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# ASYMPTOTIC EXPANSION AT A CORNER FOR THE CAPILLARY PROBLEM

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**Consider the solution of capillary surface equation over domains with corners. It is shown that there exists an asymptotic expansion of the solution at the corner if the corner angle $2\alpha$ satisfies $0 < 2\alpha < \pi$ and $\alpha + \gamma > \pi/2$ where $0 < \gamma \leq \pi/2$ is the contact angle between the surface and the container wall. It is assumed that the corner is bounded by lines. The leading terms of the expansion are calculated and properties of the remainder are given.**

**1. Introduction and results.** We consider the non-parametric capillary problem in the presence of gravity. One seeks a surface $S\colon u = u(x)$, defined over a bounded base domain $\Omega \subset \mathbf{R}^2$, such that $S$ meets vertical cylinder walls over the boundary $\partial\Omega$ in a prescribed constant angle $\gamma$. This problem leads to the equations, see Finn [2],

$$\operatorname{div} Tu = \kappa u \quad \text{in } \Omega \tag{1.1}$$

$$\nu \cdot Tu = \cos\gamma \text{ on the smooth parts of } \partial\Omega, \tag{1.2}$$

where

$$Tu = \frac{Du}{\sqrt{1 + |Du|^2}},$$

$\kappa = \text{const.} > 0$ and $\nu$ is the exterior unit normal on $\partial\Omega$.

Let the origin $x = 0$ be a corner of $\Omega$ with the interior angle $2\alpha$ satisfying

$$0 < 2\alpha < \pi. \tag{1.3}$$

For simplicity, let us assume that the corner is bounded by lines near $x = 0$, see Figure 1.

We choose $0 < R_0 < 1$ small enough such that $\Omega_{R_0} = \Omega \cap B_{R_0}$ with $B_R = \{x \in \mathbf{R}^2/x_1^2 + x_2^2 < R\}$ is a circular sector. Furthermore, we assume that

$$0 < \gamma < \pi/2 \tag{1.4}$$

and

$$\alpha + \gamma > \pi/2. \tag{1.5}$$

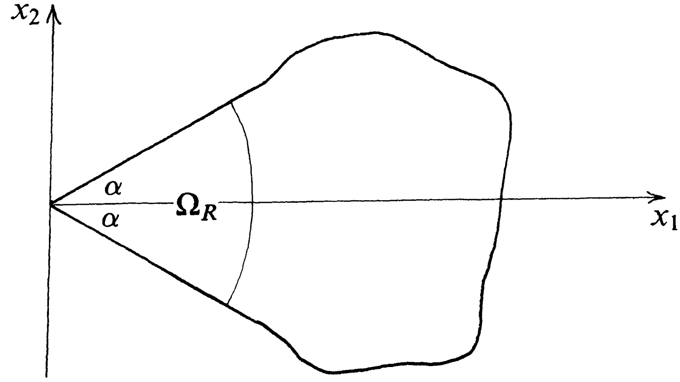


FIGURE 1

Concus and Finn [**1**] have shown that $u$ is bounded near 0 if and only if $\alpha+\gamma \geq \pi/2$. Using this result, L. Simon [**9**] proved that

$$u \in C^1(\overline{\Omega}_R) \quad \text{for each } R < R_0, \tag{1.6}$$

provided (1.4) and (1.5) are satisfied.

Similar to [**7**] we show, using the divergence structure of the left hand side of (1.1), that the assumptions (1.3)–(1.5) imply $u \in C^{1,\varepsilon}$ up to the corner. We produce an auxiliary Dirichlet problem and apply results of [**6**], see also [**7**], for the Dirichlet problem of quasilinear elliptic equations of second order over domains with corners.

Recently, Lieberman [**4**] has shown for (1.1), (1.2) that $u \in C^{1,\varepsilon}$ up to the corner. His method rests on a barrier construction for Neumann type problems. In [**7**] we have proved this result when the quantity $\kappa u$ in (1.1) is replaced by a constant. The method of [**7**] works also for (1.1), (1.2), see §2.

Once one has established that the solution of the auxiliary Dirichlet problem is in $C^{1,\varepsilon}$ then the existence of the asymptotic expansion follows from results of [**8**], see §4.

