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THE EULER CHARACTER AND CANCELLATION THEOREMS FOR WEYL MODULES

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We consider for a simple, simply connected algebraic group over an algebraically closed field of characteristic p, modules induced from characters on a Borel subgroup. We ask if the socle levels of the modules induced from characters in a general position determine the socle levels of modules induced from characters in a singular position. Technically, the question may be phrased in terms of the infinitesimal group subscheme determined by the Frobenius morphism of the global group. Qualitatively, we show that the socle levels of the global induced modules are induced from the socle levels of the infinitesimal induced modules, assuming only that Bott's Theorem applies. Quantitatively, we show that the multiplicities of the composition factors of the module induced from an infinitesimal socle layer are determined by the structure of the layer as a module for the Borel subgroup.

Introduction. Let G be a simple, simply connected algebraic group over an algebraically closed field of characteristic p, and let T be a maximal torus in a Borel subgroup B. Let G_n be the kernel of the nth power of the Forbenius map on G. Let $H^0(G/B, \lambda)$ be the G-module induced from a character λ on B, $H^0(G_nB/B, \lambda)$ the G_nB-module induced from λ, and $H^0(G/G_nB, M)$ the G-module induced from a G_nB-module M. We consider whether the socle levels of $H^0(G/B, \lambda)$ for λ in a singular position can be computed in terms of the socle levels of $H^0(G/B, \mu)$ for μ in general position. Equivalently, one can ask if the socle levels of $H^0(G/B, \lambda)$ can be computed in terms of the socle levels of $H^0(G_nB/B, \lambda)$. Qualitatively, we show that the socle levels of $H^0(G/B, \lambda)$ are induced from the G_n-socle levels of $H^0(G_nB/B, \lambda)$, assuming only that Bott's Theorem applies to the composition factors of $H^0(G_nB/B, \lambda)$. That leaves open the quantitative question of relative multiplicities of the composition factors of the socle levels of $H^0(G/B, \lambda)$ and $H^0(G_nB/B, \lambda)$. We do show that the multiplicities of the composition factors of the G-module induced from a G_n-socle layer M of $H^0(G_nB/B, \lambda)$ are determined by the G_nB-structure of the G_n-semisimple module M.

Andersen, in [4, Theorem 4.2], showed that the global socle levels $\mathrm{Soc}_G^j H^0(G/B,\lambda)$ are induced from the infinitesimal G_n-socle levels $\mathrm{Soc}_{G_n}^j H^0(G_nB/B,\lambda)$, assuming that all the highest weights μ of the G_nB-composition factors $L_n(\mu)$ of $H^0(G_nB/B,\lambda)$ are dominant and that Bott's Theorem applies to those weights. Furthermore, in that case, the socle levels agree quantitatively, i.e., the multiplicity $[\mathrm{Soc}_{G_n}^j H^0(G_nB/B,\lambda): L_n(\mu)]$ of $L_n(\mu)$ as a composition factor of $\mathrm{Soc}_{G_n}^j H^0(G_nB/B,\lambda)$ equals the multiplicity $[\mathrm{Soc}_G^j H^0(G/B,\lambda): L(\mu)]$ of the irreducible G-module $L(\mu)$ as a composition factor of $\mathrm{Soc}_G^j H^0(G/B,\lambda)$ (§3.1). In §2, we show that the global socle levels are induced from the infinitesimal G_n-socle levels, even in the singular case where some highest weights are non-dominant. To complete the picture in the singular case, we would need to know the quantitative relation between the multiplicities in the global and infinitesimal socle levels.

By the cancellation principle of Jantzen, expressed in terms of the Euler character, the multiplicity $[H^0(G/B,\lambda): L(\mu)]$ can be computed as an alternating sum of the multiplicities in $H^0(G_nB/B,\lambda)$ of the modules in the alcove transition orbit of $L_n(\mu)$ (see [7, Proposition 4.1]). From that principle, we see that singular values for $H^0(G/B,\lambda)$ occur, i.e., there are dominant weights λ, μ where

$$[H^0(G/B,\lambda): L(\mu)] < [H^0(G_nB/B,\lambda): L_n(\mu)].$$

Ideally, that cancellation in terms of alcove transition orbits would work for the socles, level by level and layer by layer. While we do not establish here that things work that way, we do show that those orbits enter into the singular values of the module $H^0(G/G_nB, M)$ induced from a G_n-socle layer

$$M = \mathrm{Soc}_{G_n}^j H^0(G_nB/B,\lambda)/\mathrm{Soc}_{G_n}^{j-1} H^0(G_nB/B,\lambda),$$

via the G_nB-structure of M (which need not be semisimple [9]). That structure does not complicate the G-structure of $H^0(G/G_nB, M)$, which is just semisimple, but it does control the values $[H^0(G/G_nB,M): L(\mu)]$. In 3.2, Theorem, we show that $[H^0(G/G_nB,M): L(\mu)]$ is determined by the position in the G_nB-structure of M of the alcove transition orbit of $L_n(\mu)$.

