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SOME RESULTS ON SPECKER'S PROBLEM

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Say that the Specker Property holds for a well ordered cardinal $\aleph$, and write this as SP($\aleph$), if the power set of $\aleph$ can be written as a countable union of sets of cardinality $\aleph$. Specker's Problem asks whether it is possible to have a model in which SP($\aleph$) holds for every $\aleph$. In this paper, we construct two models in which the Specker Property holds for a large class of cardinals. In the first model, SP($\aleph$) holds for every successor $\aleph$. In the second model, SP($\aleph$) holds for every limit $\aleph$ and for certain successor $\aleph$'s.

In 1957, Specker **[13]** stated the following question (which will henceforth be referred to as Specker's Problem): Is it consistent with the axioms of ZF to have, for each ordinal α, a countable sequence $\langle A_n : n < \omega \rangle$ of subsets of $2^{\aleph_\alpha}$ so that $|A_n| = \aleph_\alpha$ for all n and $2^{\aleph_\alpha} = \bigcup_{n\in\omega} A_n$? Since the existence of one ordinal α so that $2^{\aleph_\alpha}$ is a countable union of sets of cardinality $\aleph_\alpha$ implies that $\aleph_{\alpha+1}$ is singular, a model in which the above holds would be one in which the Axiom of Choice is false. Indeed, it can easily be seen that in such a model, AC_ω is false.

Lévy **[9]**, shortly after the invention by Cohen of forcing, constructed a model in which $2^{\aleph_0}$ is a countable union of countable sets. A later result on Specker's Problem was obtained in **[6]**, in which it was shown that, relative to the existence of a proper class of strongly compact cardinals, it is consistent for every infinite set to be a countable union of sets of smaller cardinality.

Unfortunately, we still do not know whether Specker's Problem is consistent. In this paper, we will prove the following two theorems, each of which provides a partial answer to Specker's Problem for a large class of cardinals.

THEOREM 1. Con(ZFC + *There exists a regular limit of supercompact cardinals*) $\Rightarrow$ Con(ZF + *For every successor ordinal α, $2^{\aleph_\alpha}$ is a countable union of sets of cardinality $\aleph_\alpha$*).

THEOREM 2. Con(ZFC + GCH + *There is a cardinal κ which is $2^{2^{[\kappa^{+\omega}]^{<\omega}}}$ supercompact*) $\Rightarrow$ Con(ZF + *For every limit ordinal λ, $2^{\aleph_\lambda}$ is a* *countable union of sets of cardinality* $\aleph_\lambda$ + *For every successor ordinal* α *so that* $\alpha = 3n$, $3n+1$, $\lambda+3n$, *or* $\lambda+3n+2$, *where* λ *is a limit ordinal and* $n \in \omega$, $2^{\aleph_\alpha}$ *is a countable union of sets of cardinality* $\aleph_\alpha$*).*

The techniques used in the proofs of the above two theorems can be used to establish additional results on Specker's Problem. For example, it is possible to establish the relative consistency of the theory "ZF + For every successor ordinal α, $2^{\aleph_\alpha}$ is a countable union of sets of cardinality $\aleph_\alpha$ + For every limit ordinal λ so that $\lambda = \lambda' + \omega$ where λ' is a limit ordinal, $2^{\aleph_\lambda}$ is a countable union of sets of cardinality $\aleph_\lambda$". However, as the proofs of such results involve amalgamations of the aforementioned techniques which are well illustrated by the proofs of Theorems 1 and 2, only the proofs of these theorems will be given here.

Note that some sort of strong hypotheses will be needed in order to prove the above theorems, since as previously mentioned, if $2^{\aleph_\alpha}$ is a countable union of sets of cardinality $\aleph_\alpha$, $\aleph_{\alpha+1}$ is singular with cofinality ω. Thus, if $\aleph_\alpha$ and $\aleph_{\alpha+1}$ are both so that $2^{\aleph_\alpha}$ is a countable union of sets of cardinality $\aleph_\alpha$ and $2^{\aleph_{\alpha+1}}$ is a countable union of sets of cardinality $\aleph_{\alpha+1}$, $\aleph_{\alpha+1}$ and $\aleph_{\alpha+2}$ both have cofinality ω. This implies the existence of inner models with measurable cardinals of high order.

The proofs of Theorems 1 and 2 will use the Easton iteration of partial orderings which satisfy the Prikry property developed in [7]. Before beginning the proofs of these theorems, however, we will briefly give some background information and preliminaries.

Our set theoretic notation is relatively standard. When $\alpha < \beta$ are ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation. When x is a set, $\bar{x}$ is the order type of x. For our forcing notation, however, we adopt the notation of [**12**], and say that for p and q forcing conditions, $q \Vdash p$ means that q contains more information than p. For ϕ a statement in the appropriate forcing language, $p \parallel \phi$ means that p decides ϕ.

Two partial orderings will be of particular importance in the proof of Theorems 1 and 2, namely the Lévy collapse and supercompact Prikry forcing. For $\kappa < \lambda$ regular cardinals, $\mathrm{Col}(\kappa, \lambda)$ is the Lévy collapse of λ to κ^+, i.e., $\mathrm{Col}(\kappa, \lambda) = \{f : \kappa \times \lambda \to \lambda : f$ is a function so that $|\mathrm{dmn}(f)| < \kappa$ and $f(\langle \alpha, \beta \rangle) < \beta\}$, ordered by $q \Vdash p$ iff $q \supseteq p$. The trivial condition is the empty set $\emptyset$. If $\beta_0 \in [\kappa, \lambda]$ is a regular cardinal and $p \in \mathrm{Col}(\kappa, \lambda)$, $p \restriction \beta_0 = \{\langle \langle \alpha, \beta \rangle, \gamma \rangle \in p : \beta < \beta_0\}$. $p \restriction \beta_0$ is then a condition in $\mathrm{Col}(\kappa, \beta_0)$, and for G generic on $\mathrm{Col}(\kappa, \lambda)$,

$G \upharpoonright \beta_0 = \{p \upharpoonright \beta_0 : p \in G\}$ is generic on $\mathrm{Col}(\kappa, \beta_0)$ which we may also sometimes write as $\mathrm{Col}(\kappa, \lambda) \upharpoonright \beta_0$.

Supercompact Prikry forcing is a generalization of the usual notion of Prikry forcing which was first used by Magidor in the mid 1970's. Let $\kappa < \lambda$ be such that κ is λ supercompact, and let $\mathscr{U}$ be a normal ultrafilter on $P_\kappa(\lambda)$ which satisfies the Jech-Menas partition property. (See **[11]** for a definition of this property.) For $p, q \in P_\kappa(\lambda)$, say $p \subsetneq q$ iff $p \subseteq q$ and $\bar{p} < q \cap \kappa$. Supercompact Prikry forcing $\mathrm{SC}(\kappa, \lambda)$ is then the set of all $\pi = \langle p_1, \ldots, p_n, A\rangle$ where:

1. $n \in \omega$ and $A \in \mathscr{U}$.
2. For $i = 1, \ldots, n$, $p_i \in A$.
3. For $1 \leq i < j \leq n$, $p_i \subsetneq p_j$.
4. For each $q \in A$, $p_n \subsetneq q$, and $|p_n| < |q|$.

The sequence $\langle p_1, \ldots, p_n\rangle$ is called the p-part of π and is written p-part(π).

If $\pi_1 = \langle p_1, \ldots, p_n, A\rangle$ and $\pi_2 = \langle q_1, \ldots, q_m, B\rangle$ are elements of $\mathrm{SC}(\kappa, \lambda)$ then $\pi_2 \Vdash \pi_1$ iff:

1. $n \leq m$.
2. For $i = 1, \ldots, n$, $p_i = q_i$.
3. For $i = n+1, \ldots, m$, $q_i \in A$.
4. $B \subseteq A$.

As with ordinary Prikry forcing, supercompact Prikry forcing satisfies the Prikry property, namely for ϕ a statement in the forcing language of $\mathrm{SC}(\kappa, \lambda)$ and π a condition, it is possible to shrink the measure 1 set to form a condition π' so that $\pi' \parallel \phi$.

If G is generic on $\mathrm{SC}(\kappa, \lambda)$, then the generic sequence $r = \langle p_n : n \in \omega\rangle$ (where $p_n \in r$ iff there is some $\pi \in G$ so that p_n is the nth element of p-part(π)) codes a cofinal ω sequence through λ if λ is regular. In addition, if $\alpha \in [\kappa, \lambda]$ is regular, then $r \upharpoonright \alpha = \langle p_n \cap \alpha : n \in \omega\rangle$ codes a cofinal ω sequence through α; when $\alpha = \kappa$, $r \upharpoonright \kappa$ is a Prikry sequence through κ. Also, in analogy to the Lévy collapse, for $\alpha \in [\kappa, \lambda]$ regular, we can for $\pi = \langle p_1, \ldots, p_n, A\rangle$ define $\pi \upharpoonright \alpha = \langle p_1 \cap \alpha, \ldots, p_n \cap \alpha, A \upharpoonright \alpha\rangle$, where $A \upharpoonright \alpha = \{p \cap \alpha : p \in A\}$. $\pi \upharpoonright \alpha$ is then a condition in $\mathrm{SC}(\kappa, \alpha)$ (which is defined using the restriction ultrafilter $\mathscr{U} \upharpoonright \alpha = \{A \upharpoonright \alpha : A \in \mathscr{U}\}$), and $G \upharpoonright \alpha = \{\pi \upharpoonright \alpha : \pi \in G\}$ is generic on $\mathrm{SC}(\kappa, \alpha)$ which we may also sometimes write as $\mathrm{SC}(\kappa, \lambda) \upharpoonright \alpha$.

Finally, we will say that $\aleph_\alpha$ satisfies the Specker property, written $\mathrm{SP}(\aleph_\alpha)$, if $2^{\aleph_\alpha}$ can be written as a countable union of sets of cardinality $\aleph_\alpha$.

We turn now to the proof of Theorem 1.

Proof of Theorem 1. Let $V \models$ "ZFC + There exists a regular limit of supercompact cardinals", and let α_0 be the least such limit, with $\langle \kappa_\alpha : \alpha < \alpha_0 \rangle$ the sequence of supercompact cardinals whose limit is α_0. As each κ_α is supercompact, a result in **[1]** shows that there is a supercompact ultrafilter $\mathscr{U}_\alpha$ on $P_{\kappa_\alpha}(\kappa_{\alpha+1})$ with the following property $(*)$: $\mathscr{U}_\alpha$ satisfies the Menas partition property, and there is a set $A_\alpha \in \mathscr{U}_\alpha$ so that for $q \in A_\alpha$, $q \cap \kappa_\alpha$ is an inaccessible cardinal, and if $p, q \in A_\alpha$ are so that $p \cap \kappa_\alpha = q \cap \kappa_\alpha$, then $|p| = |q|$. Let $\langle \mathscr{U}_\alpha : \alpha < \alpha_0 \rangle$ and $\langle A_\alpha : \alpha < \alpha_0 \rangle$ be such a sequence of ultrafilters $\mathscr{U}_\alpha$ and sets A_α.

Define now a sequence $\langle P_\alpha : \alpha < \alpha_0 \rangle$ of partial orderings as follows:

$$P_0 = \mathrm{Col}(\omega, \kappa_0).$$

$$P_{\alpha+1} = \mathrm{SC}(\kappa_\alpha, \kappa_{\alpha+1}),$$

where each condition in $P_{\alpha+1}$ is stronger than the trivial condition $\langle \phi, A_\alpha \rangle$.

$$P_\lambda = \mathrm{Col}\left(\left(\bigcup_{\alpha<\lambda} \kappa_\alpha\right)^+, \kappa_\lambda\right) \quad \text{for } \lambda \text{ a limit ordinal.}$$

Note that since α_0 is the least regular limit of supercompact cardinals, the definition of P_λ makes sense.