Define

$$k = \frac{\sin\alpha}{\cos\gamma} \tag{1.7}$$

and

$$a^2 = 1 - k^{-2} \tag{1.8}$$

The mapping

$$x_1 = ay_1, \qquad x_2 = y_2 \tag{1.9}$$

transforms $\Omega$ onto a new domain which we denote by $\Omega$ again. The reason for this change of variables is that the quantities $d_{ij}(Dw, DQ)$ which occur in the leading coefficients of the quasilinear equation (3.5) satisfy $d_{ij}(0,0) = 0$. Under this assumption one can start with the asymptotic expansion according to [**8**], see §4.

Let $2\omega$ be the new interior angle at the corner $y = 0$. An easy calculation shows that $0 < \omega < \pi/2$ is given by

$$\operatorname{tg}\omega = a \operatorname{tg}\alpha \tag{1.10}$$

and that $2\omega$ is the angle at the corner of the surface $S$ about $x = 0$.

We set

$$\alpha_1 = \pi/2\omega.$$

Using polar coordinates $r$, $\theta$ in the $y$-system, then for a given integer $l$, $l \geq 1$, there exist $N$ real numbers $\min(\alpha_1, 2) = \kappa_1 < \cdots < \kappa_N$ and nonnegative integers $k_j$ such that near $y = 0$ the following expansion takes place:

$$\begin{aligned} u(y) &= u(0) - k^{-1} r\cos\theta \\ &\quad + \sum_{j=1}^{N}\sum_{s=0}^{k_j} r^{\kappa_j} \ln^s r \psi_{js}(\theta) + v_l(y), \end{aligned} \tag{1.11}$$

where $v_l \in C^{l,\lambda}$, $0 < \lambda < 1$, up to the corner, $v_l = o(r^{\kappa_N})$ and the $\psi_{js}(\theta)$ are regular functions defined on $[-\omega, \omega]$, see §4.

In particular, one has for the solution $u(x)$ of (1.1), (1.2):

$$u(x) = u(0) - \frac{1}{\sqrt{k^2 - 1}} x_1 + v(x) \tag{1.12}$$

in the case $1 < \alpha_1 \leq 2$, where $v \in C^{1,\lambda}(\overline{\Omega}_R)$, $0 < \lambda < 1$, and $v = o(|x|^{\alpha_1 - \varepsilon})$.

If $\alpha_1 > 2$, then

$$u(x) = u(0) - \frac{1}{\sqrt{k^2 - 1}} x_1 + \frac{\kappa}{4a} u(0) \left(\frac{x_1^2}{a^2} + x_2^2\right) + v \tag{1.13}$$

holds, where $v \in C^{2,\lambda}(\overline{\Omega}_R)$ and $v = o(|x|^{\min(\alpha_1, 3) - \varepsilon})$.

Moreover, the inequality $d < \alpha_1 \leq d + 1$ for an integer $d \geq 2$ implies that $u \in C^{d,\lambda}(\overline{\Omega}_R)$, see Theorem 4.2.

If the inequality (1.5) is replaced by $\alpha + \gamma < \pi/2$, then a solution of (1.1), (1.2) is unbounded near the origin, see Concus and Finn [**1**] or Finn [**2**, Chapter 5.2]. Furthermore, in these works the first term of a possible asymptotic expansion was given. In the borderline case

$\alpha+\gamma=\pi/2$ Tam **[10]** obtained that the normal vector to the surface $S$ is continuous up to the corner.

In both cases, the question of existence of an asymptotic expansion is an open problem.

**2. The associated Dirichlet problem.** The following class of functions is useful for our purpose, see **[8]** and also Maz'ya and Plamenevskiĭ **[5**, p. 65**]** for a related class.

DEFINITION 2.1. For a real $\gamma$ and a nonnegative integer $m$ a function $V$ is said to be of class $A^{\gamma}_{m,\lambda}(\Omega_R)$ if $V\in C^m(\overline{\Omega}_R\backslash\{0\})$ and if there exists a constant $0<\lambda<1$ such that for $0\le |l|\le m$:

(i)

$$|D^lV|\le c|x|^{\gamma-|l|}\quad \text{for } x\in\overline{\Omega}_R\backslash\{0\},$$

(ii)

$$|D^lV(\bar{x})-D^lV(\bar{\bar{x}})|$$
$$\le c|\bar{x}-\bar{\bar{x}}|^{\lambda}\begin{cases}[\max(|\bar{x}|,|\bar{\bar{x}}|)]^{\gamma-|l|-\lambda} & \text{if } \gamma-|l|-\lambda>0,\\ [\min(|\bar{x}|,|\bar{\bar{x}}|)]^{\gamma-|l|-\lambda} & \text{if } \gamma-|l|-\lambda\le 0\\ & \text{for } \bar{x},\bar{\bar{x}}\in\overline{\Omega}_R\backslash\{0\}.\end{cases}$$

The constants $c$ may depend on $R$, $l$, $\gamma$ and $\lambda$.