Notation. $X(T)$ is the character group of T, W is the Weyl group of (G,T), and $(\ ,\ )$ is a W-invariant inner product on $X(T)\otimes_{\mathbf{Z}}\mathbf{R}$. R is the set of roots of (G,T) and γ^v is the coroot of $\gamma\in R$. S is the set of simple roots in the set R_+ of roots of the Borel subgroup opposite to B. ρ is the half-sum of the roots of R_+. For $w \in W$ and $\lambda \in X(T)$, $w(\lambda)$ denotes the ordinary action of w on λ and $w \cdot \lambda = w(\lambda + \rho) - \rho$ is the action which has $-\rho$ as fixed point, for all $w \in W$. $X(T)_+$ is the dominant chamber and $w \cdot X(T)_+$ is the w-chamber for the operation $w\cdot$.

C_0 is the fundamental alcove $\{\lambda \in X(T) | 0 < (\gamma^v, \lambda + \rho) < p,\ \gamma \in R_+\}$, $\hat{C}_0 = \{\lambda \in X(T) | 0 < (\gamma^v, \lambda+\rho) \leq p,\ \gamma \in R_+\}$ is the upper closure of C_0, and $\overline{C}_0 = \{\lambda \in X(T) | 0 \leq (\gamma^v, \lambda+\rho) \leq p,\ \gamma \in R_+\}$ is the closure of C_0.

For $\gamma \in R_+$ and $m \in \mathbf{Z}$, H_{γ, mp^n} is the p^n-hyperplane $\{\lambda \in X(T) \otimes \mathbf{R} | (\gamma^v, \lambda+\rho) = mp^n\}$. $X_n(T)$ is the n-box $\{\lambda \in X(T) | 0 < (\beta^v, \lambda+\rho) \leq p^n,\ \beta \in S\}$, a fundamental domain for the operation of $p^n X(T)$ on $X(T)$ by translation. Express $\lambda \in X(T)$ as $\lambda = \lambda^0 + p^n \lambda^1$, where $\lambda^0 \in X_n(T)$ and $\lambda^1 \in X(T)$, and call λ^0 the 0-part of λ and λ^1 the 1-part. A special n-vertex v is any intersection $\bigcap_{\beta \in S} H_{\beta, n_\beta p^n}$. The n-box with upper vertex v is $V^v = (v - (p^n - 1)\rho) + X_n(T)$; the lower vertex of V^v is $v - p^n \rho$. The lower n-vertex of the n-box V that contains the weight $\mu = \mu^0 + p^n \mu^1$ is $p^n \mu^1 - \rho$. For $w \in W$, the *alcove transition operator* $I_{w,n}$ on $X(T)$ is defined by $I_{w,n} \cdot \lambda = \lambda^0 + p^n w \cdot \lambda^1$.

G_n is the kernel of the nth power of the Frobenius map on G. $M^{(n)}$ is the nth Frobenius power of a G-module M.

$L(\mu)$ is the irreducible, G-module of highest weight $\mu \in X(T)_+$; $L_n(\mu)$ is the irreducible $G_n B$-module of highest weight $\mu \in X(T)$. $L_n(\mu)$ is isomorphic with $L(\mu^0) \otimes p^n \mu^1$.

$\mathrm{Soc}_H^m M$ is the mth level in the socle filtration of an H-module M. $\mathrm{Soc}_H^m M / \mathrm{Soc}_H^{m-1} M$ is the mth socle layer of M. If L is an irreducible H-module, $[M : L]$ is the multiplicity of L as a composition factor of M.

$H^j(G/G_n B, M)$ is the value at a $G_n B$-module M of the jth right derived functor of the induction functor $|_{G_n B}^G = H^0(G/G_n B, -)$ from $G_n B$-modules to G-modules. If M is a B-module, $H_n^0(M)$ stands for $H^0(G_n B/B, M)$ and $H^0(M)$ stands for $H^0(G/B, M)$. For a $G_n B$-module M,

$$\chi(M) = \sum_{j=0}^{\dim(G/B)} (-1)^j H^j(G/G_n B, M),$$

as an element of the Grothendieck group of G, is the *Euler character* of M.

1. Cohomology of B-modules.

1.1. *Lemmas.*

LEMMA 1. *Let $\nu \leq \mu$ be dominant weights. If w is an element of the Weyl group other than 1, then $w \cdot \mu$ is not a weight of $H^0(\nu)$.*

Proof. ν is the highest weight of $H^0(\nu)$. If $w \cdot \mu$ were a weight of $H^0(\nu)$, then $w^{-1}(w \cdot \mu) = \mu + \rho - w^{-1}(\rho)$ would be one also. For $w \neq 1$, we have $\rho > \omega^{-1}(\rho)$ and $\mu + \rho - w^{-1}(\rho) > \mu \geq \nu$. Hence, $w^{-1}(w \cdot \mu)$ is not a weight of $H^0(\nu)$.

