smallest integer such that

$$S_{k+1}(\alpha) = S_k(\alpha) + \frac{\prod_{i=1}^{k} r_i}{\alpha^{n_{k+1}}} < 1,$$

and note that $n_{k+1} > n_k$.

We construct Γ_0 as follows: Let $l_0 = 1$, and $l_i = n_{i+1} - i$, $i = 1, 2, \ldots$. Then Γ_0 is the graph indexed by the sequences $\{r_n\}, \{l_n\}$, and is denoted by $\Gamma_0(\{r_n\}, \{l_n\})$.

Let $S(x) = \lim_{k\to\infty} S_k(x)$. Note that $S(x)$ is the loop series for the graph $\Gamma_0(\{r_n\}, \{l_n\})$, and $S(\alpha) = 1$. To show that $h_G(\Gamma_0) = \log\alpha$, let R_0 be such that $h_G(\Gamma_0) = \log R_0^{-1}$. Thus $F(\Gamma_0, 0, \alpha^{-1}) = 1$ and $F(\Gamma_0, 0, R_0) \le 1$. Hence $R_0 \le \alpha^{-1}$. If in fact $R_0 < \alpha^{-1}$, then we must have $F(\Gamma_0, 0, R_0) < 1$. In this case Γ_0 is transient, hence by Theorem (2.3) $R_0 = L_0 < \alpha^{-1}$, contradicting that L_0 is the radius of convergence of the loop series. Hence $R_0 = \alpha^{-1}$ and $h_G(\Gamma_0) = \log\alpha$. By Lemma (3.2) we have $h^*(\Gamma_0) = \log\beta$. Note that if $\alpha < \beta$, then by Lemma (3.5) $\{l_n\}$ is not bounded, hence Γ_0 may be constructed to be a locally finite graph as shown in Example (3.7). Also $\{l_n\}$ is monotone nondecreasing. To show that Γ_0 may be constructed to be uniformly locally finite, we show that if for some i we have $l_i = l_{i+T}$, then $(1+T) < \alpha(m+1)/m$. Equivalently, if for some i we have $n_{i+T} = n_i + T$, then $(1+T) < \alpha(m+1)/m$.

Let T be the greatest integer such that $n_{1+T} = n_1 + T = 1 + T$. Then

$$S_{1+T}(\alpha) = \frac{1}{\alpha} + \cdots + \frac{\prod_{i=1}^{T} r_i}{a^{1+T}} \ge \frac{1}{\alpha}\left(\frac{m}{m+1}\right) + \frac{1}{\alpha}\left(T\left(\frac{m}{m+1}\right)\right).$$

Since $S_{1+T}(\alpha) < 1$, we have $(1+T) < \alpha(m+1)/m$.

In general, let k be an integer such that $n_{k+1} > n_k + 1$, and T be the greatest integer such that $n_{k+1+T} = n_{k+1} + T$. Then

$$\frac{\prod_{i=1}^{k} r_i}{\alpha^{n_{k+1}}} + \cdots + \frac{\prod_{i=1}^{k+T} r_i}{\alpha^{n_{k+1+T}}} < \frac{\prod_{i=1}^{k} r_i}{\alpha^{n_{k+1}-1}}.$$

Hence, $(1+T) < \alpha(m+1)/m$.

Let d be the greatest integer such that $d \le (m+1)/m$. Then, by construction, $\{l_n\}$ is monotone nondecreasing with the property that it cannot stay constant for more than d consecutive times. Then, by using the same idea as in Example (3.7), it is easy to see that Γ_0 may be constructed to be a uniformly locally finite graph.