For brevity we write $V^{\gamma}_{m,\lambda}$ for functions belonging to $A^{\gamma}_{m,\lambda}(\Omega_R)$.

Let $H(x)=(H_1(x),H_2(x))$ be a $C^{1,\varepsilon}(\overline{\Omega}_R)$ vector field which satisfies

$$\begin{aligned} D_iH_i&=\kappa u \quad \text{in } \Omega_R,\\ \nu_iH_i&=0 \quad\ \ \text{on } \Sigma_R,\end{aligned} \tag{2.1}$$

where $\Sigma_R=(\partial\Omega\cap B_R)\backslash\{0\}$.

Here and subsequently we use the summation convention. In §3 we show that there exists a solution of (2.1) satisfying

$$H_1=\frac{\kappa}{2}u(0)x_1+V^{2-\varepsilon}_{1,1-\delta},\quad H_2=\frac{\kappa}{2}u(0)x_2+V^{2-\varepsilon}_{1,1-\delta} \tag{2.2}$$

for arbitrary $\varepsilon>0$, $\delta>0$. We choose $\varepsilon$ and $\delta$ close to 0.

Define

$$A_i=\frac{D_iu}{\sqrt{1+|Du|^2}};$$

then (1.1) implies $D_i(A_i-H_i)=0$ in $\Omega_R$. Consequently, we may introduce a stream function $\psi$ by setting

$$d\psi=(-A_2+H_2)\,dx_1+(A_1-H_1)\,dx_2.$$

Let

$$\psi(x) = \int_{\mathfrak{C}} d\psi,$$

where $\mathfrak{C}$ denotes a curve in $\overline{\Omega}_R$ connecting the origin with $(x_1, x_2) \in \overline{\Omega}_R$.

Since $\nu_i H_i = 0$ on $\Sigma_R$, the boundary condition (1.2) implies

$$\psi(x) = -k^{-1}x_2 \quad \text{on } \Sigma_R, \tag{2.3}$$

where $k$ is given by (1.7).

Thus, the nonlinear boundary condition (1.2) becomes a linear Dirichlet condition with a known right hand side. This is a point which simplifies the calculation of the asymptotic expansion, see §4.

Subsequently, subscripts are used to indicate partial derivatives, for example $\psi_{,i} = \partial\psi/\partial x_i$.

We set

$$F_1 = -\psi_{,1} + H_2, \quad F_2 = \psi_{,2} + H_1$$

and

$$F^2 = F_1^2 + F_2^2.$$

Because of

$$u \in C^1(\overline{\Omega}_{R_0}) \quad \text{and} \quad F^2 = \frac{|Du|^2}{1 + |Du|^2}$$

one has

$$\max_{\overline{\Omega}_R} F^2 < 1 \quad \text{for each } R \leq R_0. \tag{2.4}$$

The equations

$$\psi_{,1} = -A_2 + H_2, \quad \psi_{,2} = A_1 - H_1$$

imply

$$u_{x_1} = (1 - F^2)^{-1/2}F_2, \quad u_{x_2} = (1 - F^2)^{-1/2}F_1. \tag{2.5}$$

Hence, since $u_{x_1x_2} = u_{x_2x_1}$ in $\Omega_R$, one has in $\Omega_R$

$$[(1 - F^2)^{-1/2}F_1]_{x_1} = [(1 - F^2)^{-1/2}F_2]_{x_2}$$

or

$$\int_{\Omega_R} a_i(H(x), D\psi)D_i v \, dx = 0 \quad \text{for all } v \in C_0^1(\Omega_R), \tag{2.6}$$