LEMMA 2. *Let N be a B-module. If $\mathrm{Soc}_B N$ contains no weight of $H^0(\nu)$ for each dominant weight ν of N, then $H^0(N) = (0)$.*

Proof. By the hypothesis, the evaluation map $H^0(N) \to N$ is zero; hence, $H^0(N)$ is zero.

Let N be a B-module and let μ be a weight. Let $N_{\leq \mu}$ denote the submodule $\sum_{\nu \leq \mu} N_\nu$ of N, where N_ν is the T-eigenspace of weight ν in N.

DEFINITION. Let K be a set of weights. $\sim$ is the equivalence relation on K generated by the pairs (μ, δ) such that $\mathrm{Ext}^1_B(\mu, \delta) \neq (0)$. The elements of the equivalence class $[\mu]$ of μ will be said to be *linked to μ by extensions through K.*

REMARK. $\nu \in [\mu]$ exactly when either $\nu = \mu$ or there is a sequence $\mu = \mu_0, \mu_1, \ldots, \mu_m = \nu$ where for each i, either $\mathrm{Ext}^1_B(\mu_i, \mu_{i+1}) \neq (0)$ or $\mathrm{Ext}^1_B(\mu_{i+1}, \mu_i) \neq (0)$.

Let K be the set of weights in a fixed B-module N. For each μ, there is the B-submodule of N, $N_{[\mu]} = \sum_{\nu \in [\mu]} N_\nu$. N is the direct sum $\sum_{[\mu]} N_{[\mu]}$. One can compute that decomposition of N, knowing that $\mathrm{Ext}^1_B(\mu, \delta) \neq (0)$ if and only if $\delta = \mu - p^j \beta$ for some $\beta \in S$ and $j \geq 0$ [**2**, Corollary 2.4].

A weak form of Bott's Theorem is valid in characteristic p [**1**, Corollary 2.4]. Let μ be a weight in $\overline{C}_0$. Then,

(a) If μ is dominant, then

$$H^j(G/B, w \cdot \mu) = \begin{cases} L(\mu) & \text{if } j = l(w), \\ (0) & \text{if } j \neq l(w). \end{cases}$$

(b) If μ is not dominant, then $H^j(G/B, w \cdot \mu) = (0)$ for all j and all $w \in W$.

The following proposition and corollary show how the position of the weights of $W \cdot \mu$ in N controls the multiplicity of $L(\mu)$ in $H^0(N)$, in the case where all the weights of N lie in $W \cdot \overline{C}_0 = \bigcup_{w \in W} w \cdot \overline{C}_0$.

PROPOSITION. *Let μ be a dominant weight and let N be a B-module whose dominant weights lie in $\hat{C}_0$. If the weights of* $\mathrm{Soc}_B N$ *lie in $W \cdot \mu$, then* $[H^0(N): L(\mu)] = \dim_k(\mathrm{Soc}_B N)_\mu$.

Proof. Since μ is not a weight of $N/N_{\leq \mu}$, $[H^0(N/N_\mu): L(\mu)]$ equals zero by Bott's Theorem; hence, $[H^0(N): L(\mu)] = [H^0(N_{\leq \mu}): L(\mu)]$. Since $(\mathrm{Soc}_B N)_\mu = (\mathrm{Soc}_B N_{\leq \mu})_\mu$, the proposition will follow from showing that $H^0(N_{\leq \mu})$ is the sum of $\dim_k(\mathrm{Soc}_B N_{\leq \mu})_\mu$-copies of $L(\mu)$. Since $\mathrm{Ext}^1_B(\nu, \mu) = (0)$ for weights $\nu \leq \mu$, we have $N_{\leq \mu} = N_0 \oplus (\mathrm{Soc}_B N_{\leq \mu})_\mu$ for some submodule N_0 whose socle consists of weights from $\{w \cdot \mu\}_{w \neq 1}$. By Lemmas 1 and 2, $H^0(N_{\leq \mu}) = H^0((\mathrm{Soc}_B N_{\leq \mu})_\mu)$, the sum of $\dim_k(\mathrm{Soc}_B N_{\leq \mu})_\mu$-copies of $L(\mu)$.

COROLLARY. *Let μ be a dominant weight and let N be a B-module whose weights lie in $W \cdot \overline{C}_0$. Let $N^1_{[\mu]}$ be the maximal quotient of $N_{[\mu]}$ whose socle consists only of weights from $W \cdot \mu$, i.e., the quotient of $N_{[\mu]}$ by the maximal submodule N_1 which contains no weight from $W \cdot \mu$. Then* $[H^0(N): L(\mu)] = \dim_k(\mathrm{Soc}_B N^1_{[\mu]})_\mu$.