Finally, we show that if $\alpha = \beta$ then Γ_0 is positive recurrent. Note that for every n we have

$$\prod_{i=1}^{n} r_i = f_0^{(n+l_n)}.$$

Thus

$$\begin{aligned}\operatorname{Log} L_0^{-1} &= \overline{\lim_n} \frac{1}{n+l_n} \log f_0^{(n+l_n)} \\ &= \frac{1}{1+C} \log W_0^{-1}, \quad \text{where } C = \underline{\lim_n} \frac{l_n}{n}.\end{aligned}$$

Thus we have $L_0 = W_0^{1/(1+C)}$. Now, for every n we have $l_n \geq n/d$, $C = \underline{\lim}_n (l_n/n) \geq 1/d > 0$, and $W_0 < L_0$. Since $\alpha = \beta$, $R_0 = W_0$, we have $R_0 < L_0$ and Γ_0 is positive recurrent. □

This result shows that in the Countable-alphabet case, h_G and h^* can be anything, unlike the finite-alphabet case, where the possible entropies are exactly the Perron numbers (see [7]).

4. Computational examples. In this section we present some computational results of the topological entropy for certain classes of examples of chains. Consider a chain X over $Z \times Z$ with the following transitions: $(x_1, y_1) \to (x_2, y_2)$ if either $x_2 = x_1 \pm 1$ or $y_2 = y_1 \pm 1$. We picture X as follows:

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ \ldots & 0 & \leftrightarrow & 0 & \leftrightarrow & 0 & \ldots \\ & \updownarrow & & \updownarrow & & \updownarrow & \\ \ldots & 0 & \leftrightarrow & 0 & \leftrightarrow & 0 & \ldots \\ & \updownarrow & & \updownarrow & & \updownarrow & \\ \ldots & 0 & \leftrightarrow & 0 & \leftrightarrow & 0 & \ldots \\ & \vdots & & \vdots & & \vdots & \end{array}$$

We compute the entropy of some subchains of X whose states are contained in some nice region of R^2. The following result will be used in the computation.

Let X be a chain with state space S, and $f\colon S \to C$. Let f_∞ be the corresponding map of X generated by f and $Y = f_\infty(X)$. For every $n \geq 1$, let $BX(n,s)$ $(BY(n,c))$ be the set of n-blocks in X (Y) starting with $s \in S$ $(c \in C)$; and let f_n be the corresponding n-block map generated by f. For $B \in BY(n,c)$ let $f_n^{-1}(B) = \{B' \mid B' \in BX(n,s)$ and $f_n(B') = B\}$.

With these notations we have:

PROPOSITION (4.1). *Let X and $Y = f_\infty(X)$ be connected chains. If for $i = 1, 2$ there exists $c_i \in C$ and $s_i \in f^{-1}(c_i)$ such that*

(i) *for every $n \geq 1$, $f_n(BX(n, s_1)) = BY(n, c_1)$, and*

(ii) $\{\#f_n^{-1}(B)\}$ *is bounded as* B *varies over all blocks in* Y *starting with* c_2, *then* $h^*(X) = h^*(Y)$.

Proof. For $s \in S$ and $c \in C$, let $T_s(n)$ ($T'_c(n)$) be the number of blocks in X (Y) starting with s (c). Then

$$h^*(Y) = \overline{\lim_n} \frac{1}{n} \log T'_{c_1}(n) \le \overline{\lim_n} \frac{1}{n} \log T_{s_1}(n) = h^*(X).$$

On the other hand, if the bound on $\#f_n^{-1}(B)$ is k, then we have

$$h^*(X) = \overline{\lim_n} \frac{1}{n} \log T_{s_2}(n) \le \overline{\lim_n} \frac{1}{n} \log(k \cdot T'_{c_2}(n)) = h^*(Y). \qquad \square$$

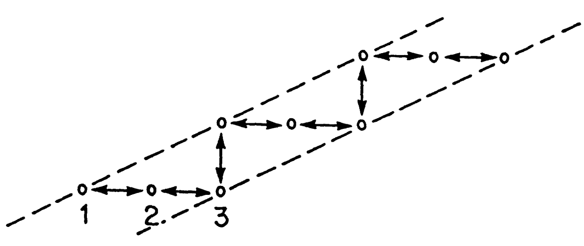
REMARK (4.2). Proposition (4.1) holds for the following two more general cases:

(i) f is an m-block map of X.