We are now in a position to define the partial ordering P which will be used in the proof of Theorem 1. P consists of all elements $p = \langle p_\alpha : \alpha < \alpha_0 \rangle$ of $\prod_{\alpha<\alpha_0} P_\alpha$ so that the support of p is some ordinal $< \alpha_0$, i.e., so that $\exists \beta < \alpha_0\ \forall \gamma \geq \beta\ [p_\gamma$ is the trivial condition$]$. The ordering is the componentwise one.

Let G be V-generic on P. The model for Theorem 1 will be a certain submodel N of $V[G]$. The intuition behind the construction of N will be as follows. We wish to define N in a manner so that the κ_α's and the $(\bigcup_{\alpha<\lambda} \kappa_\alpha)^+$'s are the successor cardinals and so that each of these cardinals satisfies the Specker property. Thus, we will place in N just enough information to be able to collapse each of the above cardinals, preserve the fact that they indeed remain cardinals in N, and define the sequence which witnesses the fact that they satisfy the Specker property. For any cardinal δ which becomes a successor cardinal in N, we will place in N for each $n \in \omega$, roughly speaking, the partial collapse map to δ^+ restricted to the nth element of the Prikry sequence through the least $\kappa_\alpha > \delta$, together with the partial collapse map to γ^+ restricted to the nth element of the Prikry sequence through the least $\kappa_\alpha > \gamma$ for every γ in a certain set of cardinals below δ.

Getting specific, let for each $\alpha < \alpha_0$, G_α be the projection of G onto P_α. Let r_0 be the collapse map of κ_0 to ω_1 generated by G_0. For λ a limit ordinal, let r_λ be the collapse map of κ_λ to $(\bigcup_{\alpha<\lambda} \kappa_\alpha)^{++}$ generated by G_λ, and for $\beta = \alpha + 1$ a successor ordinal, let $r_\beta = \langle r_\alpha^n \colon n \in \omega \rangle$ be the ω sequence generated by G which codes a cofinal ω sequence through each regular cardinal in the interval $[\kappa_\alpha, \kappa_{\alpha+1}]$. We can now define, for each $n < \omega$ and each $\beta < \alpha_0$, $s_n^\beta = \langle r_\alpha \restriction (r_\alpha^n \cap \kappa_\alpha) \colon \alpha \leq \beta \rangle$. N will then be defined as $R(\alpha_0)$ of the least model M of ZF extending V which contains, for every $n < \omega$ and every $\beta < \alpha_0$, the set s_n^β. More precisely, let L_1 be a ramified sublanguage of the forcing language L associated with P which contains symbols $\underline{v}$ for each $v \in V$, a predicate symbol $\underline{V}$ (to be interpreted as $\underline{V}(\underline{v}) \Leftrightarrow v \in V$), and all symbols of the form $\underline{s_n^\beta}$ for $n < \omega$ and $\beta < \alpha_0$. As usual, we can assume that each $\underline{v}$ is invariant under any automorphism of P. We can also assume that each $\tau \in L_1$ which mentions only $\underline{s_n^\beta}$ is invariant under any automorphism $\pi = \langle \pi_\alpha \colon \alpha < \alpha_0 \rangle$ of P such that π_α is generated by a function which is the identity on the ordinal determined by $r_\alpha^n \cap \kappa_\alpha$ for $\alpha \leq \beta$ if there is enough information to determine all such ordinals.

Working in $V[G]$, we define an inner model M as follows.

$$M_0 = \varnothing.$$
$$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha \text{ if } \lambda \text{ is a limit ordinal.}$$
$$M_{\alpha+1} = \{x \subseteq M_\alpha \colon x \text{ is definable over } M_\alpha \text{ by a term } \tau \in L_1 \text{ of rank} \leq \alpha\}.$$
$$M = \bigcup_{\alpha \in \text{Ordinals}^V} M_\alpha.$$

The standard arguments will show that for $N = R(\alpha_0)^M$, since α_0 is a limit ordinal and $M \models \text{ZF}$, N satisfies all axioms of ZF with the possible exception of Replacement.

We now prove a sequence of lemmas which shows that N is the desired model for Theorem 1.

LEMMA 1.1. *Assume that $x \in M$ is a set of ordinals. Then:*
(a) *$x \in V[s_n^\delta]$ for some $n < \omega$ and $\delta < \alpha_0$.*
(b) *If $x \subseteq \omega$, $x \in V[s_n^0]$ for some $n < \omega$.*
(c) *If $\alpha < \alpha_0$ and $x \subseteq \kappa_\alpha$, $x \in V[s_n^\delta]$ for some $n < \omega$ and $\delta = \alpha + 1$.*
(d) *If $\lambda < \alpha_0$ is a limit ordinal and $x \subseteq (\bigcup_{\alpha<\lambda} \kappa_\alpha)^+$, $x \in V[s_n^\delta]$ for some $n < \omega$ and $\delta = \lambda$.*

Proof of Lemma 1.1. We will first prove (a) and then show how (b), (c), and (d) all follow from (a). Let $\tau \in L_1$ and $p \in P$ be such that τ denotes x and $p \Vdash$"$\tau \subseteq \gamma_0$" for some ordinal γ_0. As $\tau \in L_1$, τ contains only finitely many symbols of the form $\underline{s_n^\delta}$ for $n < \omega$ and $\delta < \alpha_0$; using standard coding tricks, we can assume that τ mentions only one symbol of the form s_n^δ. We show that $p \Vdash$ "$x \in V[s_n^\delta]$."

Let $p = \langle p_\alpha \colon \alpha < \alpha_0 \rangle$, where $\gamma < \alpha_0$ is such that p_α is trivial for $\alpha \geq \gamma$. First, since $\delta < \alpha_0$, we can assume without loss of generality that $\gamma \geq \delta$ and for every $\alpha \leq \delta$, r_α^n is determined. (Simply extend p_α for $\alpha \leq \delta + 1$ so that the finite portion of the Prikry sequence determined by p_α has length at least n.) Next, define a function $f \colon \alpha_0 \to \alpha_0$ by $f(\beta) = r_\beta^n \cap \kappa_\beta$ for $\beta \leq \delta$, $f(\lambda) = (\bigcup_{\alpha<\lambda} \kappa_\alpha)^+$ for $\lambda > \delta$ a limit ordinal, and for $\beta = \alpha + 1 > \delta$ a successor ordinal, $f(\beta) = \kappa_\alpha$. Our first claim is that if $q = \langle q_\alpha \colon \alpha < \alpha_0 \rangle$, $s = \langle s_\alpha \colon \alpha < \alpha_0 \rangle$, $q \Vdash p$, $s \Vdash p$ are such that $\forall \alpha < \alpha_0$ $[q_\alpha \restriction f(\alpha) = s_\alpha \restriction f(\alpha)]$, then for any $\beta_0 < \gamma_0$, if $q \Vdash$ "$\beta_0 \in \tau$", $s \Vdash$"$\beta_0 \in \tau$."

The proof of this claim is very similar to the proofs of Lemmas 1.1 and 1.6 of **[1]**. Specifically, if the claim is false, then let $r^0 = \langle u_\alpha^0 \colon \alpha < \alpha_0 \rangle$ be such that $r^0 \Vdash s$ and $r^0 \Vdash$ "$\beta_0 \notin \tau$." For each successor $\alpha = \beta + 1$, $\alpha < \alpha_0$, let $u_\alpha^1 \in P_\alpha$ be such that $u_\alpha^1 \Vdash q_\alpha$ and so that for $\langle t_1^\alpha, \ldots, t_k^\alpha \rangle$ the p-part of u_α^1 and $\langle \hat{t}_1^\alpha, \ldots, \hat{t}_k^\alpha \rangle$ the p-part of u_α^0, $t_j^\alpha \cap f(\alpha) = \hat{t}_j^\alpha \cap f(\alpha)$ for $j = 1, \ldots, k$. Form a condition $r^2 = \langle u_\alpha^2 \colon \alpha < \alpha_0 \rangle$ by $u_\alpha^2 = u_\alpha^1$ if α is a successor ordinal and $u_\alpha^2 = q_\alpha$ if α is a limit ordinal or $\alpha = 0$. Clearly, $r^2 \Vdash q$ and $r^2 \Vdash$ "$\beta_0 \in \tau$."

We define now an automorphism $\pi = \langle \pi_\alpha \colon \alpha < \alpha_0 \rangle$ of P so that $\pi(r^2)$ is compatible with r^0 and $\pi(r^2) \Vdash$"$\beta_0 \in \tau$". If $\lambda = 0$ or λ is a limit ordinal, then by the homogeneity of the Lévy collapse, we can let π_λ be any automorphism of P_λ so that $\pi_\lambda(u_\lambda^2)$ is compatible with u_λ^0 and π_λ is generated by a function which is the identity on $f(\lambda)$. If $\alpha = \beta + 1$ is a successor ordinal, then as in **[3]** or **[1]**, Lemma 1.6, let π_α be an automorphism so that $\pi_\alpha(u_\alpha^2)$ is compatible with u_α^0 and π_α is generated by a function which is the identity on $f(\alpha)$. $\pi = \langle \pi_\alpha \colon \alpha < \alpha_0 \rangle$ is thus an automorphism of P so that $\pi(r^2)$ is compatible with r^0, and by the invariance properties of τ, $\pi(r^2) \Vdash$ "$\beta_0 \in \tau$". Since $\pi(r^2)$ is compatible with r^0 and $r^0 \Vdash$ "$\beta_0 \notin \tau$", this is a contradiction. Thus, if $q = \langle q_\alpha \colon \alpha < \alpha_0 \rangle$, $s = \langle s_\alpha \colon \alpha < \alpha_0 \rangle$, $q \Vdash p$, $s \Vdash p$ are such that $\forall \alpha < \alpha_0$ $[q_\alpha \restriction f(\alpha) = s_\alpha \restriction f(\alpha)]$, then if $q \Vdash$ "$\beta_0 \in \tau$", $s \Vdash$ "$\beta_0 \in \tau$". Now, if we define $y = \{\rho < \gamma_0 \colon \exists q \Vdash p[q = \langle q_\alpha \colon \alpha < \alpha_0 \rangle$, $q_\alpha \restriction f(\alpha) \in G_\alpha \restriction f(\alpha)$, and $q \Vdash$ "$\rho \in \tau$"$]\}$, then using the preceding fact, we can argue as in Lemma 1.1 of [1] and show that $x = y$. Since y is definable in $V[\prod_{\alpha<\alpha_0} G_\alpha \restriction f(\alpha)]$, this shows that $x \in V[\prod_{\alpha<\alpha_0} G_\alpha \restriction f(\alpha)]$.

We next show that $x \in V[\prod_{\alpha\leq\delta} G_\alpha \restriction f(\alpha)]$. Since each $G_\alpha \restriction f(\alpha)$ is recoverable from $r_\alpha \restriction (r_\alpha^n \cap \kappa_\alpha)$, this will show that $x \in V[s_n^\delta]$. To show that $x \in V[\prod_{\alpha\leq\delta} G_\alpha \restriction f(\alpha)]$ we again argue as in Lemma 1.6 of [1]. Specifically, we know that for any successor $\alpha = \beta + 1 > \delta$, $P_\alpha \restriction f(\alpha)$ is supercompact Prikry forcing on $P_{\kappa_\beta}(\kappa_\beta)$ defined using the measure $\mathscr{U}_\beta \restriction \kappa_\beta = \{A \restriction \kappa_\beta : A \in \mathscr{U}_\beta\}$. It is well known that this partial ordering is canonically isomorphic to ordinary Prikry forcing P'_α on κ_β defined using the canonical normal ultrafilter on κ_β generated by $\mathscr{U}_\beta \restriction \kappa_\beta$. Call this normal ultrafilter $\mathscr{U}'_\beta$. If we let $P''_\alpha = \{\langle s, S\rangle : s \in [\kappa_\beta]^{<\omega} \,\&\, S \in \mathscr{U}'_\beta\}$ ordered as in ordinary Prikry forcing, i.e., if we define P''_α as in ordinary Prikry forcing except that it is not necessarily the case that $\bigcup s < \bigcap S$, then as in [5] and Lemma 1.6 of [1] we can without loss of generality replace P'_α with P''_α.