Proof. By Bott's Theorem, $[H^k(N_1): L(\mu)] = 0$ for $k \geq 0$; that implies that $[H^0(N_{[\mu]}): L(\mu)] = [H^0(N^1_{[\mu]}): L(\mu)]$, which equals $\dim_k(\mathrm{Soc}_B N^1_{[\mu]})_\mu$ by the proposition. Furthermore, by Bott's Theorem again, $[H^0(N): L(\mu)] = [H^0(N_{[\mu]}): L(\mu)]$.

1.2 *G_nB-modules.* Let M be a semisimple G_n-module. For $\mu^0 \in X_n(T)$, $M(\mu^0)$ denotes the sum of all submodules of M isomorphic with $L(\mu^0)$.

LEMMA 1. *Let M be a G_nB-module. If M is G_n-semisimple, then $M = \bigoplus_{\mu^0 \in X_n(T)} M(\mu^0)$ as G_nB-modules.*

Proof. $M(\mu^0)$ is a G_nB-submodule of M. In fact,

$$\mathrm{Hom}_{G_n}(L(\mu^0), M(\mu^0)) = \mathrm{Hom}_{G_n}(L(\mu^0), M)$$

is a G_nB-module via the G_nB-structures of $L(\mu^0)$ and M. Since the images of elements of $\mathrm{Hom}_{G_n}(L(\mu^0), M(\mu^0))$ span $M(\mu^0)$, $M(\mu^0)$ is G_nB-stable.

REMARK. $M(\mu^0)$ is isomorphic canonically with $L(\mu^0)\otimes N^{(n)}$, where $N^{(n)} = \operatorname{Hom}_{G_n}(L(\mu^0), M(\mu^0)) = \operatorname{Hom}_{G_n}(L(\mu^0), M)$. The map $N_0 \mapsto L(\mu^0) \otimes N_0^{(n)}$ from the lattice of B-submodules of N to the lattice of G_nB-submodules of $L(\mu^0) \otimes N^{(n)}$ is an isomorphism with inverse map $T \mapsto \operatorname{Hom}_{G_n}(L(\mu^0), T)^{(-n)}$.

The paragraphs (i)–(iii) below give versions for G_nB-modules of results from 1.1.

(i) Bott's Theorem [1]. Let $\mu = \mu^0 + p^n\mu^1$ be a weight with $\mu^1 \in \overline{C}_0$.

(a) If μ is dominant, then

$$H^j(G/G_nB, L_n(\mu^0 + p^n w \cdot \mu^1)) = \begin{cases} L(\mu) & \text{if } j = l(w), \\ (0) & \text{if } j \neq l(w). \end{cases}$$

(b) If μ is not dominant, then $H^j(G/G_nB, L_n(\mu^0 + p^n w \cdot \mu^1)) = (0)$ for all j.

(ii) Let K be a set of weights, and let μ^0 be an element of $X_n(T)$. Let $L_n(\mu^0 + p^nK)$ be the set of irreducible G_nB-modules $\{L_n(\mu^0 + p^n\nu) | \nu \in K\}$.

DEFINITION. $\sim_L$ is the equivalence relation on $L_n(\mu^0 + p^nK)$ generated by the pairs $(L_n(\mu^0 + p^n\nu), L_n(\mu^0 + p^n\delta))$ such that

$$\operatorname{Ext}^1_{G_nB}(L_n(\mu^0 + p^n\nu),\ L_n(\mu_0 + p^n\delta)) \neq (0).$$

If $L_n(\mu^0 + p^n\nu) \sim_L L_n(\mu^0 + p^n\delta)$, then we will say that they are *linked by extensions through* $L_n(\mu^0 + p^nK)$.

By [2, Corollary 2.4] and [3, Lemma 3.2],

$$\operatorname{Ext}^1_{G_nB}(L_n(\mu^0 + p^n\nu),\ L_n(\mu^0 + p^n\delta)) \quad \text{and} \quad \operatorname{Ext}^1_B(\nu, \delta)$$

are isomorphic (except when $p = 2$ and G has type C_l, which was pointed out to me by M. Kaneda); hence

$$L_n(\mu^0 + p^n\nu) \sim_L L_n(\mu^0 + p^n\delta)$$

if and only if $\nu \sim \delta$, with that one exceptional case.

Let K be the set of weights of a B-module N, and let $\mu^0 \in X_n(T)$.