(ii) X and $Y = f_\infty(X)$ are both transitive symbolic dynamical systems.

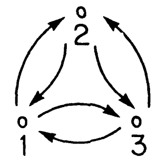
EXAMPLE (4.3). *Chains contained in the domain* $D(m,n,w)$. Let $D(m,n,w)$ be the domain in R^2 bounded by two parallel lines of slope m/n with horizontal distance w. We assume that one of these lines passes through $(0,0)$. Let $\Gamma(m,n,w)$ be the largest subchain of X contained in $D(m,n,w)$. To compute the h^*-entropy of $\Gamma(m,n,w)$, we note that it consists of a "fundamental" finite chain Γ_0 which repeats itself periodically in an obvious way giving $\Gamma(m,n,w)$. Γ_0 may be chosen to include the states (x,y) in $\Gamma(m,n,w)$ such that $y = 1,2,\ldots,m$. Let c be the number of states in Γ_0 and consider the following ordering or labelling of these states by $1,2,\ldots,c$ as follows: $(x_1,y_1) < (x_2,y_2)$ if $y_1 = y_2$ and $x_1 < x_2$ or $y_1 > y_2$. If we repeat this labelling periodically in an obvious way to cover all states of $\Gamma(m,n,w)$, we obtain a one block map, f, from the states of $\Gamma(m,n,w)$ to the set $\{1,\ldots,c\}$. The map f_∞ generated by f gives a finite chain $\Gamma_F(m,n,w)$ over $\{1,\ldots,c\}$ and f_∞ satisfies the condition of Proposition (4.1). Hence we have $h^*(\Gamma(m,n,w) = h^*(\Gamma_F(m,n,w))$.

For example consider $\Gamma(1,2,2)$:



the fundamental chain is given by $\circ \rightleftarrows \circ \rightleftarrows \circ$ and may be labelled by 1, 2 and 3. $\Gamma_F(1,2,2)$ is given by

$\Gamma_F(1,2,2)$:



In general, the transition matrix for $\Gamma(1,n,w)$ is given by the $(w+1)\times(w+1)$ matrix $J_{w+1}^{(n)}$, where

$$J_{w+1}^{(n)}(ij) = 1 \quad \text{if } |i-j| = 1 \text{ or } n, w \geq n > 1.$$

We note that for $w < n$, $h^*(\Gamma(1,n,w)) = 0$. Also, for $n = 1$, we have

$$J_{w+1}^{(1)}(ij) = 2 \quad \text{for } |i-j| = 1.$$

Computation of h_G for $\Gamma(m,n,w)$ is generally harder. If we like to compute h_G from the definition, then we should be able to identify h_G for suitable subchains of $\Gamma(m,n,w)$. This may be easily done in two cases: $\Gamma(1,1,w)$ and the infinite vertical rectangular strip denoted here by $\Gamma(1,0,w)$.

EXAMPLE (4.4). *h_G for $\Gamma(1,1,w)$.* Consider the chain $\Gamma_l(1,n)$ given by

$\Gamma_l(1,n)$:

Let $\Gamma_l(n\otimes m)$ be the Cartesian product of $\Gamma_l(1,n)$ and $\Gamma_l(1,m)$. That is, $\Gamma_l(n\otimes m)$ is the chain over $\{1,\dots,n\}\times\{1,\dots,m\}$ with the following transitions: $(x_1,y_1)\to(x_2,y_2)$ if $x_1\to x_2$ in $\Gamma_l(1,n)$ and $y_1\to y_2$ in $\Gamma_l(1,m)$. If T_n and T_m are the transition matrices for $\Gamma_l(1,n)$ and $\Gamma_l(1,m)$, respectively, then $T_{n\otimes m} = T_n \otimes T_m$ is the transition matrix for $\Gamma_l(n\otimes m)$ (in some order of the state space of $\Gamma_l(n\otimes m)$), and $h_G(\Gamma_l(n\otimes m)) = h_G(\Gamma_l(1,n)) + h_G(\Gamma_l(1,m))$. Note that $\Gamma_l(n\otimes m)$ is the disjoint union of two chains, and the entropy of $\Gamma_l(n\otimes m)$ is the maximum of the entropy of these two chains. Since any finite subgraph of $\Gamma(1,1,w)$ is contained in a component of $\Gamma_l((w+1)\otimes m)$ for some integer m, we have

$$h_G(\Gamma(1,1,w)) = \log\left(2\times 2\cos\frac{\pi}{w+2}\right).$$

An application of Theorem (2.1) shows that $\Gamma(1,1,w)$ is transient.

EXAMPLE (4.5). *h_G for $\Gamma(1,0,w)$ and the Cartesian sum of two chains.* The chain $\Gamma(1,0,w)$ is given as

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
0 & \leftrightarrow & 0 & \leftrightarrow \dots \leftrightarrow & 0 \\
\updownarrow & & \updownarrow & & \updownarrow \\
0 & \leftrightarrow & 0 & \leftrightarrow \dots \leftrightarrow & 0 \\
(0,0) & & (1,0) & & (W-1,0) \\
\vdots & & \vdots & & \vdots
\end{array}
$$

In order to compute $h_G(\Gamma(1,0,w))$ we find an easy way to compute the topological entropies of some finite rectangular chains, that is chains of the form

$$
\begin{array}{ccccc}
(1,m) & & (2,m) & & (n,m) \\
0 & \leftrightarrow & 0 & \leftrightarrow \dots \leftrightarrow & 0 \\
\updownarrow & & \updownarrow & & \updownarrow \\
\vdots & & \vdots & & \vdots \\
0 & \leftrightarrow & 0 & \leftrightarrow \dots \leftrightarrow & 0 \\
\updownarrow & & \updownarrow & & \updownarrow \\
0 & \leftrightarrow & 0 & \leftrightarrow \dots \leftrightarrow & 0 \\
(1,1) & & (2,1) & & (n,1)
\end{array}
$$

This is done using an idea we introduce as follows. Let Γ_1 and Γ_2 be finite chains with state space U and V respectively. We define the *Cartesian sum* $\Gamma_1 \oplus \Gamma_2$ to be the chain over $U \times W$ with the following transitions; $(u_1, v_1) \to (u_2, v_2)$ if $u_1 \to u_2$ in Γ_1 and $v_1 = v_2$ or $v_1 \to v_2$ in Γ_2 and $u_1 = u_2$.

Let T_1 and T_2 be the transition matrices for Γ_1 and Γ_2 respectively. Order $U \times V$ according to the following: $(u_1, v_1) < (u_2, v_2)$ if $v_1 < v_2$ or $v_1 = v_2$ and $u_1 < u_2$. According to this order, the transition matrix T of $\Gamma_1 \oplus \Gamma_2$ is given by $T = T_1 \otimes I_m + I_n \otimes T_2$. The matrix T satisfies the following two properties: (1) The eigenvalues of T are given by $\alpha+\beta$ where α and β are eigenvalues for T_1 and T_2 respectively, (2) the eigenvectors of T are given by $X \otimes Y$, where X and Y are eigenvectors for T_1 and T_2 respectively.