Now let σ be a canonical term for x in the forcing language associated with

$$Q = \prod_{\alpha\leq\delta} P_\alpha \restriction f(\alpha) \times \prod_{\{\alpha\in[\delta+1,\alpha_0):\ \alpha \text{ is a successor ordinal}\}} P''_\alpha \times \prod_{\{\lambda\in[\delta+1,\alpha_0):\ \lambda \text{ is a limit ordinal}\}} P_\lambda \restriction f(\lambda).$$

Define a term η in the forcing language associated with $\prod_{\alpha\leq\delta} P_\alpha \restriction f(\alpha)$ by $p = \langle p_\alpha : \alpha \leq \delta\rangle \Vdash$ "$\rho \in \eta$" iff $\langle p_\alpha : \alpha < \alpha_0\rangle \Vdash$ "$\rho \in \sigma$", where for $\alpha \geq \delta + 1$, p_α is the trivial condition. Clearly, η will denote a set in $V[\prod_{\alpha\leq\delta} G_\alpha \restriction f(\alpha)]$ which is a subset of x. The proof will be complete if we can show that $\Vdash_Q$"$\sigma \subseteq \eta$".

To this end, let $q = \langle q_\alpha : \alpha < \alpha_0\rangle \Vdash$ "$\rho \in \sigma$". It suffices to show that $r = \langle q_\alpha : \alpha \leq \delta\rangle \times \langle \hat{r}_\alpha : \alpha > \delta\rangle \Vdash$ "$\rho \in \sigma$", where for $\alpha > \delta$, $\hat{r}_\alpha$ is the trivial condition. If this is not the case, then let $s = \langle s_\alpha : \alpha < \alpha_0\rangle \Vdash r$ be such that $s \Vdash$ "$\rho \notin \sigma$", and let $\beta \in (\delta, \alpha_0)$ be such that for all $\gamma \geq \beta$, s_γ and q_γ are the trivial condition. Without loss of generality, assume that for all successors $\delta < \gamma < \beta$, the p-parts of s_γ and q_γ have the same length.

We construct now an automorphism $\psi = \langle \psi_\alpha : \alpha < \alpha_0\rangle$ of Q as follows. For ordinals $\alpha \geq \beta$, ordinals $\alpha \leq \delta$, and limit ordinals $\alpha \in (\delta, \beta)$ let ψ_α be the identity. (Note that for $\alpha \in (\delta, \beta)$ a limit ordinal, $P_\alpha \restriction f(\alpha)$ is the trivial partial ordering.) For $\alpha \in (\delta, \beta)$ a successor ordinal, as in Lemma 1.6 of [1] let ψ_α be an automorphism of P''_α so that $\psi_\alpha(s_\alpha)$ is compatible with q_α and ψ_α is generated by a function which is a permutation of κ_γ for $\alpha = \gamma+1$. $\psi = \langle \psi_\alpha \colon \alpha < \alpha_0 \rangle$ is then an automorphism of Q so that $\psi(s)$ is compatible with q, and since we can assume that σ is invariant under any automorphism generated by ψ_α's as just defined, $\psi(s) \Vdash$ "$\rho \notin \sigma$" and $q \Vdash$"$\rho \in \sigma$". This contradiction shows that $x \in V[s_n^\delta]$.

To show (b), (c), and (d), let σ be either ω, κ_α, or $(\bigcup_{\alpha<\lambda} \kappa_\alpha)^+$, and let γ be such that $x \subseteq \sigma$ and $x \in V[s_n^\gamma]$. If $\gamma \leq \delta$ for δ as defined in (b), (c), and (d), then the proof is complete, since $V[s_n^\gamma] \subseteq V[s_n^\delta]$. Thus, assume that $\gamma > \delta$. As in part (a), we know that $x \in V[\prod_{\alpha \leq \gamma} G_\alpha \restriction f(\alpha)]$ for f as defined previously. We show that $x \in V[s_n^\delta]$ by showing that $V[\prod_{\alpha \leq \gamma} G_\alpha \restriction f(\alpha)]$ and $V[\prod_{\alpha \leq \delta} G_\alpha \restriction f(\alpha)]$ contain the same subsets of σ and then using the identification of $V[\prod_{\alpha \leq \delta} G_\alpha \restriction f(\alpha)]$ with $V[s_n^\delta]$.

To this end, we need to show that forcing over $V[\prod_{\alpha \leq \delta} G_\alpha \restriction f(\alpha)]$ with $\prod_{\alpha \in [\delta+1, \gamma]} P_\alpha \restriction f(\alpha)$ adds no new subsets of σ. Write

$$\prod_{\alpha \in [\delta+1, \gamma]} P_\alpha \restriction f(\alpha)$$

as $Q' \times Q''$, where

$$Q' = \prod_{\{\alpha \colon \alpha \in [\delta+1, \gamma] \text{ and } \alpha \text{ is a successor ordinal}\}} P_\alpha \restriction f(\alpha)$$

and

$$Q'' = \prod_{\{\lambda \colon \lambda \in [\delta+1, \gamma] \text{ and } \lambda \text{ is a limit ordinal}\}} P_\lambda \restriction f(\lambda).$$

This factorization generates a factorization of $\prod_{\alpha \in [\delta+1, \gamma]} G_\alpha \restriction f(\alpha)$ into $G' \times G''$. Since each λ so that $P_\lambda \restriction f(\lambda)$ is a component partial ordering of Q'' is $> \delta$, the closure properties of the Lévy collapse and the definition of Q'' ensure that the subsets of σ in $V[G'']$ and V are the same. Further, by the definition of f and each P_η, for $\alpha = \beta+1$ a fixed but arbitrary successor ordinal in $[\delta + 1, \gamma]$, $|\prod_{\eta \leq \beta} P_\eta \restriction f(\eta)| < \kappa_\beta$. Also, if $\lambda > \alpha$ is a limit ordinal, the closure properties of the Lévy collapse and the definition of P ensure that each $P_\lambda \restriction f(\lambda)$ is (at least) $2^{2^{f(\alpha)}}$ closed. Thus, since $P_\alpha \restriction f(\alpha)$ is a supercompact Prikry partial ordering on $P_{\kappa_\beta}(f(\alpha))$ with $|P_\alpha \restriction f(\alpha)| < 2^{2^{f(\alpha)}}$, an application of the closure properties of

$$\prod_{\{\lambda \colon \lambda \in [\alpha, \gamma] \text{ and } \lambda \text{ is a limit ordinal}\}} P_\lambda \restriction f(\lambda) = Q'''$$

followed by an application of the Lévy-Solovay results **[10]** shows that $V^{Q''' \times \prod_{\eta \leq \beta} P_\eta \restriction f(\eta)} \models$ "$P_\alpha \restriction f(\alpha)$ is a partial ordering which satisfies the Prikry property and adds no new bounded subsets to κ_β". Thus, Q' can be regarded in $V[G'']$ as a full support iteration of partial orderings each of which satisfies the Prikry property and adds no new bounded subsets to κ_δ, so since α_0 is the least regular limit of supercompact cardinals, the result of [7] shows that forcing over $V[G'']$ with Q' adds no new bounded subsets to κ_δ, i.e., since $\sigma < \kappa_\delta$, $V[G''][G'] = V[G'][G''] = V[\prod_{\alpha \in [\delta+1,\gamma]} G_\alpha \restriction f(\alpha)] \models$ "The subsets of σ are the same as those in V". Thus, any new subsets of σ in $V[\prod_{\alpha \leq \gamma} G_\alpha \restriction f(\alpha)]$ are generated by forcing over $V[\prod_{\alpha \in [\delta+1,\gamma]} G_\alpha \restriction f(\alpha)]$ with $\prod_{\alpha \leq \delta} P_\alpha \restriction f(\alpha)$, i.e., since

$$V\left[\prod_{\alpha \in [\delta+1,\gamma]} G_\alpha \restriction f(\alpha)\right]\left[\prod_{\alpha \leq \delta} G_\alpha \restriction f(\alpha)\right] = V\left[\prod_{\alpha \leq \delta} G_\alpha \restriction f(\alpha)\right]\left[\prod_{\alpha \in [\delta+1,\gamma]} G_\alpha \restriction f(\alpha)\right],$$

forcing over $V[\prod_{\alpha \leq \delta} G_\alpha \restriction f(\alpha)]$ with $\prod_{\alpha \in [\delta+1,\gamma]} P_\alpha \restriction f(\alpha)$ adds no new subsets of σ. Thus, $x \in V[s_n^\delta]$. This proves Lemma 1.1. □

LEMMA 1.2. *For* $\sigma = \kappa_\alpha$ *or* $\sigma = (\bigcup_{\alpha<\lambda} \kappa_\alpha)^+$, λ *a limit ordinal,* $N \models$ "*σ is a cardinal*".

Proof of Lemma 1.2. For δ as in Lemma 1.1, since $N \subseteq M$ and $\sigma < \alpha_0$, Lemma 1.1 shows that if $x \subseteq \sigma$ and $x \in N$ then $x \in V[s_n^\delta]$ for some $n \in \omega$. Let f be as in Lemma 1.1. By the identification of $V[s_n^\delta]$ with $V[\prod_{\alpha \leq \delta} G_\alpha \restriction f(\alpha)]$, view $V[s_n^\delta]$ as $V[G_\delta \restriction f(\delta)][\prod_{\alpha<\delta} G_\alpha \restriction f(\alpha)]$.

If $\sigma = \kappa_\alpha$, then $\delta = \alpha + 1$ and $P_\delta \restriction f(\delta)$ is a supercompact Prikry ordering on $P_{\kappa_\alpha}(f(\delta))$. This means that $V[G_\delta \restriction f(\delta)] \models$ "κ_α is a cardinal and $|\prod_{\beta<\delta} P_\beta \restriction f(\beta)| < \kappa_\alpha$", so $V[s_n^\delta] \models$ "κ_α is a cardinal". Thus, no subset of κ_α in N can code a collapsing map of κ_α, i.e., $N \models$ "κ_α is a cardinal". If $\sigma = (\bigcup_{\alpha<\lambda} \kappa_\alpha)^+$, then $\delta = \lambda$, and $P_\delta \restriction f(\delta)$ is $\mathrm{Col}((\bigcup_{\alpha<\lambda} \kappa_\alpha)^+, f(\lambda))$. Therefore, by the definition of P and f, $V[G_\lambda \restriction f(\lambda)] \models$ "σ is a regular cardinal and for each $\alpha < \lambda$, $P_\alpha \restriction f(\alpha)$ is $\kappa(\alpha)$-c.c. for some $\kappa(\alpha) < \bigcup_{\alpha<\lambda} \kappa_\alpha$ which depends on $P_\alpha \restriction f(\alpha)$", so by the definition of $P_\alpha \restriction f(\alpha)$, $V[G_\lambda \restriction f(\lambda)] \models$ "All antichains in $\prod_{\alpha<\lambda} P_\alpha \restriction f(\alpha)$ have size $\leq \bigcup_{\alpha<\lambda} \kappa_\alpha$". This means that $V[s_n^\delta] \models$ "σ is a cardinal". The exact same reasoning as before shows $N \models$ "σ is a cardinal". This proves Lemma 1.2. □

LEMMA 1.3. *Every successor cardinal in N is either a κ_α or a $(\bigcup_{\alpha<\lambda}\kappa_\alpha)^+$ for some limit ordinal $\lambda < \alpha_0$.*

Proof of Lemma 1.3. Since $N = R(\alpha_0)^M$, it suffices to show that any successor cardinal κ^+ in M below α_0 is either κ_α or a $(\bigcup_{\alpha<\lambda}\kappa_\alpha)^+$ for some limit $\lambda < \alpha_0$. To show this, we argue by contradiction. Assume κ^+ is the least successor cardinal in M below α_0 which does not satisfy this property. Consider two cases.