$$\psi = -k^{-1}x_2 \quad \text{on } \Sigma_R,$$

where

$$a_1 = -(1 - F^2)^{-1/2}F_1 \quad \text{and} \quad a_2 = (1 - F^2)^{-1/2}F_2.$$

A simple calculation shows that

$$(1-F^2)^{-1/2}|\xi|^2 \le a_{ij}\xi_i\xi_j \le (1-F^2)^{-3/2}|\xi|^2$$

for all $\xi \in \mathbf{R}^2$, where $a_{ij} = \partial a_i/\partial q_j$, $q_j = \psi_{x_j}$, and the $a_{ij}$ are given by

$$a_{11} = (1-F^2)^{-3/2}(1-F_2^2), \quad a_{22} = (1-F^2)^{-3/2}(1-F_1^2) \quad \text{and}$$
$$a_{12} = a_{21} = -(1-F^2)^{-3/2}F_1F_2.$$

LEMMA 2.1. *There exists a positive constant $\lambda < 1$ such that $\psi \in C^{1,\lambda}(\overline{\Omega}_R)$ for each fixed $R < R_0$.*

*Proof*. We use the barrier function of [7] for

$$M\psi \equiv a_{ij}(H, D\psi)\psi_{,ij} + a_{i,H_j}(H, D\psi)H_{j,i} = 0 \tag{2.7}$$

instead as in [7] for $M_0\psi \equiv a_{ij}\psi_{,ij} = 0$.

In [7] the right hand side of (1.1) was replaced by a constant.

Since $H$ satisfies (2.2), we infer that for a constant $\mu > 0$

$$|\psi + k^{-1}x_2| \le c|x|^{1+\mu} \quad \text{in } \overline{\Omega}_R$$

by using (2.6) and (2.4), see [6] or [8, §3.1]. Then, the lemma follows from this inequality as in [6] or [8, §3.2].

**3. The transformed problem.** Define for $\omega$ of (1.10) $C = \{y \in \mathbf{R}^2/|\theta| < \omega\}$ and consider for a sufficiently regular function $h(\theta)$ defined on $[-\omega, \omega]$, for a nonnegative constant $\beta$ and a nonnegative integer $s$ the boundary value problem

$$\begin{aligned} \Delta Q &= h(\theta)r^\beta \ln^s r \quad \text{in } C, \\ \frac{\partial Q}{\partial \nu} &= 0 \quad \text{on } \partial C\setminus\{0\}. \end{aligned} \tag{3.1}$$

As in [8, Lemma 4.5] for the case of the corresponding Dirichlet problem, one shows also for (3.1)

LEMMA 3.1. (i) *If $(\beta+2) \ne n\alpha_1$ is satisfied for all integers $n \ge 1$, then a solution of (3.1) is given by*

$$Q = r^{\beta+2}\sum_{l=0}^{s} f_l(\theta)\ln^l r,$$

*where the functions $f_l$ are regular on $[-\omega, \omega]$ and satisfy $f_l'(-\omega) = f_l'(\omega) = 0$.*

(ii) *Let* $(\beta + 2) = n\alpha_1$ *for an integer* $n \geq 1$ *and*

$$\varphi_n(\theta) = \frac{1}{\sqrt{\omega}} \cos[n\alpha_1(\theta + \omega)].$$

*Then there exist constants* $c_l$ *such that a solution of* (3.1) *is given by*

$$Q = r^{\beta+2} \left[ \sum_{l=0}^{s} f_l(\theta) \ln^l r + \varphi_n(\theta) \sum_{l=0}^{s} c_l \ln^{1+l} r \right]$$

*with regular functions* $f_l$ *on* $[-\omega, \omega]$ *satisfying* $f_l'(-\omega) = f_l'(\omega) = 0$.

REMARK 3.1. If the right hand side of (3.1) is radially symmetric, that is $h = \text{const.}$, then a solution of (3.1) is given by the function $Q$ of (i) with constants $f_l$ since each radially symmetric function satisfies the boundary condition of (3.1).

REMARK 3.2. If the right hand side of (3.1) is a polynomial in $y_1$, $y_2$ of degree $q$ with $q + 2 < \alpha_1$, then there is a polynomial $P_{q+2}$ of degree $q + 2$ which is a solution of (3.1) and satisfies $P_{q+2} = O(|y|^2)$, see [**8**, §5] for the case of the related Dirichlet problem.