LEMMA. (a) *$L(\mu^0) \otimes N^{(n)}$ is the direct sum $\sum_{[\nu]} L(\mu^0) \otimes N^{(n)}_{[\nu]}$.*

(b) *Suppose that $p \neq 2$ or G does not have type C_l. The composition factors of $L(\mu^0)\otimes N^{(n)}_{[\nu]}$ are the composition factors of $L(\mu^0)\otimes N^{(n)}$ which are linked by extensions to $L_n(\mu^0 + p^n\nu)$ through $L_n(\mu^0 + p^nK)$.*

(iii) Let M be a G_nB-module, which is semisimple as a G_n-module. By Lemma 1 and the remark above, $M = \bigoplus_{\nu^0 \in X_n(T)} L(\nu^0) \otimes N(\nu^0)^{(n)}$

where $N(\nu^0)^{(n)} = \mathrm{Hom}_{G_n}(L(\nu^0), M)$. For μ dominant, let $M_{[\mu]} = L(\mu^0) \otimes N(\mu^0)^{(n)}_{[\mu^1]}$ and let $M^1_{[\mu]}$ be the maximal quotient of $M_{[\mu]}$ whose socle contains only factors from $\{L_n(\mu^0 + p^n w \cdot \mu^1)\}_{w \in W}$, i.e. the quotient of $M_{[\mu]}$ by the maximal submodule which has no composition factor from $\{L_n(\mu^0 + p^n w \cdot \mu^1)\}_{w \in W}$. By the remark above, we have $M^1_{[\mu]} = L(\mu^0) \otimes (N(\mu^0)^1_{[\mu^1]})^{(n)}$.

Suppose that $N(\mu^0)$ has its weights within $W \cdot \overline{C}_0$.

Proposition. $[H^0(G/G_nB, M) : L(\mu)] = [\mathrm{Soc}_{G_nB} M^1_{[\mu]} : L_n(\mu)]$.

Proof. Let $N = N(\mu^0)$. Since $H^0(G/G_nB, M) = \sum_{\nu^0 \in X_n(T)} L(\nu^0) \otimes H^0(N(\nu^0))^{(n)}$, we have $[H^0(G/G_nB, M) : L(\mu)] = [H^0(N) : L(\mu^1)]$. By the corollary in 1.1, that equals $\dim_k(\mathrm{Soc}_B N^1_{[\mu^1]})_{\mu^1} = [\mathrm{Soc}_{G_nB} M^1_{[\mu]} : L_n(\mu)]$.

1.3. Take $n = 1$. The elements of the orbit $\{L_1(\mu^0 + p^1 w \cdot \mu^1)\}_{w \in W}$ that occur as composition factors of $H^0_1(\lambda)$ are linked by extensions in $H^0_1(\lambda)$, in the sense given by the proposition below.

Lemma. *Let γ be a root. If $L_1(\mu)$ and $L_1(\mu - mp\gamma)$ are composition factors of $H^0_1(\lambda)$, then so is $L_1(\mu - jp\gamma)$, for $0 < j < m$.*

Proof. By a Theorem of Ye [11] and its extension to p-irregular weights [7], the set of highest weights of composition factors of $H^0_1(\lambda)$ equals the W-linkage class $WL \cdot \lambda = \bigcap_{w \in W} I_{w^{-1}} \cdot \mathrm{SL}(w \cdot \lambda)$, where $\mathrm{SL}(w \cdot \lambda)$ is the set of weights strongly linked to $w \cdot \lambda$. Hence, we need to show that $I_w \cdot (\mu - jp\gamma) = \mu^0 + pw \cdot \mu^1 - jpw(\gamma)$ is strongly linked to $w \cdot \lambda$, whenever $\mu^0 + pw \cdot \mu^1$ and $\mu^0 + pw \cdot \mu^1 - mpw(\gamma)$ are strongly linked to $w \cdot \lambda$ for all $w \in W$. In fact, that implication holds for w taken one at a time. If $w(\gamma) \in R^+$, then $\mu^0 + pw \cdot \mu^1 - jpw(\gamma) \uparrow \mu^0 + pw \cdot \mu^1 \uparrow w \cdot \lambda$; if $w(\gamma) \in -R^+$, then $\mu^0 + pw \cdot \mu^1 - jpw(\gamma) \uparrow \mu^0 + pw \cdot \mu^1 - mpw(\gamma) \uparrow w \cdot \lambda$. (Here, $\uparrow$ is the relation of strong linkage.)

Proposition. *Let λ and μ be dominant weights and let $w \in W$. If $L_1(\mu^0 + pw \cdot \mu^1)$ is a composition factor of $H^0_1(\lambda)$, then there is a sequence $\{L_1(\delta_i)\}_{i=1}^t$ of composition factors of $H^0_1(\lambda)$ where $\delta_1 = \mu^0 + pw \cdot \mu^1$, $\delta_t = \mu$ and $\mathrm{Ext}^1_{G_1B}(L_1(\delta_{i+1}), L_1(\delta_i)) \neq (0)$, for $1 \leq i < t$.*

Proof. By induction on $l(w)$. Suppose that $l(s_\beta w) = l(w) - 1$ for some $\beta \in S$. $L_1(\mu^0 + ps_\beta w \cdot \mu^1)$ is a composition factor of $H^0_1(\lambda)$ since