Case 1. $\kappa = (\delta^+)^M$ for some cardinal $\delta < \alpha_0$. By the leastness of κ^+, κ is either a κ_α or a $(\bigcup_{\alpha<\lambda}\kappa_\alpha)^+$ for some limit $\lambda < \alpha_0$. If $\kappa = \kappa_\alpha$ for some $\alpha < \alpha_0$, then by the definition of M for each $n \in \omega$ $V[r_{\alpha+1} \restriction (r^n_{\alpha+1} \cap \kappa_{\alpha+1})] \subseteq M$. As in **[1]** and **[3]**, by the fact that $r_{\alpha+1}$ is generic for supercompact Prikry forcing on $P_{\kappa_\alpha}(\kappa_{\alpha+1})$, $V[r_{\alpha+1} \restriction (r^n_{\alpha+1} \cap \kappa_{\alpha+1})] \models$ "There are no cardinals in the interval $(\kappa_\alpha, (r^n_{\alpha+1} \cap \kappa_{\alpha+1})]$". Since $\langle (r^n_{\alpha+1} \cap \kappa_{\alpha+1}) \colon n \in \omega\rangle$ is cofinal in $\kappa_{\alpha+1}$, $M \models$ "There are no cardinals in the interval $(\kappa_\alpha, \kappa_{\alpha+1})$". By Lemma 1.2, $M \models$ "$\kappa_{\alpha+1}$ is a cardinal", so $M \models$ "$\kappa_{\alpha+1} = \kappa_\alpha^+$". If $\kappa = (\bigcup_{\alpha<\lambda}\kappa_\alpha)^+$ for some limit $\lambda < \alpha_0$, then again by the definition of M for each $n \in \omega$ $V[r_\lambda \restriction (r^n_\lambda \cap \kappa_\lambda)] \subseteq M$. Since r_λ is generic for $\mathrm{Col}((\bigcup_{\alpha<\lambda}\kappa_\alpha)^+, \kappa_\lambda)$ and $\langle r^n_\lambda \cap \kappa_\lambda \colon n \in \omega\rangle$ is cofinal in κ_λ, $V[r_\lambda \restriction (r^n_\lambda \cap \kappa_\lambda)] \models$ "There are no cardinals in the interval $((\bigcup_{\alpha<\lambda}\kappa_\alpha)^+, (r^n_\lambda \cap \kappa_\lambda))$" and $M \models$ "There are no cardinals in the interval $((\bigcup_{\alpha<\lambda}\kappa_\alpha)^+, \kappa_\lambda)$". By Lemma 1.2, $M \models$ "κ_λ is a cardinal", so $M \models$ "$\kappa_\lambda = ((\bigcup_{\alpha<\lambda}\kappa_\alpha)^+)^+$". Thus, if $\kappa = (\delta^+)^M$ for some cardinal $\delta < \alpha_0$, then $(\kappa^+)^M = \kappa_\alpha$ for some $\alpha < \alpha_0$.

Case 2. $\kappa < \alpha_0$ is a limit cardinal in M. There must be unboundedly many κ_α's below κ, for if $\sigma < \kappa$ is a bound on the κ_α's, then the cardinal (in V or M) $(\sigma^{++})^M$ is below κ and is neither a κ_α nor a $(\bigcup_{\alpha<\lambda}\kappa_\alpha)^+$, contradicting the leastness of κ^+. We can thus write, in V, $\kappa = \bigcup_{\alpha<\lambda}\kappa_\alpha$ for some $\lambda < \alpha_0$. By Lemma 1.2, $(\bigcup_{\alpha<\lambda}\kappa_\alpha)^+$ remains a cardinal in M. Since $V \subseteq M$, $M \models$ "$(\bigcup_{\alpha<\lambda}\kappa_\alpha)^+ = \kappa^+$". Thus, if κ is a limit cardinal in M and $\kappa < \alpha_0$, $M \models$ "There is a limit ordinal $\lambda < \alpha_0$ so that $\kappa^+ = (\bigcup_{\alpha<\lambda}\kappa_\alpha)^{+V}$". This fact, together with Case 1, proves Lemma 1.3. □

LEMMA 1.4. $N \models$ "$\mathrm{SP}(\omega)$, *for every* $\alpha < \alpha_0$, $\mathrm{SP}(\kappa_\alpha)$, *and for every limit ordinal* $\lambda < \alpha_0$, $\mathrm{SP}((\bigcup_{\alpha<\lambda}\kappa_\alpha)^+)$".

Proof of Lemma 1.4. Let us first consider the cardinal κ_α. Working in N, let $X_n = \{x \subseteq \kappa_\alpha \colon x \text{ is definable using } \underline{s^{\alpha+1}_n}\}$. By Lemma 1.1 and the fact that $N = R(\alpha_0)^M$, $X_n \in V[s_n^{\alpha+1}]$. The cardinality of X_n in $V[s_n^{\alpha+1}]$ is some ordinal $\delta < (r_{\alpha+1}^{n+1} \cap \kappa_{\alpha+1}) < \kappa_{\alpha+1}$. Since in $V[s_{n+1}^{\alpha+1}] \subseteq M$ δ is collapsed to κ_α (this is shown by an argument similar to the ones given in **[3]** and **[1]**, Lemma 1.3), κ_α is a cardinal in $V[s_{n+1}^{\alpha+1}]$, and $X_n \in V[s_{n+1}^{\alpha+1}]$ ($V[s_n^{\alpha+1}] \subseteq V[s_{n+1}^{\alpha+1}]$), $V[s_{n+1}^{\alpha+1}] \models$ "$|X_n| = \kappa_\alpha$". Since any function which is a bijection between X_n and κ_α must be an element of $R(\alpha_0)^{V[s_{n+1}^{\alpha+1}]} \subseteq R(\alpha_0)^M = N$, $N \models$ "$|X_n| = \kappa_\alpha$". Finally, as Lemma 1.1 and the fact that $N = R(\alpha_0)^M$ show that any $x \subseteq \kappa_\alpha$ so that $x \in N$ is in X_n for some n, $N \models \mathrm{SP}(\kappa_\alpha)$.

Turning now to the cardinal $\delta_\lambda = (\bigcup_{\alpha<\lambda} \kappa_\alpha)^+$, we can again define in N $X_n = \{x \subseteq \delta_\lambda : x$ is definable using $\underline{s_n^\lambda}\}$. As before, Lemma 1.1 implies that $X_n \in V[s_n^\lambda]$, and $|X_n|$ in $V[s_n^\lambda]$ is some $\delta < (r_\lambda^{n+1} \cap \kappa_\lambda) < \kappa_\lambda$. Again, since in $V[s_{n+1}^\lambda] \subseteq M$ δ is collapsed to δ_λ by the Lévy collapse map generated by $r_\lambda \restriction (r_\lambda^{n+1} \cap \kappa_\lambda)$, $V[s_{n+1}^\lambda] \models$ "δ_λ is a cardinal", and $V[s_n^\lambda] \subseteq V[s_{n+1}^\lambda]$, $V[s_{n+1}^\lambda]$ and M both satisfy "$|X_n| = \delta_\lambda$". Lemma 1.1, the fact that $N = R(\alpha_0)^M$, and the fact that $R(\alpha_0)^{V[s_{n+1}^\lambda]} \subseteq R(\alpha_0)^M = N$ then again yield that $N \models \mathrm{SP}(\delta_\lambda)$.

The proof of Lemma 1.4 is completed by noting that the argument for $\mathrm{SP}(\kappa_\alpha)$ works for ω by letting $\kappa_{-1} = \omega$. □

LEMMA 1.5. $N \models \mathrm{ZF}$.

Proof of Lemma 1.5. The proof that $N \models$ Replacement will show that $N \models \mathrm{ZF}$. We mimic the proof of Theorem 4.2 of **[8]**. If Replacement fails in N, then for some set $X \in N$ there is a class function f on X such that $f''X \notin N$. Since $N \models$ Aussonderung, we can assume without loss of generality that range$(f) \subseteq \alpha_0$ and range(f) is unbounded in α_0. Again without loss of generality we can assume that $X = R(\alpha)$ for some $\alpha < \alpha_0$, and we fix α the least such ordinal. This α cannot be a limit ordinal, for if it were then we could construct a function $f' \colon \alpha \to \alpha_0$ whose range was unbounded in α_0. Since $N \subseteq M$ and $M \models \mathrm{ZF}$, $f' \in M$, so $M \models$ "α_0 is singular". By Lemma 1.1a, $f' \in V[s_n^\delta]$ for some $\delta < \alpha_0$. The model $V[s_n^\delta]$ is obtained by forcing with a partial ordering Q so that $|Q| < \alpha_0$, so since $V \models$ "α_0 is inaccessible", $V^Q = V[s_n^\delta] \models$ "α_0 is inaccessible", contradicting $V[s_n^\delta] \models$ "α_0 is singular". Thus, $\alpha = \beta + 1$ for some β, and in M there is a function $f \colon p(R(\beta)) = R(\alpha) \to \alpha_0$ whose range is unbounded in α_0.

We make now the following

Claim. For each ordinal $\gamma < \alpha_0$, $R(\omega + \gamma)^N = R(\omega + \gamma)^M \subseteq R(\omega + \gamma)^{V[\prod_{\sigma<\gamma} G_\sigma]}$.

Proof of Claim. Letting $V[\prod_{\sigma<0} G_\sigma] = V$, the claim is true for $\gamma = 0$ by the absoluteness of $R(\omega)$. If γ is a limit ordinal, then by hypothesis, for every $\sigma < \gamma$, $R(\omega+\sigma)^M \subseteq R(\omega+\sigma)^{V[\prod_{\delta<\sigma} G_\sigma]}$. It is also true that for every $\sigma < \gamma$, $V[\prod_{\delta<\sigma} G_\delta] \subseteq V[\prod_{\delta<\gamma} G_\delta]$; therefore, for every $\sigma < \gamma$,

$$R(\omega+\sigma)^N = R(\omega+\sigma)^M \subseteq R(\omega+\sigma)^{V[\prod_{\delta<\sigma} G_\delta]} \subseteq R(\omega+\sigma)^{V[\prod_{\delta<\gamma} G_\delta]},$$

so

$$\begin{aligned}\bigcup_{\sigma<\gamma} R(\omega+\sigma)^N &= \bigcup_{\sigma<\gamma} R(\omega+\sigma)^M = R(\omega+\gamma)^M \\ &\subseteq \bigcup_{\sigma<\gamma} R(\omega+\sigma)^{V[\prod_{\delta<\sigma} G_\delta]} \subseteq \bigcup_{\sigma<\gamma} R(\omega+\sigma)^{V[\prod_{\delta<\gamma} G_\delta]} \\ &= R(\omega+\gamma)^{V[\prod_{\delta<\gamma} G_\delta]}.\end{aligned}$$

If $\gamma = \sigma + 1$ is a successor ordinal, then by hypothesis we have $R(\omega+\sigma)^N = R(\omega+\sigma)^M \subseteq R(\omega+\sigma)^{V[\prod_{\delta<\sigma} G_\delta]}$. Since $|\prod_{\delta<\sigma} P_\delta| < \kappa_\sigma$, $V[\prod_{\delta<\sigma} G_\delta] \models$ "κ_σ is inaccessible", so

$$V\left[\prod_{\delta<\sigma} G_\delta\right] \models \text{“}|R(\omega+\sigma)^{V[\prod_{\delta<\sigma} G_\delta]}| < \kappa_\sigma\text{”}.$$