REMARK 3.3. In order to illustrate (ii) of Lemma 3.1, let $h(\theta)r^{n\alpha_1-2}$ be the right hand side of (3.1), where $n \geq 2$ is an integer. Then logarithmic terms occur in $Q$ if and only if

$$\int_{-\omega}^{\omega} h(\theta)\varphi_n(\theta)\, d\theta \neq 0.$$

A solution of (3.1) is given by

$$Q = r^{n\alpha_1}[f_0(\theta) + c_0\varphi_n(\theta) \ln r], \quad c_0 \neq 0.$$

We consider the problem

$$\begin{aligned} \Delta Q &= f(y) \quad \text{in } \Omega_{R_0}, \\ \frac{\partial Q}{\partial \nu} &= 0 \quad \text{on } \Sigma_{R_0}. \end{aligned} \tag{3.2}$$

As in [**8**, Lemma 4.4] for the corresponding Dirichlet problem, one proves

LEMMA 3.2. *Suppose that* $f \in A^{\gamma}_{m,\lambda}(\Omega_{R_0})$ *and* $\gamma > -1$. *Then there exists a solution* $Q \in A^{\gamma+2-\varepsilon}_{m+2,\lambda}(\Omega_R)$ *of* (3.2) *for each* $R$, $0 < R < R_0$, *and arbitrary* $\varepsilon > 0$.

From $u \in C^1(\overline{\Omega}_{R_0})$ one concludes that

$$u(y) - u(0) \in A^1_{0,1-\delta}(\Omega_{R_0})$$

for each $0 < \delta < 1$. Then, Lemmas 3.1 and 3.2 imply that

$$\begin{aligned} \Delta Q &= \kappa u \quad \text{in } \Omega_{R_0}, \\ \frac{\partial Q}{\partial \nu} &= 0 \quad \text{on } \Sigma_{R_0} \end{aligned} \tag{3.3}$$

possesses a solution

$$Q = \frac{\kappa}{4} u(0)(y_1^2 + y_2^2) + V, \quad \text{where } V \in A^{3-\varepsilon}_{2,1-\delta}(\Omega_R) \tag{3.4}$$

and $R < R_0$ may be chosen arbitrarily close to $R_0$.

We define $w$ by $\psi = w - k^{-1}x_2$ and set $H_1 = aQ_{y_1}$ and $H_2 = Q_{y_2}$. Multiplying the differential equation (2.7) by $(1-F^2)^{3/2}$, introducing $\psi$ and $H$ from above into (2.7), then, in conjunction with (2.3), we get after the mapping (1.9) the problem

$$\begin{aligned} \Delta w &= d_{ij}(Dw, DQ)w_{,ij} + G(Dw, DQ, D^2Q) \quad \text{in } \Omega_R, \\ w &= 0 \quad \text{on } \Sigma_R, \end{aligned} \tag{3.5}$$

where

$$d_{11} = \frac{1}{a^2}\left[-\frac{2}{k}(w_{,2} + aQ_{,1}) + (w_{,2} + aQ_{,1})^2\right],$$

$$d_{22} = \left(-\frac{1}{a}w_{,1} + Q_{,2}\right)^2,$$

$$d_{12} = d_{21} = -\frac{1}{ak}\left(-\frac{1}{a}w_{,1} + Q_{,2}\right) + \frac{1}{a}\left(-\frac{1}{a}w_{,1} + Q_{,2}\right)(w_{,2} + aQ_{,1})$$

and

$$\begin{aligned} G = aQ_{,12}&\left[\left(-\frac{1}{a}w_{,1} + Q_{,2}\right)^2 + \frac{2}{a^2k}(w_{,2} + aQ_{,1}) - \frac{1}{a^2}(w_{,2} + aQ_{,1})^2\right] \\ &+ (Q_{,11} - Q_{,22})\left[-\frac{1}{k}\left(-\frac{1}{a}w_{,1} + Q_{,2}\right)\right. \\ &\qquad\qquad \left. + \left(-\frac{1}{a}w_{,1} + Q_{,2}\right)(w_{,2} + aQ_{,1})\right]. \end{aligned}$$

After the mapping (1.9) we obtain from (2.5)

$$u_{y_1} = a(1-F^2)^{-1/2}F_2, \quad u_{y_2} = (1-F^2)^{-1/2}F_1, \tag{3.6}$$

where $F_1$ and $F_2$ are given by

$$F_1 = -\frac{1}{a}w_{,1} + Q_{,2} \quad \text{and} \quad F_2 = -\frac{1}{k} + w_{,2} + aQ_{,1}.$$

**4. Existence of the asymptotic expansion.** Under the assumptions of §1 the asymptotic expansion follows from (3.3), (3.5) and (3.6) by using results of [8].