$L_1(\mu^0+pw\cdot\mu^1)$ is one, by Proposition 2.8 of [7]. Since $\mu^0+pw\cdot\mu^1 = \mu^0+ps_\beta w\cdot\mu^1+(w^{-1}(\beta)^v,\mu^1+\rho)p\beta$, where $-(w^{-1}(\beta)^v,\mu^1+\rho)\geq 0$, the elements of $\{L_1(\mu^0+pw\cdot\mu^1-jp\beta)\}_{j=0}^{-(w^{-1}(\beta)^v,\mu^1+\rho)}$ are composition factors of $H_1^0(\lambda)$, by the lemma, and

$$\mathrm{Ext}^1_{G_1B}(L_1(\mu^0+pw\cdot\mu^1-jp\beta),L_1(\mu^0+pw\cdot\mu^1-(j+1)p\beta))\neq(0),$$

by [3, Lemma 3.2].

REMARK. Suppose that all the composition factors $\{L_1(\delta_i)\}_{i=1}^n$ of $H_1^0(\lambda)$ which appear in the proposition above occur in the G_1-socle layer $M=\mathrm{Soc}^j_{G_1}H_1^0(\lambda)/\mathrm{Soc}^{j-1}_{G_1}H_1^0(\lambda)$. In the terms of 1.2, $M(\mu^0) = L(\mu^0)\otimes N^{(1)}$, where $N=\mathrm{Hom}_{G_1}(L(\mu^0),M)$; let K be the set of weights of N. By the proposition, $L_1(\mu)$ and $L_1(\mu^0+pw\cdot\mu^1)$ are linked by extensions through $L_1(\mu^0+pK)$; hence, they occur as composition factors of $M^1_{[\mu]}$. Thus, the elements of the orbit $\{L_1(\mu^0+pw\cdot\mu^1)\}_w$ that occur in M may be examined within the restricted setting of $M^1_{[\mu]}$, whenever the factors of the linking extensions all occur in M. For instance, that applies to the weight μ in Figure 2, for the group of type G_2.

2. Induction from infinitesimal to global socle levels. Let M be a G_nB-module. The mth G_n-socle level $\mathrm{Soc}^m_{G_n}M$ is a G_nB-submodule of M.

LEMMA 1. $\mathrm{Soc}^m_G H^0(G/G_nB,M)\subset H^0(G/G_nB,\mathrm{Soc}^m_{G_n}M)$.

Proof. Induction on m.
$m=1$. Let L be any irreducible G-module. We have

$$\mathrm{Hom}_G(L,H^0(G/G_nB,M))=\mathrm{Hom}_{G_nB}(L,M),$$

and

$$\begin{aligned}&\mathrm{Hom}_G(L,H^0(G/G_nB,\mathrm{Soc}_{G_n}M))\\&\quad=\mathrm{Hom}_{G_nB}(L,\mathrm{Soc}_{G_n}M)=\mathrm{Hom}_{G_n}(L,\mathrm{Soc}_{G_n}M)^B.\end{aligned}$$

Since L is semisimple as a G_n-module,

$$\mathrm{Hom}_{G_n}(L,M)=\mathrm{Hom}_{G_n}(L,\mathrm{Soc}_{G_n}M).$$

Hence, $\mathrm{Soc}_G H^0(G/G_nB,M)=\mathrm{Soc}_G(H^0(G/G_nB,\mathrm{Soc}_{G_n}M))$ as submodules of $H^0(G/G_nB,M)$.

Suppose that $\mathrm{Soc}_G^{m-1}(H^0(G/G_nB, M)) \subset H^0(G/G_nB, \mathrm{Soc}_{G_n}^{m-1}M)$, as submodules of $H^0(G/G_nB, M)$. We need to show that in the exact sequence

$$\begin{aligned} 0 &\to H^0(G/G_nB, \mathrm{Soc}_{G_n}^m M) \\ &\to H^0(G/G_nB, M) \xrightarrow{\pi} H^0(G/G_nB, M/\mathrm{Soc}_{G_n}^m M), \end{aligned}$$

we have $\pi = 0$ at the submodule $\mathrm{Soc}_G^m H^0(G/G_nB, M)$. Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
 & & H^0(G/G_nB, \mathrm{Soc}_{G_n}^m M/\mathrm{Soc}_{G_n}^{m-1}M) \arrow[d] \\
\mathrm{Soc}_G^m(H^0(G/G_nB, M)) \arrow[r, hook] & H^0(G/G_nB, M) \arrow[r, "k"] \arrow[dr, "\pi"'] & H^0(G/G_nB, M/\mathrm{Soc}_G^{m-1}M) \arrow[d] \\
 & & H^0(G/G_nB, M/\mathrm{Soc}_{G_n}^m M),
\end{tikzcd}$$

where the vertical sequence is exact. $\pi = 0$ on $\mathrm{Soc}_G^m H^0(G/G_nB, M)$ if and only if k maps $\mathrm{Soc}_G^m H^0(G/G_nB, M)$ into