If $x \subseteq R(\omega+\sigma)^M$ is a set in M then x can be viewed as a subset of κ_σ in the analogue of M constructed by forcing over $V[\prod_{\delta<\sigma} G_\delta]$ with $\prod_{\delta\in[\sigma,\alpha_0)} P_\delta$. The analogue of Lemma 1.1a then implies that for some n, $x \in V[\prod_{\delta<\sigma} G_\delta][G_\sigma^n] \subseteq V[\prod_{\delta<\sigma+1} G_\delta]$, i.e., that $x \in V[\prod_{\delta<\sigma+1} G_\delta]$. Thus, all subsets of $R(\omega+\sigma)^M$ in M are elements of $V[\prod_{\delta<\sigma+1} G_\delta]$. Since

$$R(\omega+\sigma)^M \subseteq R(\omega+\sigma)^{V[\prod_{\delta<\sigma} G_\delta]} \subseteq R(\omega+\sigma)^{V[\prod_{\delta<\sigma+1} G_\delta]},$$

this immediately implies that

$$\begin{aligned}R(\omega+\sigma+1)^M &= R(\omega+\gamma)^M = R(\omega+\gamma)^N \\ &\subseteq R(\omega+\sigma+1)^{V[\prod_{\delta<\sigma+1} G_\delta]} = R(\omega+\gamma)^{V[\prod_{\delta<\gamma} G_\delta]}. \quad \square\end{aligned}$$

Using the Claim and working in $V[\prod_{\sigma<\gamma} G_\sigma]$ for the γ so that $\omega+\gamma = \alpha$, for δ the cardinal $< \alpha_0$ so that $V[\prod_{\sigma<\gamma} G_\gamma] \models$ "$|R(\alpha)^{V[\prod_{\sigma<\gamma} G_\sigma]}| = \delta$" we can get in the analogue of M, M^*, obtained by forcing over $V[\prod_{\sigma<\gamma} G_\sigma]$ with $\prod_{\sigma\in[\gamma,\alpha_0)} P_\sigma$ a function $g\colon \delta \to \alpha_0$ whose range is unbounded in α_0. Since the analogue of Lemma 1.1a will be true about M^*, α_0 will be a regular cardinal in M^*. (In V, $|\prod_{\sigma<\gamma} P_\sigma| < \alpha_0$, and any subset of α_0 in M^* will be in a generic extension of $V[\prod_{\sigma<\gamma} G_\sigma]$ obtained by forcing with a partial ordering of cardinality $< \alpha_0$.) Since α_0 is thus both regular and singular in M^*, this contradiction establishes that $N \models$ Replacement and proves Lemma 1.5. $\square$

Lemmas 1.1–1.5 complete the proof of Theorem 1.

We turn now to the proof of Theorem 2. For convenience, we restate the theorem here.

THEOREM 2. Con(ZFC + GCH + *There exists a cardinal* κ *which is* $2^{2^{[\kappa^{+\omega}]^{<\omega}}}$ *supercompact*) $\Rightarrow$ Con(ZF + *For every limit ordinal* λ, $\mathrm{SP}(\aleph_\lambda)+$ *For every successor ordinal* α *of the form* $\alpha = 3n$, $3n+1$, $\lambda + 3n$, *or* $\lambda + 3n + 2$ *where* λ *is a limit ordinal and* $n \in \omega$, $\mathrm{SP}(\aleph_\alpha)$).

Proof of Theorem 2. Let $V \models$"ZFC + κ is λ_0 supercompact for $\lambda_0 = 2^{2^{[\kappa^{+\omega}]^{<\omega}}}$", and let $\mathcal{U}$ be a normal measure on $P_\kappa(\lambda_0)$. The proof of Theorem 2 will use a modification of the models N_A described in **[8]** and **[2]**. The construction of the model M which will witness the conclusions of Theorem 2, as in **[8]** and **[2]**, will use supercompact Radin forcing. We describe the forcing conditions below.

Let $j: V \to \hat{M}$ be the elementary embedding associated with $\mathcal{U}$, where $\hat{M}$ is an inner model so that $\hat{M}^{\lambda_0} \subseteq \hat{M}$. Using the embedding j, we define a Radin sequence of measures $\mu_{<\kappa^+} = \langle \mu_\alpha : \alpha < \kappa^+ \rangle$ on $R(\kappa^{+\omega})$ by $\mu_0(x) = 1$ iff $\langle j(\beta) : \beta < \kappa^{+\omega} \rangle \in j(x)$, and for $0 < \alpha < \kappa^+$, $\mu_\alpha(x) = 1$ iff $\langle \mu_\beta : \beta < \alpha \rangle \in j(x)$. $R_{<\kappa^+}$ is supercompact Radin forcing defined using $\mu_{<\kappa^+}$, i.e., $R_{<\kappa^+}$ consists of all finite sequences of the form $\langle \langle p_1, u_1, C_1 \rangle, \ldots, \langle p_n, u_n, C_n \rangle, \langle \mu_{<\kappa^+}, C \rangle \rangle$ with the following properties:

1. For $1 \le i < j \le n$, $p_i \subsetneqq p_j$.
2. For $i \le n$, $p_i \cap \kappa$ is an inaccessible cardinal.
3. $\bar{p}_i = (p_i \cap \kappa)^{+\omega}$.
4. For $i \le n$, u_i is a Radin sequence of measures on $R(\bar{p}_i)$ with $(u_i)_0$ a supercompact measure on $P_{p_i \cap \kappa}(\bar{p}_i)$.
5. C_i is a sequence of measure 1 sets for u_i.
6. C is a sequence of measure 1 sets for $\mu_{<\kappa^+}$.
7. For each $p \in (C)_0$, where $(C)_0$ is the coordinate of C so that $(C_0) \in \mu_0$, $\bigcup_{i=1}^n p_i \subsetneqq p$.
8. For each $p \in (C)_0$, $\bar{p} = (p \cap \kappa)^{+\omega}$.

Properties (1) and (7) both follow from the fact that μ_0 is a supercompact measure on $P_\kappa(\lambda_0)$. Properties (4), (5), and (6) are all standard properties of Radin forcing. Properties (2), (3), and (8) all follow since μ_0 is generated by j or, equivalently, by $\mathscr{U} \upharpoonright \kappa^{+\omega}$, so we can assume that each p_i and each $p \in (C)_0$ is an element of $\{p \in P_\kappa(\lambda_0)\colon p \cap \kappa$ is an inaccessible cardinal and $\overline{p \cap \kappa^{+\omega}} = (p \cap \kappa)^{+\omega}\} \upharpoonright \kappa^{+\omega}$.

If $\pi_0 = \langle\langle p_1, u_1, C_1\rangle, \ldots, \langle p_n, u_n, C_n\rangle, \langle \mu_{<\kappa^+}, C\rangle\rangle$ and $\pi_1 = \langle\langle q_1, v_1, D_1\rangle, \ldots, \langle q_m, v_m, D_m\rangle, \langle \mu_{<\kappa^+}, D\rangle\rangle$ then $\pi_1 \Vdash \pi_0$ if the following conditions hold.

1. For each $\langle p_j, u_j, C_j\rangle$ which appears in π_0 there is a $\langle q_i, v_i, D_i\rangle$ which appears in π_1 so that $\langle q_i, v_i\rangle = \langle p_j, u_j\rangle$ and $D_i \subseteq C_j$.
2. $D \subseteq C$.
3. $n \leq m$.
4. If $\langle q_i, v_i, D_i\rangle$ does not appear in π_0, let $\langle p_j, u_j, C_j\rangle$ (or $\langle \mu_{<\kappa^+}, C\rangle$) be the first element of π_0 so that $p_j \cap \kappa > q_i \cap \kappa$. Then
   (a) q_i is order isomorphic to some $q \in (C_j)_0$.
   (b) There exists an $\alpha < \gamma_j$, where γ_j is the length of u_j, so that v_i is isomorphic "in a natural way" to an ultrafilter sequence $v \in (C_j)_\alpha$.
   (c) For β_i the length of v_i, there is a function $f\colon \beta_i \to \gamma_j$ so that for $\beta < \beta_i$, $(D_i)_\beta$ is a set of ultrafilter sequences so that for some subset $(D_i)'_\beta$ of $(C_j)_{f(\beta)}$, each ultrafilter sequence in $(D_i)_\beta$ is isomorphic "in a natural way" to an ultrafilter sequence in $(D_i)'_\beta$.

For a further explanation of the above ordering (including what "in a natural way" means) or other facts about supercompact Radin forcing, see **[8]**, **[2]**, or **[4]**.

We now define a partial ordering P by

$$P = R_{<\kappa^+} \times \prod_{\{(\alpha,\beta)\colon\ \alpha<\beta<\kappa \text{ and } \alpha \text{ and } \beta \text{ are inaccessible}\}} \operatorname{Col}(\alpha^{+(\omega+1)}, \beta) \times \prod_{\{(\omega,\alpha)\colon\ \alpha<\kappa \text{ is inaccessible}\}} \operatorname{Col}(\omega, \alpha)$$

ordered componentwise. Let G be V-generic on P. The model M for Theorem 2 will be a submodel of $V[G]$ and will be a modification of the model N_A as described in **[8]** and **[2]**. We describe this model in more detail below.

Let G_0 be the projection of G onto $R_{<\kappa^+}$. For any condition $\pi = \langle\langle p_1, u_1, C_1\rangle, \ldots, \langle p_n, u_n, C_n\rangle, \langle \mu_{<\kappa^+}, C\rangle\rangle \in R_{<\kappa^+}$, in analogy with supercompact Prikry forcing call $\langle p_1, \ldots, p_n\rangle$ the p-part of π. Let $R = \{p\colon \exists\pi \in G_0[p \in p\text{-part}(\pi)]\}$ and let $R_l = \{p\colon p \in R$ and p is a limit point of $R\}$. We define three sets E_0, E_1, and E_2 by $E_0 = \{\alpha$: For some $\pi \in G_0$ and some $p \in p\text{-part}(\pi)$, $p \cap \kappa = \alpha\}$, $E_1 = \{\alpha\colon \alpha$ is a limit point of $E_0\}$, and $E_2 = E_1 \cup \{\beta\colon \exists \alpha \in E_1[\beta = \alpha^{+\omega}$ or $\beta = \alpha^{+(\omega+1)}]\} \cup \{\omega\}$. Let $\langle \alpha_\nu\colon \nu < \kappa\rangle$ be the continuous increasing enumeration of E_2, and let $\nu = \nu' + n$ for some $n \in \omega$. Sets $C_i^m(\alpha_\nu)$ for $m \in \omega$ are then defined in the following manner.