Instead of these equations, we consider the more general system (4.1)–(4.3):

$$\text{(4.1)} \qquad \Delta w = d_{ij}(Dw, DQ)w_{,ij} + G(Dw, DQ, D^2Q) \quad \text{in } \Omega_R,$$
$$w = 0 \quad \text{on } \Sigma_R,$$
$$\text{(4.2)} \qquad \Delta Q = f(y, u) \quad \text{in } \Omega_R,$$
$$\frac{\partial Q}{\partial \nu} = 0, \quad \text{on } \Sigma_R,$$
$$\text{(4.3)} \qquad u_{y_1} = \chi_1(Dw, DQ)$$
$$u_{y_2} = \chi_2(Dw, DQ) \quad \text{in } \overline{\Omega}_R.$$

We assume that $d_{ij}(0,0) = 0$ and that the functions $d_{ij}$, $G$, $f$, $\chi_1$ and $\chi_2$ are sufficiently regular. For finite many real numbers $0 \le \kappa_1 < \kappa_2 < \cdots < \kappa_N$ and nonnegative integers $k_j$ we set

$$P^{(\kappa_1)} = \sum_{j=1}^{N} \sum_{s=0}^{k_j} r^{\kappa_j} \ln^s r \psi_{js}(\theta),$$

where $\psi_{js}$ denote regular functions defined on $[-\omega, \omega]$. From (1.6) we infer that

$$\text{(4.4)} \qquad u = P^{(0)} + V^1_{0,\lambda_1},$$

where $0 < \lambda_1 < 1$ may be chosen arbitrarily close to 1. Lemmas 3.1 and 3.2 imply the existence of a solution $Q$ of (4.2) such that

$$\text{(4.5)} \qquad Q = P^{(2)} + V^{3-\varepsilon}_{2,\lambda_1}.$$

Under the assumption $w \in C^{1,\lambda}$, $0 < \lambda < 1$, up to the corner we conclude from (4.1) that, see [8, Lemma 3.4],

$$\text{(4.6)} \qquad w \in A^{\min(\alpha_1, 2)-\varepsilon}_{2,\lambda}.$$

We set

$$\alpha_j = j\alpha_1 \quad \text{and} \quad v_j(\theta) = \frac{1}{\sqrt{\omega}} \sin[\alpha_j(\theta + \omega)]$$

for integers $j \ge 1$.

*The case* $1 < \alpha_1 \le 2$. Introducing $Q$ and $w \in A^{\alpha_1-\varepsilon}_{2,\lambda}$ into (4.1), we obtain

$$\Delta w = P^{(0)} + V^{2\alpha_1-3-\varepsilon}_{0,\lambda}.$$

Since $Q$ satisfies (4.5), there exist constants $c_j$ such that, see [**8**, §4.2],

$$\begin{aligned} w &= P^{(2)} + \sum_{\alpha_j < 2\alpha_1 - 1} c_j r^{\alpha_j} v_j(\theta) + V_{2,\lambda}^{2\alpha_1 - 1 - \varepsilon} \\ &= P^{(2)} + c_1 r^{\alpha_1} v_1(\theta) + V_{2,\lambda}^{2\alpha_1 - 1 - \varepsilon} \\ &= P^{(\alpha_1)} + V_{2,\lambda}^{2\alpha_1 - 1 - \varepsilon}. \end{aligned}$$

Now we introduce this $w$ and $Q$ of (4.5) into the equation (4.3) and see that $u = P^{(0)} + V_{2,\lambda}^{2\alpha_1 - 1 - \varepsilon}$. Using this $u$ and considering (4.2), Lemmas 3.1 and 3.2 yield

$$Q = P^{(2)} + V_{4,\lambda}^{2\alpha_1 + 1 - \varepsilon}. \tag{4.7}$$

Introducing this $Q$ and $w$ from above into (4.1), one obtains

$$w = P^{(\alpha_1)} + V_{4,\lambda}^{2\alpha_1 - 1 - \varepsilon}, \tag{4.8}$$

see [**8**, Lemma 3.4].