$$H^0(G/G_nB, \mathrm{Soc}_{G_n}^m M/\mathrm{Soc}_{G_n}^{m-1}M).$$

Since k factors through $H^0(G/G_nB, M)/H^0(G/G_nB, \mathrm{Soc}_{G_n}^{m-1}M)$, it factors also through $H^0(G/G_nB, M)/\mathrm{Soc}_G^{m-1}H^0(G/G_nB, M)$, by the inductive hypothesis. Thus, we need to show that k maps

$$\begin{aligned} &\mathrm{Soc}_G^m H^0(G/G_nB, M)/\mathrm{Soc}_G^{m-1}H^0(G/G_nB, M) \\ &\quad = \mathrm{Soc}_G(H^0(G/G_nB, M)/\mathrm{Soc}_G^{m-1}H^0(G/G_nB, M)) \end{aligned}$$

into

$$H^0(G/G_nB, \mathrm{Soc}_{G_n}^m M/\mathrm{Soc}_{G_n}^{m-1}M) = H^0(G/G_nB, \mathrm{Soc}_{G_n}(M/\mathrm{Soc}_{G_n}^{m-1}M)).$$

But k maps $\mathrm{Soc}_G(H^0(G/G_nB, M)/\mathrm{Soc}_G^{m-1}H^0(G/G_nB, M))$ into $\mathrm{Soc}_G H^0(G/G_nB, M/\mathrm{Soc}_{G_n}^{m-1}M)$, which equals

$$\mathrm{Soc}_G(H^0(G/G_nB, \mathrm{Soc}_{G_n}(M/\mathrm{Soc}_{G_n}^{m-1}M)))$$

by the case $m = 1$.

LEMMA 2. *Let λ be a dominant weight. Suppose that those highest weights of composition factors of the G_nB-module $H_n^0(\lambda)$ which are dominant have 1-parts in $\hat{C}_0$. Then*

$$H^0(G/G_nB, \mathrm{Soc}_{G_n}^j H_n^0(\lambda)/\mathrm{Soc}_{G_n}^{j-1}H_n^0(\lambda))$$

is G-semisimple, for $j \geq 1$.

Proof. This lemma is proved in [4] in essense. Let

$$M = \mathrm{Soc}^j_{G_n} H^0_n(\lambda)/\mathrm{Soc}^{j-1}_{G_n} H^0_n(\lambda).$$

In the terms of 1.2,

$$\begin{aligned} H^0(G/G_nB, M) &= \bigoplus_{\mu^0 \in X_n(T)} H^0(G/G_nB, M(\mu^0)) \\ &= \bigoplus_{\mu^0 \in X_n(T)} L(\mu^0) \otimes H^0(N(\mu^0))^{(n)}. \end{aligned}$$

$H^0(N(\mu^0))$ is a semisimple G-module, since $\mathrm{Ext}^1_G(L(\nu), L(\delta)) = (0)$ when ν and δ are weights in $\hat{C}_0$, by strong linkage.

Andersen in [4, Theorem 4.2] showed that the socle levels of the global module $H^0(\lambda)$ are induced from the G_n-socle levels of the infinitesimal module $H^0_n(\lambda)$, on condition that the highest weights of the composition factors of the G_nB-module $H^0_n(\lambda)$ have 1-parts in $\hat{C}_0$. The following proposition gives an extension of that result to singular cases.

PROPOSITION. *Let λ be a dominant weight. Suppose that those highest weights of composition factors of the G_nB-module $H^0_n(\lambda)$ which are dominant have 1-parts in $\hat{C}_0$. Then, for each $j \geq 1$,*

$$\mathrm{Soc}^j_G H^0(\lambda) = H^0(G/G_nB, \mathrm{Soc}^j_{G_n} H^0_n(\lambda)).$$

Proof. By induction on j. For $j = 1$,

$$\mathrm{Soc}^1_G H^0(\lambda) = L(\lambda) = H^0(G/G_nB, \mathrm{Soc}^1_{G_n} H^0_n(\lambda)).$$

Suppose that $\mathrm{Soc}^{j-1}_G H^0(\lambda) = H^0(G/G_nB, \mathrm{Soc}^{j-1}_{G_n} H^0_n(\lambda))$, and consider the exact sequence

$$\begin{aligned} 0 &\to H^0(G/G_nB, \mathrm{Soc}^{j-1}_{G_n} H^0_n(\lambda)) \to H^0(G/G_nB, \mathrm{Soc}^j_{G_n} H^0_n(\lambda)) \\ &\to H^0(G/G_nB, \mathrm{Soc}^j_{G_n} H^0_n(\lambda)/\mathrm{Soc}^{j-1}_{G_n} H^0_n(\lambda)). \end{aligned}$$

From Lemma 2, we see that $H^0(G/G_nB, \mathrm{Soc}^j_{G_n} H^0_n(\lambda))/\mathrm{Soc}^{j-1}_G H^0(\lambda)$ is semisimple, so that $H^0(G/G_nB, \mathrm{Soc}^j_{G_n} H^0_n(\lambda)) \subset \mathrm{Soc}^j_G H^0(\lambda)$. By Lemma 1, they are equal.