1. $\nu' = \nu \neq 0$ and $n = 0$. Let $p(\alpha_\nu)$ be the element p of R such that $p \cap \kappa = \alpha_\nu$, and let $h_{p(\alpha_\nu)}\colon p(\alpha_\nu) \to \overline{p(\alpha_\nu)}$ be the order isomorphism between $p(\alpha_\nu)$ and $\overline{p(\alpha_\nu)}$. $C_1^m(\alpha_\nu) = \{h_{p(\alpha_\nu)}{}'' p \cap \alpha_\nu^{+m}\colon p \in R_l,\ p \subseteq p(\alpha_\nu)$, and $h^{-1}_{p(\alpha_\nu)}(\alpha_\nu^{+m}) \in p\}$.
2. $\nu' = 0$ and $n = 3k + 1$. Let $C_2^m(\alpha_\nu) = \{h_{p(\alpha_\nu)}``p \cap \alpha_\nu^{+m}\colon p \in R$, and if $k \geq 1$, $p(\alpha_{\nu'+3(k-1)+1}) \subsetneq p \subseteq p(\alpha_\nu)\}$.
3. ν' is a limit ordinal and for $k > 0$, $n = 3k$. Let $C_3^m(\alpha_\nu) = \{h_{p(\alpha_\nu)}{}'' p \cap \alpha_\nu^{+m}\colon p \in R$ and $p(\alpha_{\nu'+3(k-1)}) \subsetneq p \subseteq p(\alpha_\nu)\}$.
4. $\nu' = 0$ and $n = 3k$. Let $r_{3k+1} = \langle r_{3k+1}^m\colon m \in \omega\rangle$ be the ω sequence generated by G_0 which codes a cofinal sequence through each cardinal in the interval $[\alpha_{3k+1}, \alpha_{3k+1}^{+\omega}]$. (Note that since $\mu_{<\kappa^+}$ has length κ^+ the cofinal sequence through each cardinal in the interval $[\alpha_{3k+1}, \alpha_{3k+1}^{+\omega}]$ and the Radin sequences $C_i^m(\alpha_\nu)$ discussed in 1, 2, and 3 above will all have length ω.) Let $H(\alpha_{3k}, \alpha_{3k+1})$ be the projection of G onto $\mathrm{Col}(\alpha_{3k}, \alpha_{3k+1})$. Then $C_4^m(\alpha_\nu) = H(\alpha_{3k}, \alpha_{3k+1}) \restriction (r_{3k+1}^m \cap \alpha_{3k+1})$.
5. ν' is a limit ordinal and $n = 3k+2$. Let $r_{\nu'+3k+3} = \langle r_{\nu'+3k+3}^m\colon m \in \omega\rangle$ be the ω sequence generated by G_0 which codes a cofinal ω sequence through each cardinal in the interval $[\alpha_{\nu'+3k+3}, \alpha_{\nu'+3k+3}^{+\omega}]$. Let $H(\alpha_{\nu'+3k+2}, \alpha_{\nu'+3k+3})$ be the projection of G onto $\mathrm{Col}(\alpha_{\nu'+3k+2}, \alpha_{\nu'+3k+3})$. Then

$$C_5^m(\alpha_\nu) = H(\alpha_{\nu'+3k+2}, \alpha_{\nu'+3k+3}) \restriction (r_{\nu'+3k+3}^m \cap \alpha_{\nu'+3k+3}).$$

Intuitively, M is $R(\kappa)$ of the least model of ZF extending V which contains, for each $\beta < \kappa$ and each $m < \omega$, $\prod_{\nu \leq \beta} C_i^m(\alpha_\nu)$, where i takes on the values 1 through 5 depending upon which of the above categories ν is and $C_i^m(\alpha_\nu) = \{\varnothing\}$ if ν is not in any of categories 1 through 5. As with Theorem 1, the uniform manner in which the collapsing maps have been placed into M will ensure that the desired cardinals remain cardinals in M and satisfy the Specker property.

To define M more precisely, it is necessary to define canonical names $\underline{\alpha_\nu}$ for the α_ν's and canonical names $\underline{C_i^m(\nu)}$ for the sets $C_i^m(\alpha_\nu)$. Recall that if $\alpha_\nu \in E_1$ it is possible to decide $p(\alpha_\nu)$ (and hence $\overline{p(\alpha_\nu)}$) by writing $\omega \cdot \nu = \omega^{\sigma_0} \cdot n_0 + \omega^{\sigma_1} \cdot n_1 + \cdots + \omega^{\sigma_k} \cdot n_k$ (where $\sigma_0 > \sigma_1 > \cdots > \sigma_k > 0$ are ordinals, $n_0, \ldots, n_k$ are integers, and $+$, $\cdot$, and exponentiation are as in ordinal arithmetic), letting $\pi = \langle\langle p_{ij_i}, u_{ij_i}, D_{ij_i}\rangle_{i\le k, 1\le j_i\le n_i}, \langle\mu_{<\kappa^+}, D\rangle\rangle$ be such that $\min(p_{i1}\cap\kappa, \omega^{\mathrm{length}(u_{i1})}) = \sigma_i$ and $\mathrm{length}(u_{ij_i}) = \min(p_{i1}\cap\kappa, \mathrm{length}(u_{i1}))$ for $1 \le j_i \le n_i$ and letting $p(\alpha_\nu)$ be p_{kn_k}. Further $D_\nu = \{r \in P : r \restriction R_{<\kappa^+}$ extends a condition π of the above form$\}$ is a dense open subset of P, and any element of D_ν, besides determining α_ν, determines in addition $\alpha_\nu^{+\omega}$ and $\alpha_\nu^{+(\omega+1)}$. Then for *any* $\alpha_\nu \in E_2$, $\underline{\alpha_\nu}$ is the name of the α_ν determined by any element of $D_\nu \cap G$; in the notation of **[8]**, for $\alpha_\nu \in E_1$, $\underline{\alpha_\nu} = \{\langle r, \check{\alpha}_\nu(r)\rangle : r \in D_\nu\}$, where $\alpha_\nu(r)$ is the α_ν determined by the condition r. For α_ν so that $\alpha_\nu = \beta_\gamma^{+\omega}$ or $\alpha_\nu = \beta_\gamma^{+(\omega+1)}$, where $\beta_\gamma \in E_1$, $\underline{\alpha_\nu} = \{\langle r, \check{\beta}_\gamma^{+\zeta}(r)\rangle : r \in D_\nu\}$, where $\zeta = \omega$ or $\omega + 1$ depending on whether $\alpha_\nu = \beta_\gamma^{+\omega}$ or $\alpha_\nu = \beta_\gamma^{+(\omega+1)}$, γ is the (unique) ordinal so that $\alpha_\nu = \beta_\gamma^{+\zeta}$, and $\beta_\gamma(r)$ is the β_γ determined by the condition r.

The canonical names $\underline{C_i^m(\nu)}$ for the sets $C_i^m(\alpha_\nu)$ are defined in a manner so as to be invariant under the appropriate group of automorphisms. Specifically, there are five cases to consider. We again write $\nu = \nu' + n$, and in analogy to **[8]** and **[2]**, assume without loss of generality that for any $\alpha_\nu \in E_1$, D_ν determines $\alpha_{\nu-1}$ (if $\nu - 1$ exists), α_ν, $\alpha_{\nu+1} = \alpha_\nu^{+\omega}$, $\alpha_{\nu+2} = \alpha_\nu^{+(\omega+1)}$, and $\alpha_{\nu+3}$.

1. $\nu' = \nu \neq 0$ and $n = 0$. $\underline{C_1^m(\nu)}$ is then the name for $\{h_{p(\alpha_\nu)(r)}{}''p \cap \alpha_\nu^{+m} : \exists r \in P[r \in D_\nu \cap G, p \in p\text{-part}(r \restriction R_{<\kappa^+}), p \subseteq p(\alpha_\nu)(r), p \in R_l \restriction r$, and $h^{-1}_{p(\alpha_\nu)(r)}(\alpha_\nu^{+m}) \in p]\}$ where $p(\alpha_\nu)(r)$ and $h_{p(\alpha_\nu)(r)}$ are the $p(\alpha_\nu)$ and $h_{p(\alpha_\nu)}$ determined by the condition r and $R_l \restriction r$ is the portion of R_l determined by r. Note that this definition is unambiguous, since for any r and r' so that $r, r' \in D_\nu \cap G$, $p(\alpha_\nu)(r) = p(\alpha_\nu)(r')$. In the notation of **[8]**, $\underline{C_1^m(\nu)} = \{\langle r, (\check{r} \restriction R_{<\kappa^+}) \restriction (\alpha_\nu(r), \alpha_\nu^{+m}(r))\rangle : r \in D_\nu\}$, where for $r \in P$, $\pi = r \restriction R_{<\kappa^+}$, $\pi \restriction (\alpha_\nu(r), \alpha_\nu^{+m}(r)) = \{h_{p(\alpha_\nu)(r)}{}''p \cap \alpha_\nu^{+m} : p \in p\text{-part}(\pi)$, $p \subseteq p(\alpha_\nu)(r)$, $p \in R_l \restriction \pi$, and $h^{-1}_{p(\alpha_\nu)(r)}(\alpha_\nu^{+m}) \in p\}$.

2. $\nu' = 0$ and $n = 3k + 1$. We have assumed without loss of generality that for $k \ge 1$, D_ν determines $\alpha_{\nu'+3(k-1)+1}$. $\underline{C_2^m(\nu)}$ is then the name for $\{h_{p(\alpha_\nu)(r)}{}''p \cap \alpha_\nu^{+m} : \exists r \in P[r \in D_\nu \cap G, p \in p\text{-part}(r \restriction R_{<\kappa^+}), p(\alpha_{\nu'+3(k-1)+1}) \subseteq p \subseteq p(\alpha_{\nu'+3k+1})$, $p \in R \restriction r$, and $h^{-1}_{p(\alpha_\nu)(r)}(\alpha_\nu^{+m}) \in p]\}$, where $R \restriction r$ is the portion of R determined by r. The unambiguity of this definition again follows from the fact that for $r, r' \in D_\nu \cap G$, $p(\alpha_{\nu'+3(k-1)+1})(r) = p(\alpha_{\nu'+3(k-1)+1})(r')$ and $p(\alpha_{\nu'+3k+1})(r) = p(\alpha_{\nu'+3k+1})(r')$. In the notation of **[8]**, $\underline{C_2^m(\nu)} = \{\langle r, (\check{r} \restriction R_{<\kappa^+}) \restriction (\alpha_\nu(r), \alpha_\nu^{+m}(r))\rangle : r \in D_\nu\}$, where this time for $r \in P$,

$\pi = r \restriction R_{<\kappa^+}$, $\pi \restriction (\alpha_\nu(r), \alpha_\nu^{+m}(r)) = \{h_{p(\alpha_\nu)(r)}{}''p \cap \alpha_\nu^{+m} \colon p \in p$-part$(\pi)$, $p \in R \restriction \pi$, $p(\alpha_{\nu'+3(k-1)+1})(r) \subseteq p \subseteq p(\alpha_{\nu'+3k+1})(r)$, and $h^{-1}_{p(\alpha_\nu)(r)}(\alpha_\nu^{+m}) \in p\}$. For $k = 0$, the definition of $\underline{C_2^m(\nu)}$ is the same as just stated, dropping the proviso that

$$p(\alpha_{\nu'+3(k-1)+1})(r) \subseteq p \subseteq p(\alpha_{\nu'+3k+1})(r)$$

and

$$p(\alpha_{\nu'+3(k-1)+1}) \subseteq p \subseteq p(\alpha_{\nu'+3k+1}).$$

3. ν' is a limit ordinal and for $k > 0$, $n = 3k$. Again assume without loss of generality that D_ν determines $\alpha_{\nu'+3(k-1)}$. $\underline{C_3^m(\nu)}$ is then the name for $\{h_{p(\alpha_\nu)(r)}{}''p \cap \alpha_\nu^{+m} \colon \exists r \in P[r \in D_\nu \cap G, p \in p$-part$(r \restriction R_{<\kappa^+})$, $p(\alpha_{\nu'+3(k-1)}) \subseteq p \subseteq p(\alpha_{\nu'+3k})$, $p \in R \restriction r$, and $h^{-1}_{p(\alpha_\nu)(r)}(\alpha_\nu^{+m}) \in p]\}$. In the notation of **[8]**, $\underline{C_3^m(\nu)} = \{\langle r, (\check{r} \restriction R_{<\kappa^+}) \restriction (\alpha_\nu(r), \alpha_\nu^{+m}(r))\rangle \colon r \in D_\nu\}$, where for $r \in P$, $\pi = r \restriction R_{<\kappa^+}$, $\pi \restriction (\alpha_\nu(r), \alpha_\nu^{+m}(r)) = \{h_{p(\alpha_\nu)(r)}{}''p \cap \alpha_\nu^{+m} \colon p \in p$-part$(\pi)$, $p \in R \restriction \pi$, $p(\alpha_{\nu'+3(k-1)})(r) \subseteq p \subseteq p(\alpha_{\nu'+3k})(r)$, and $h^{-1}_{p(\alpha_\nu)(r)}(\alpha_\nu^{+m}) \in p\}$.