From (4.1) in conjunction with (4.7) and (4.8) it follows that

$$w = P^{(0)} + V_{2,\lambda}^{3\alpha_1 - 4 - \varepsilon}.$$

Then, according to [**8**, §4.2] we see that

$$\begin{aligned} w &= P^{(2)} + \sum_{\alpha_j < 3\alpha_1 - 1} c_j r^{\alpha_j} v_j(\theta) + V_{4,\lambda}^{3\alpha_1 - 2 - \varepsilon} \\ &= P^{(\alpha_1)} + V_{4,\lambda}^{3\alpha_1 - 2 - \varepsilon}. \end{aligned}$$

Finally, by induction it follows that

$$u = P^{(0)} + V_{2(n-1),\lambda}^{n(\alpha_1 - 1) + 1 - \varepsilon}.$$

*The case* $\alpha_1 > 2$. Using (4.4) and (4.5) in (4.1), we get from (4.6) in $\Omega_R$ the equation

$$w = P^{(0)} + V_{0,\lambda}^{1-\varepsilon}.$$

Again, from [**8**, §4.2] it follows that

$$w = P^{(2)} + V_{2,\lambda}^{\min(\alpha_1, 3) - \varepsilon}.$$

Introducing (4.5) and this $w$ into (4.3), we obtain in the case $\alpha_1 > 3$ that $u = P^{(0)} + V_{2,\lambda}^{3-\varepsilon}$. Thus, $Q = P^{(2)} + V_{4,\lambda}^{5-\varepsilon}$ is a solution of (4.2). This implies, see [**8**, Lemma 3.4], $w = P^{(2)} + V_{4,\lambda}^{3-\varepsilon}$. Substituting this $w$ and $Q$ into (4.1), we see that

$$w = P^{(0)} + V_{2,\lambda}^{2-\varepsilon}$$

which yields $w = P^{(2)} + V_{4,\lambda}^{\min(\alpha_1, 4) - \varepsilon}$.

Repeating this argument, we obtain

$$(4.9) \qquad Q = P^{(2)} + V^{d+2-\varepsilon}_{2(d-1),\lambda} \quad \text{and} \quad w = P^{(2)} + V^{\alpha_1-\varepsilon}_{2(d-1),\lambda},$$

provided the inequality $d < \alpha_1 \leq d+1$ is satisfied for an integer $d \geq 2$.

We now introduce $Q$ and $w$ from (4.9) into (4.1) and find that

$$w = P^{(0)} + V^{\alpha_1-1-\varepsilon}_{2(d-1)-2,\lambda}.$$

Using again a result of [**8**, §4.2], it follows that

$$\begin{aligned} w &= P^{(2)} + \sum_{\alpha_j < \alpha_1+1} c_j r^{\alpha_j} V_j(\theta) + V^{\alpha_1+1-\varepsilon}_{2(d-1),\lambda} \\ &= P^{(2)} + c_1 r^{\alpha_1} v_1(\theta) + V^{\alpha_1+1-\varepsilon}_{2(d-1),\lambda} \\ &= P^{(2)} + V^{\alpha_1+1-\varepsilon}_{2(d-1),\lambda}. \end{aligned}$$

From (4.3) we infer that

$$u = P^{(0)} + V^{\alpha_1+1-\varepsilon}_{2(d-1),\lambda}.$$

Again, using Lemmas 3.1 and 3.2, one sees that

$$Q = P^{(2)} + V^{\alpha_1+3-\varepsilon}_{2(d-1)+2,\lambda}$$

is a solution of (4.2). Taking this $Q$ into account in (4.1), one finds, see [**8**, Lemma 3.4],

$$w = P^{(2)} + V^{\alpha_1+1-\varepsilon}_{2(d-1)+2n,\lambda}$$

for each integer $n \geq 1$.

Collecting the above results, we have under the assumption $w \in C^{1,\lambda}(\overline{\Omega}_{R_0})$

THEOREM 4.1. *For a given integer $p \geq 0$ there exist $N$ real numbers $\kappa_j$, $\min(\alpha_1, 2) = \kappa_1 < \kappa_2 < \cdots < \kappa_N < \bar{\kappa}_p$, where*

$$\bar{\kappa}_p = \begin{cases} (p+2)(\alpha_1 - 1) + 1 & \text{if } \alpha_1 \leq 2, \\ \alpha_1 + p + 1 & \text{if } \alpha_1 > 2 \end{cases}$$