Now, the finite rectangular chain given above is in fact $\Gamma_l(1,n) \oplus \Gamma_l(1,m)$ and is denoted here by $\Gamma_l(n \oplus m)$. It follows from the previous discussion that

$$h_G(\Gamma_l(n \oplus m)) = \log\left(2\cos\frac{\pi}{n+1} + 2\cos\frac{\pi}{m+1}\right).$$

Direct application of the definition of h_G yields:

$$h_G(\Gamma(1,0,w)) = \log\left(2 + 2\cos\frac{\pi}{w+2}\right).$$

REMARK (4.6). Let D be a region in R^2 such that D contains arbitrarily large rectangles, and let Γ_D be the largest subchain of X contained in D. Then $h_G(\Gamma_D) = \log 4$. We also note that for any region D, we have $h_G(\Gamma_D) = h^*(\Gamma_D)$, as indicated in Petersen **[9]**. We note that the computation for the entropy of Γ_D, where D is a region bounded by two parallel lines of irrational slope, is hard. We feel that the methods introduced here are not going to work in this case. One possible approach may be through studying the continuity properties of h^* or h_G.

Finally, the entropies of the following examples may be computed by first constructing chains which factor onto them. For details see **[10]**.

EXAMPLE (4.7). *The sum-bounded systems* $X(n,m)$. Consider the closed shift-invariant set

$$X(n,m) = \left\{ \{x_i\}_{-\infty}^{\infty} \mid x_i = \pm 1, -n \le \sum_{l=k}^{K} x_l \le m, \right.$$

$$\left. k \text{ and } K \text{ are integers and } n, m > 0 \right\}.$$

Then for $n, m > 0$, the system $X(n,m)$ is sofic with topological entropy

$$h(X(n,m)) = \log\left(2\cos\frac{\pi}{\min(n,m)+2}\right).$$

EXAMPLE (4.8). *The sum-bounded-above systems* $X(-\infty, m)$. Consider the system $X(-\infty, m)$, where

$$X(-\infty,m) = \left\{ \{x_i\}_{-\infty}^{\infty} \mid x_i = \pm 1, \sum_{l=k}^{K} x_i \le m, \right.$$

$$\left. k \text{ and } K \text{ are integers and } m > 0 \right\}.$$

Then $h(X(-\infty,m)) = \log x^*$, where x^* is the largest root of the equation $(1-x)f_{m+1}(x) + f_{m-1}(x) = 0$, and

$$f_m(x) = (-1)^m \sum_{k=0}^{[m/2]} (-1)^k \binom{n-k}{k} x^{n-2k}.$$

EXAMPLE (4.9). *The sum-bounded run-length-limited systems* $X(n, m; l, k)$. Let $x \in \{-1, 1\}^Z$ and $B_r = x_1 \cdots x_r$ be an r-block in x. If $x_1 = \cdots = x_r$, then B_r is said to be a run of length r of x. B_r is a positive (negative) run if $x_1 = \cdots = x_r = 1$ ($x_1 = \cdots = x_r = -1$). A subblock B of B_r is said to be an end run for B_r if $B_r = x_1 \cdots x_i B$, and B is a run of maximal length, that is B is a run by $x_i B$ is not a run.

For n, m, l and k positive integers we define

$$X(n, m; l, k) = \{x = \{x_i\}_{-\infty}^{\infty} \mid x_i = \pm 1, n \leq \text{ the length of a positive run in } x \leq m, l \leq \text{ the length of a negative run in } x \leq k\}.$$

Then, $h(X(n, m; l, k)) = \log x^*$, where x^* is the largest root of the equation $\sum_{i=n+l}^{m+k} a_i/x^i = 1$ (see Petersen [9]), and the a_i's are given as follows:

Let $M = \min((k-l)+1, (m-n)+1)$. Then

$$a_i = \begin{cases} i - (n+l) + 1 & \text{if } n + l \leq i \leq (n+1) + (M-2), \\ M & \text{if } (n+l) + (M-1) \leq i \leq (m+k) - (M-1), \\ a_{(l+n)+(k+n)-i} & \text{if } (m+k) - (M-2) \leq i \leq m + k, \\ 0 & \text{otherwise.} \end{cases}$$

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NORTH CAROLINA CENTRAL UNIVERSITY
DURHAM, NC 27707