4. $\nu' = 0$ and $n = 3k$. Let $\underline{r_{3k+1}^{0,m}}$ be the canonical name for the mth element of $C_2^0(\alpha_{3k+1})$, $r_{3k+1}^{0,m}$, defined using $\underline{C_2^0(3k+1)}$. $\underline{C_4^m(\nu)}$ is then the name for $\{p \restriction \underline{r_{3k+1}^{0,m}} \colon \exists q \in P[q \in D_{\nu+1} \cap G, p \in q \restriction \mathrm{Col}(\alpha_\nu(q), \alpha_{\nu+1}(q))]\}$. In the notation of **[8]**,

$$\underline{C_4^m(\nu)} = \{\langle q, (\check{q} \restriction \mathrm{Col}(\alpha_\nu(q), \alpha_{\nu+1}(q)) \restriction \underline{r_{3k+1}^{0,m}}\rangle \colon q \in D_{\nu+1}\}.$$

5. ν' is a limit ordinal and $n = 3k+2$. Let $\underline{r_{\nu'+3k+3}^{0,m}}$ be the canonical name for the mth element of $C_3^0(\alpha_{\nu'+3k+3})$, $r_{\nu'+3k+3}^{0,m}$, defined using $\underline{C_3^0(\nu'+3k+3)}$. $\underline{C_5^m(\nu)}$ is then the name for $\{p \restriction \underline{r_{\nu'+3k+3}^{0,m}} \colon \exists q \in P[q \in D_{\nu+1} \cap G, p \in q \restriction \mathrm{Col}(\alpha_\nu(q), \alpha_{\nu+1}(q))]\}$. In the notation of **[8]**, $\underline{C_5^m(\nu)} = \{\langle q, (\check{q} \restriction \mathrm{Col}(\alpha_\nu(q), \alpha_{\nu+1}(q))) \restriction \underline{r_{\nu'+3k+3}^{0,m}} \colon q \in D_{\nu+1}\}$.

Using the canonical names $\underline{C_i^m(\nu)}$, define for fixed m the canonical names $\underline{E_\nu^m}$, $\nu < \kappa$, as the name for $(\prod_{\beta \leq \nu} \underline{C_i^m(\beta)})$, where i takes on the values 1 through 5 depending upon in which of the categories β is, and where $\underline{C_i^m(\beta)}$ is a term for $\{\varnothing\}$ if β is in none of these five categories. Let $\mathscr{G}$ be the group of automorphisms of **[8]**, and let $\underline{C(G)} = \bigcup_{m \in \omega}(\bigcup_{\nu < \kappa}(\bigcup_{\pi \in \mathscr{G}}\{\pi(\underline{E_\nu^m})\}))$, where for each $\pi \in \mathscr{G}$, $\pi(\underline{E_\nu^m})$ is found by taking the action of π on each component of $\underline{E_\nu^m}$, $\underline{C_i^m(\beta)}$, for $\beta \leq \nu$. $C(G) = \bigcup_{m \in \omega}(\bigcup_{\nu < \kappa}(\bigcup_{\pi \in \mathscr{G}}\{i_G(\pi(\underline{E_\nu^m}))\})) = i_G(\underline{C(G)})$. M is then the set of all sets of rank $< \kappa$ of the model consisting of all sets which are hereditarily V definable from $C(G)$, i.e., $M = R(\kappa)^{\mathrm{HVD}(C(G))}$.

By the definition of M, we know that for any set $x \subseteq \alpha_\nu$ in M, α_ν arbitrary, a term τ for x can be found which mentions only finitely many of the $\underline{E^m_\nu}$. By letting m_0 be the sup of all of the m's appearing in τ, and letting ν_0 be the sup of all of the ν's appearing in τ, τ can be rewritten using a term τ' which mentions only $\underline{E^{m_0}_{\nu_0}}$. The following weak analogue of Theorem 3.2.11 of [**8**] then holds: For any $x \subseteq \alpha_\nu$ (or indeed, any ordinal $\delta \in M$) as just mentioned, $x \in V[\langle \alpha_\beta : \beta \leq \nu_0 \rangle, E^{m_0}_{\nu_0}]$. Then, using again the arguments of Theorem 3.2.11 of [**8**], together with the fact that the subsets of α_ν in $V[\langle \alpha_\beta : \beta \leq \nu_0 \rangle, E^{m_0}_{\nu_0}]$ are the same as the subsets of α_ν in $V[\langle \alpha_\beta : \beta \leq \nu \rangle, E^{m_0}_\nu]$ (since as in Lemma 1.1, the part of the forcing above ν which determines the portion of $E^{m_0}_{\nu_0}$ above ν, $F^{m_0}_{\nu_0} = \langle C^{m_0}_\beta : \nu < \beta \leq \nu_0 \rangle$, can be taken as a full support iteration of partial orderings which satisfy the Prikry property of short enough length (the length is short enough since the full Radin forcing has length κ^+, so any ordinal in the Radin sequence will be singular) so that the result of [7] again can be applied to show that $V[\langle \alpha_\beta : \nu < \beta \leq \nu_0 \rangle, F^{m_0}_{\nu_0}]$ contains the same subsets of α_ν as V, so any new subsets of α_ν come from $V[\langle \alpha_\beta : \beta \leq \nu \rangle, \langle C^{m_0}_\beta : \beta \leq \nu \rangle])$ we have that for any $x \subseteq \alpha_\nu$ with $x \in M$, $x \in V[\langle \alpha_\beta : \beta \leq \nu \rangle, E^{m_0}_\nu]$. This fact will be the key fact used in completing the proof of Theorem 2.

LEMMA 2.1. *Each α_ν is a cardinal in M.*

Proof of Lemma 2.1. By the preceding remarks, for any $x \subseteq \alpha_\nu$ with $x \in M$ there is some $m \in \omega$ so that $x \in V[\langle \alpha_\beta : \beta \leq \nu \rangle, E^m_\nu]$. Since $\langle \alpha_\beta : \beta \leq \nu \rangle$ is a Radin sequence together with the ωth and $\omega + 1$st successors of each element of the sequence, α_ν remains a cardinal in $V[\langle \alpha_\beta : \beta \leq \nu \rangle]$.

Assume now that ν is a successor ordinal. Write $E^m_\nu = \prod_{\beta < \nu} C^m_\beta \times C^m_\nu$. If α_ν is an element of the Radin sequence, then C^m_ν is generated by a supercompact Radin ordering which, because the length of the original Radin sequence of measures is κ^+, is isomorphic to a supercompact Prikry ordering. It is therefore the case that α_ν is a cardinal in $V[C^m_\nu]$. The partial ordering which generates $\langle \langle \alpha_\beta : \beta < \nu \rangle, \prod_{\beta < \nu} C^m_\beta \rangle$ has cardinality $< \alpha_\nu$ by GCH; hence, α_ν is a cardinal in

$$V[C^m_\nu][\langle \alpha_\beta : \beta < \nu \rangle, \prod_{\beta < \nu} C^m_\beta] = V[\langle \alpha_\beta : \beta \leq \nu \rangle, E^m_\nu].$$

If α_ν is the $\omega + 1$st successor of an element of the Radin sequence, then C^m_ν is generated by a Lévy collapse ordering, so it is again the case that α_ν is a cardinal in $V[C^m_\nu]$. The same argument as stated in the next to last sentence can be applied again to show that α_ν is a cardinal in $V[\langle\alpha_\beta\colon \beta \le \nu\rangle, E^m_\nu]$. If α_ν is the ωth successor of an element of the Radin sequence, then $\langle\langle\alpha_\beta\colon \beta \le \nu\rangle, E^m_\nu\rangle$ is generated by a partial ordering of cardinality $< \alpha_\nu$ by GCH, so again, α_ν is a cardinal in $V[\langle\alpha_\beta\colon \beta \le \nu\rangle, E^m_\nu]$.

If ν is a limit ordinal, assume that α_ν is not a cardinal in $V[\langle\alpha_\beta\colon \beta \le \nu\rangle, E^m_\nu]$. There must then be a subset of α_{β_0} for $\beta_0 < \nu$ which codes this fact. This subset of α_{β_0} is an element of $V[\langle\alpha_\beta\colon \beta \le \beta_0\rangle, E^{m'}_{\beta_0}]$ for some $m' \in \omega$. By GCH, the set $\langle\langle\alpha_\beta\colon \beta \le \beta_0\rangle, E^{m'}_{\beta_0}\rangle$ is generated by a partial ordering of cardinality $< \alpha_\nu$, an impossibility. Thus, for any value of ν, no subset $x \subseteq \alpha_\nu$ can code a function f so that for any m having the property that $x \in V[\langle\alpha_\beta\colon \beta \le \nu\rangle, E^m_\nu]$ f witnesses that α_ν is not a cardinal. This means that α_ν is a cardinal in M. $\square$

LEMMA 2.2. *$M \vDash$ "$\langle\alpha_\nu\colon \nu \in \mathrm{Ord}^M\rangle = \langle\aleph_\nu\colon \nu \in \mathrm{Ord}^M\rangle$".*

Proof of Lemma 2.2. By the fact that all E^m_ν code collapse maps, the definition of the sequence $\langle\alpha_\nu\colon \nu < \kappa\rangle$, and the definition of M, it is inductively the case that $M \vDash$ "$\forall\nu\ [\alpha_\nu \le \aleph_\nu]$". Since each α_ν is a cardinal in M, this immediately yields that $M \vDash$ "$\forall\nu\ [\alpha_\nu = \aleph_\nu]$". $\square$

LEMMA 3.3. *If α_ν is an element of the Radin sequence, or if α_ν is the $\omega+1$st successor of an element of the Radin sequence, then $M \vDash \mathrm{SP}(\aleph_\nu)$.*

Proof of Lemma 2.3. Working in M, for α_ν as above, let $X_n = \{x \subseteq \alpha_\nu\colon x$ is definable using $E^n_\nu\}$. By our earlier remarks, $X_n \in V[\langle\alpha_\beta\colon \beta \le \nu\rangle, E^n_\nu] \subseteq V[\langle\alpha_\beta\colon \beta \le \nu\rangle, E^{n+1}_\nu]$, and each $x \subseteq \alpha_\nu$ so that $x \in M$ is an element of X_m for some m. As in Lemma 1.4, $V[\langle\alpha_\beta\colon \beta \le \nu\rangle, E^{n+1}_\nu] \vDash$ "$|X_n| = \alpha_\nu$", i.e., $V[\langle\alpha_\beta\colon \beta \le \nu\rangle, E^{n+1}_\nu]$ contains a bijection f between X_n and α_ν. This bijection f is such that

$$f \in R(\kappa)^{V[\langle\alpha_\beta\colon \beta\le\nu\rangle, E^{n+1}_\nu]} \subseteq R(\kappa)^{\mathrm{HVD}(C(G))} = M, \quad \text{so } M \vDash \text{“}|X_n| = \alpha_\nu\text{”}.$$

Thus, as in Lemma 1.4, $M \vDash$ "$\bigcup_{n\in\omega} X_n = 2^{\alpha_\nu}$ and for each n, $|X_n| = \alpha_\nu$", i.e., $M \vDash \mathrm{SP}(\alpha_\nu)$. $\square$

It inductively follows, by the definition of M and Lemmas 2.1 and 2.2, that the α_ν's as in the statement of Lemma 2.3 become the required $\aleph_\nu$'s of the statement of Theorem 2.

As with Theorem 1 and in analogy with the model N_A of **[8]** and **[2]**, M will satisfy all of the axioms of ZF with the possible exception of Replacement. Thus, the following lemma will complete the proof of Theorem 2.

LEMMA 2.4. $M \models \mathrm{ZF}$.