*such that*

$$u(y) = u(0) + u_{y_i}(0) y_i + \sum_{j=1}^{N} \sum_{s=0}^{k_j} r^{\kappa_j} \ln^s r \psi_{js}(\theta) + V,$$

*where $V \in A^{\bar{\kappa}_p-\varepsilon}_{\bar{m},\lambda}(\Omega_R)$ with arbitrary $0 < R < R_0$ and $\bar{m}$ is given by*

$$\bar{m} = \begin{cases} 2(p+1) & \text{if } \alpha_1 \leq 2, \\ 2(d-1) + 2(p+1) & \text{if } \alpha_1 > 2. \end{cases}$$

As sketched in [**8**, §5] for the corresponding Dirichlet problem in two-dimensional domains with corners, one obtains

THEOREM 4.2. *Let $d \geq 1$ be an integer such that $d < \alpha_1 \leq d+1$ is satisfied. Then $u(y) = P_d + V_{m,\lambda}^{\alpha_1-\varepsilon}$, where $m \geq d+1$ and $P_d$ denotes a polynomial of degree $d$ in $y_1, y_2$.*

*In particular, one has $u \in C^{d,\mu}$, $0 < \mu < 1$, up to the corner since $\lambda > 0$ may be chosen close to* 0.

REMARK 4.1. In Grisvard [**3**, Chapter 6.4] it was proved that the asymptotic expansion in Hölder spaces exists, provided the right hand side of $\Delta u = f$ is Hölder continuous up to the corner. In the case $1 < \alpha_1 < 2$ the sum $d_{ij}(Dw, DQ)w_{,ij}$ on the right of (4.1) belongs to the class $A_{0,\lambda}^{3\alpha_1-3-\varepsilon}$. In particular, one cannot assume that the right hand side of the differential equation (4.1) is Hölder continuous up to the corner, provided $2\alpha_1 - 3 \leq 0$ is satisfied. Therefore, the results of Grisvard [**3**, Chapter 6.4] do not cover our case. Moreover, the results of [**8**, Chapter 4] imply a simplified proof of the existence of the asymptotic expansion in Hölder-type spaces for solutions to (3.2) or to the corresponding Dirichlet problem.

**5. Leading terms for the capillary problem.** In the case $1 < \alpha_1 \leq 2$ we obtain from (3.5), (3.3), (3.6) and (3.4) after a simple calculation

$$u(y) = u(0) - \frac{1}{k} r\cos\theta + c_1 \frac{1}{a^2\sqrt{\omega}} r^{\alpha_1} \cos[\alpha_1(\theta+\omega)] + \frac{\kappa}{4a} u(0) r^2 + V, \tag{5.1}$$

where $c_1$ is a constant and $V \in A_{3,\lambda}^{2\alpha_1-1-\varepsilon}$.

In particular, we have in $x$-coordinates:

$$u(x) = u(0) - \frac{1}{\sqrt{k^2-1}} x_1 + V_{3,\lambda}^{\alpha_1-\varepsilon}.$$

Hence, since $\lambda$ may be chosen close to 0, the expansion (1.12) of §1 follows.

If $\alpha_1 > 2$, then (5.1) holds with $V \in A_{3,\lambda}^{3-\varepsilon}$. This implies

$$u(x) = u(0) - \frac{1}{\sqrt{k^2-1}} x_1 + \frac{\kappa}{4a} u(0) \left(\frac{x_1^2}{a^2} + x_2^2\right) + V, \tag{5.2}$$

where $V \in A_{3,\lambda}^{\min(\alpha_1,3)-\varepsilon}$. From this expansion (1.13) follows in view of the definition of the class $A_{m,\lambda}^{\gamma}$ and the fact that we may choose $\lambda$ close to 0.

An easy calculation shows that the sum of the first three items on the right hand side of (5.2) is equal to the expansion up to second order at $x = 0$ of the lower hemisphere of radius $2/\kappa u(0)$ touching the plane

$$z = u(0) - \frac{1}{\sqrt{k^2 - 1}} x_1 \quad \text{at } x = 0.$$

If $\alpha_1 = 2$, that is $2\omega = \pi/2$, then the expansion (5.1) implies $u \in C^{2,\mu}(\overline{\Omega}_R)$, $0 < \mu < 1$. That means we have more regularity near the corner than Theorem 4.2 asserts for the general problem (4.1)–(4.3).

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