Proof of Lemma 2.4. The proof of Lemma 2.4 is analogous to the proofs of Lemma 1.5 and Theorem 4.2 of [**8**]. If Replacement fails in M, then for some set $X \in M$ there is a class function f on X such that $f''X \notin M$. As before, we can assume that $X = R(\beta)$ for some $\beta < \kappa$ and range$(f) \subseteq \kappa$ with range(f) unbounded in κ. If β is the minimal such ordinal, then again, β must be a successor ordinal, for if β were a limit ordinal, then we could construct a function $f'\colon \beta \to \kappa$ whose range was unbounded in κ. Since $M \subseteq \mathrm{HVD}(C(G))$ and $\mathrm{HVD}(C(G)) \models \mathrm{ZF}$, $f' \in \mathrm{HVD}(C(G))$. By our earlier remarks, for some $m \in \omega$ and $\nu < \kappa$, $f' \in V[\langle \alpha_\gamma\colon \gamma \le \nu\rangle, E^m_\nu]$. Since $\langle\langle \alpha_\gamma\colon \gamma \le \nu\rangle, E^m_\nu\rangle$ is generated by a partial ordering of cardinality $< \kappa$ and κ is inaccessible in V, $V[\langle \alpha_\gamma\colon \gamma \le \nu\rangle, E^m_\nu] \models$ "κ is inaccessible", a contradiction to the fact that we have just shown $V[\langle \alpha_\gamma\colon \gamma \le \nu\rangle, E^m_\nu] =$"$\kappa$ is singular". Thus, $\beta = \delta + 1$ for some δ, and in $\mathrm{HVD}(C(G))$ there is a function $f\colon p(R(\delta)) = R(\beta) \to \kappa$ whose range is unbounded in κ.

Define now, for $\nu < \kappa$, $\underline{E_\nu}$ as a term for the product of the full collapse maps for the relevant elements of the sequence $\langle \alpha_\beta\colon \beta < \kappa\rangle$; in other words, $\underline{E_\nu}$ is a term for $\prod_{\beta \le \nu} \underline{C_i(\beta)}$, where as before, $\underline{C_i(\beta)}$ is a term for $\{\varnothing\}$ for the appropriate values of β, and for those values of β for which $\underline{C_i(\beta)}$ is not a term for $\{\varnothing\}$, $\underline{C_i(\beta)}$ is as in the definition of $\underline{C^m_i(\beta)}$ except that full and not partial collapse maps are taken. Using this definition, we make now the following

Claim. For each ordinal $\gamma < \kappa$,

$$R(\omega+\gamma)^M = R(\omega+\gamma)^{\mathrm{HVD}(C(G))} \subseteq R(\omega+\gamma)^{V[\langle \alpha_\delta\colon \delta \le \gamma + n(\gamma)\rangle, E_{\gamma+n(\gamma)}]}$$

for some $n(\gamma) \in \omega$.

Proof of Claim. Since $R(\omega)$ is absolute, the claim is true for $\gamma = 0$. If γ is a limit ordinal, then by hypothesis, for every $\sigma < \gamma$, $R(\omega+\sigma)^M \subseteq R(\omega+\sigma)^{V[\langle \alpha_\delta\colon \delta \le \sigma + n(\sigma)\rangle, E_{\sigma+n(\sigma)}]}$. It is also true that for every $\sigma < \gamma$, $V[\langle \alpha_\delta\colon \delta \le \sigma + n(\sigma)\rangle, E_{\sigma+n(\sigma)}] \subseteq V[\langle \alpha_\delta\colon \delta \le \gamma\rangle, E_\gamma]$; therefore, for every $\sigma < \gamma$,

$$\begin{aligned} R(\omega+\sigma)^M = R(\omega+\sigma)^{\mathrm{HVD}(C(G))} &\subseteq R(\omega+\sigma)^{V[\langle \alpha_\delta\colon \delta \le \sigma + n(\sigma)\rangle, E_{\sigma+n(\sigma)}]} \\ &\subseteq R(\omega+\sigma)^{V[\langle \alpha_\delta\colon \delta \le \gamma\rangle, E_\gamma]}, \end{aligned}$$

so

$$\begin{aligned}\bigcup_{\sigma<\gamma} R(\omega+\sigma)^M &= \bigcup_{\sigma<\gamma} R(\omega+\sigma)^{\mathrm{HVD}(C(G))} \\ &\subseteq \bigcup_{\sigma<\gamma} R(\omega+\sigma)^{V[\langle\alpha_\delta:\ \delta\le\sigma+n(\sigma)\rangle, E_{\sigma+n(\sigma)}\rangle]} \\ &\subseteq \bigcup_{\sigma<\gamma} R(\omega+\sigma)^{V[\langle\alpha_\delta:\ \delta\le\gamma\rangle, E_\gamma]} = R(\omega+\gamma)^{V[\langle\alpha_\delta:\ \delta\le\gamma\rangle, E_\gamma]};\end{aligned}$$

note that $n(\gamma) = 0$.

If $\gamma = \sigma + 1$ is a successor ordinal, then by hypothesis we have $R(\omega+\sigma)^M = R(\omega+\sigma)^{\mathrm{HVD}(C(G))} \subseteq R(\omega+\sigma)^{V[\langle\alpha_\delta:\ \delta\le\sigma+n(\sigma)\rangle, E_{\sigma+n(\sigma)}]}$. For Q the partial ordering which generates $\langle\langle\alpha_\delta\colon \delta \le \sigma + n(\sigma)\rangle, E_{\sigma+n(\sigma)}\rangle$, we can find some $m \in \omega$ such that $\alpha_{\sigma+n(\sigma)+m}$ is inaccessible and $|Q| < \alpha_{\sigma+n(\sigma)+m}$, so

$$\begin{aligned}&V[\langle\alpha_\delta\colon \delta \le \sigma + n(\sigma)\rangle, E_{\sigma+n(\sigma)}] \\ &\qquad \models \text{“}|R(\omega+\sigma)^{V[\langle\alpha_\delta:\ \delta\le\sigma+n(\sigma)\rangle, E_{\sigma+n(\sigma)}]}| < \alpha_{\sigma+n(\sigma)+m}\text{”}.\end{aligned}$$

If $x \subseteq R(\omega+\sigma)^M$ is a set in M then x can be viewed as a subset of $\alpha_{\sigma+n(\sigma)+m}$ in the analogue of M constructed by forcing over $V[\langle\alpha_\delta\colon \delta \le \sigma + n(\sigma)\rangle, E_{\sigma+n(\sigma)}]$ with the partial ordering consisting of the cartesian product of the portion of the Radin forcing $R_{<\kappa^+}$ above $\alpha_{\sigma+n(\sigma)}$ with

$$\prod_{\{(\alpha,\beta)\colon\ \alpha_{\sigma+n(\sigma)}<\alpha<\beta<\kappa \text{ and } \alpha \text{ and } \beta \text{ are inaccessible}\}} \mathrm{Col}(\alpha^{+(\omega+1)}, \beta).$$

An analogue of our remarks before Lemma 2.1 which takes into account that the successor of an element in the sequence $\langle\alpha_\nu\colon \nu < \kappa\rangle$ is not necessarily an inaccessible cardinal yields that $x \in V[\langle\alpha_\delta\colon \delta \le \sigma + n(\sigma)\rangle, E_{\sigma+n(\sigma)}]\,[\langle\alpha_\delta\colon \sigma+n(\sigma) < \delta \le \sigma+n(\sigma)+m\rangle, \langle C_i(\delta)\colon \sigma+n(\sigma) < \delta \le \sigma+n(\sigma)+m\rangle]\,[\langle\alpha_\delta\colon \sigma+n(\sigma)+m < \delta \le \beta\rangle, \langle C_i^k(\delta)\colon \sigma+n(\sigma)+m < \delta \le \beta\rangle]$ where $\beta < \kappa$ and $k \in \omega$. As in our remarks before Lemma 2.1, the subsets of $\alpha_{\sigma+n(\sigma)+m}$ in this model are the same as those in $V[\langle\alpha_\delta\colon \delta \le \sigma + n(\sigma) + m\rangle, E_{\sigma+n(\sigma)+m}]$, i.e., $x \in V[\langle\alpha_\delta\colon \delta \le \sigma + n(\sigma)\rangle, E_{\sigma+n(\sigma)}]\,[\langle\alpha_\delta\colon \sigma + n(\sigma) < \delta \le \sigma + n(\sigma) + m\rangle, \langle C_i(\delta)\colon \sigma + n(\sigma) < \delta \le \sigma + n(\sigma) + m\rangle] = V[\langle\alpha_\delta\colon \delta \le \sigma + n(\sigma) + m\rangle, E_{\sigma+n(\sigma)+m}]$. Thus, all subsets of $R(\omega+\sigma)^M$ in M are elements of $V[\langle\alpha_\delta\colon \delta \le \sigma + n(\sigma) + m\rangle, E_{\sigma+n(\sigma)+m}]$. Since

$$\begin{aligned}R(\omega+\sigma)^M &\subseteq R(\omega+\sigma)^{V[\langle\alpha_\delta:\ \delta\le\sigma+n(\sigma)\rangle, E_{\sigma+n(\sigma)}]} \\ &\subseteq R(\omega+\sigma)^{V[\langle\alpha_\delta:\ \delta\le\sigma+n(\sigma)+m\rangle, E_{\sigma+n(\sigma)+m}]}\end{aligned}$$

this immediately yields that

$$\begin{aligned} R(\omega+\sigma+1)^M &= R(\omega+\gamma)^M = R(\omega+\gamma)^{\mathrm{HVD}(C(G))} \\ &\subseteq R(\omega+\sigma+1)^{V[\langle\alpha_\delta:\ \delta\leq\sigma+n(\sigma)+m\rangle, E_{\sigma+n(\sigma)+m}]} \\ &= R(\omega+\gamma)^{V[\langle\alpha_\delta:\ \delta\leq\gamma+n(\gamma)\rangle, E_{\gamma+n(\gamma)}]}. \end{aligned}$$

□

Using the Claim and working in $V[\langle\alpha_\sigma : \sigma \leq \gamma+n(\gamma)\rangle, E_{\gamma+n(\gamma)}]$ for the γ so that $\omega+\gamma=\beta$, for δ the cardinal $<\kappa$ so that

$$V[\langle\alpha_\sigma : \sigma \leq \gamma+n(\gamma)\rangle, E_{\gamma+n(\gamma)}] \models \text{“}|R(\beta)^{V[\langle\alpha_\sigma:\ \sigma\leq\gamma+n(\gamma)\rangle, E_{\gamma+n(\gamma)}]}| = \delta\text{”}$$

we can get in the analogue of M, M^*, obtained by forcing over $V[\langle\alpha_\sigma : \sigma \leq \gamma+n(\gamma)\rangle, E_{\gamma+n(\gamma)}]$ with the partial ordering consisting of the cartesian product of the portion of the Radin forcing $R_{<\kappa^+}$ above $\alpha_{\gamma+n(\gamma)}$ with

$$\prod_{\{(\alpha,\beta):\ \alpha_{\gamma+n(\gamma)}<\alpha<\beta<\kappa \text{ and } \alpha \text{ and } \beta \text{ are inaccessible}\}} \mathrm{Col}(\alpha^{+(\omega+1)}, \beta)$$

a function $g : \delta \to \kappa$ whose range is unbounded in κ. Since the same analogue of Lemma 2.1 that was true in the preceding paragraph will be true about M^*, κ will be a regular cardinal in M^*. (In V, the partial ordering which generates $\langle\langle\alpha_\sigma : \sigma \leq \gamma+n(\gamma)\rangle, E_{\gamma+n(\gamma)}\rangle$ has cardinality $<\kappa$, and any subset of κ in M^* will be in a generic extension of $V[\langle\alpha_\sigma : \sigma \leq \gamma+n(\gamma)\rangle, E_{\gamma+n(\gamma)}]$ obtained by forcing with a partial ordering of cardinality $<\kappa$.) Since κ is thus both regular and singular in M^*, this contradiction establishes that $M \models$ Replacement and proves Lemma 2.4. □

Lemmas 2.1–2.4 complete the proof of Theorem 2.

In conclusion, we remark that the assumption of GCH shows that for all cardinals δ in the preceding models for which $\mathrm{SP}(\delta)$ is false, 2^δ can be written as a countable union of sets of cardinality δ^+.

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