REMARK. Quantitatively, the proposition is deficient, since it does not relate the multiplicities $[\mathrm{Soc}^j_{G_n} H^0_n(\lambda) \colon L_n(\mu)]$ and $[\mathrm{Soc}^j_G H^0(\lambda) \colon L(\mu)]$. In the next section, we tie that quantitative question in with the G_nB-structure of the G_n-socle layers of $H^0_n(\lambda)$.

3. Cancellation in socle levels. Throughout this section, we consider only dominant weights λ such that the highest weights of the composition factors of $H_n^0(\lambda)$ have 1-parts in $\bigcup_{w\in W} w\cdot \overline{C}_0$, in order that Bott's Theorem be effective. The systematic use of the Euler character below is motivated by a talk of H. H. Andersen at the 1986 A.M.S. Summer Institute of Arcata.

Definition. $\mathrm{Soc}_G^j H^0(\lambda)$ can be computed from $\mathrm{Soc}_{G_n}^j H_n^0(\lambda)$ via cancellation if $H^0(G/G_nB, \mathrm{Soc}_{G_n}^j H_n^0(\lambda)) = \chi(\mathrm{Soc}_{G_n}^j H_n^0(\lambda))$, in the Grothendieck group of G.

Since $\mathrm{Soc}_G^j H^0(\lambda) = H^0(G/G_nB, \mathrm{Soc}_{G_n}^j H_n^0(\lambda))$, by the proposition of §2, and since

$$\chi(\mathrm{Soc}_{G_n}^j H_n^0(\lambda)) = \sum_{\mu\in X(T)_+}\left(\sum_{w\in W}(-1)^{l(w)}[\mathrm{Soc}_{G_n}^j H_n^0(\lambda): L_n(\mu^0 + p^n w\cdot \mu^1)]\right) L(\mu),$$

by Bott's Theorem and the additivity of the Euler character, computation via cancellation amounts to the equality

$$\mathrm{Soc}_G^j H^0(\lambda) = \sum_{\mu\in X(T)_+}\left(\sum_{w\in W}(-1)^{l(w)}[\mathrm{Soc}_{G_n}^j H_n^0(\lambda): L_n(\mu^0 + p^n w\cdot \mu^1)]\right) L(\mu),$$

in the Gorthendieck group. When that holds, it determines $\mathrm{Soc}_G^j H^0(\lambda)$ in terms of the multiplicities of the composition factors of the jth infinitesimal G_n-socle level.

$H^0(\lambda)$ itself can be computed from $H_n^0(\lambda)$ by cancellation, since $H^i(G/G_nB, H_n^0(\lambda)) = H^i(\lambda) = (0)$ for $i > 0$, by Kempf's Vanishing Theorem.

3.1. Let μ be a dominant weight. If $\{L_n(\mu^0 + p^n w\cdot\mu^1)\}_{w\neq 1}$ are not composition factors of $H_n^0(\lambda)$, then

$$[\mathrm{Soc}_{G_n}^j H_n^0(\lambda): L_n(\mu)] = [\mathrm{Soc}_G^j H^0(\lambda): L(\mu)],$$

for each j, i.e., no cancellation takes place. In fact, by the hypothesis, $[H^0(G/G_nB, \mathrm{Soc}_{G_n}^j H_n^0(\lambda)): L(\mu)]$ and $[\mathrm{Soc}_{G_n}^j H_n^0(\lambda): L_n(\mu)]$ each equals the coefficient of $L(\mu)$ in $\chi(\mathrm{Soc}_{G_n}^j H_n^0(\lambda))$. The conclusion follows from the proposition of §2.

For $n = 1$, the proposition below shows that the lengths of the socle series of the global and infinitesimal induced modules coincide whenever the top factor of $H_1^0(\lambda)$ had dominant highest weight. Let $\lambda = \lambda^0 + p\lambda^1, \lambda^0 \in X_n(T)$.

PROPOSITION. *If $\langle \beta^\nu, \lambda^1 \rangle \geq 1$ for all $\beta \in S$, then the lengths of the socle series of $H^0(\lambda)$ and $H_1^0(\lambda)$ coincide.*

Proof. λ lies in the 1-box V with lower vertex $v = p\lambda^1 - \rho$. Let $y_0 = T_{v+\rho}w_0T_{-(v+\rho)}$ for $T_{v+\rho}$ the operator of translation by $v+\rho = p\lambda^1$. By the hypothesis, the weight $\mu = y_0 \cdot \lambda$ is dominant. From **[10]**, it follows that the top factor of $H_1^0(\lambda)$ is $L_1(y^0 \cdot \lambda)$, a composition factor of multiplicity one. Since $H^0(G/G_1B, \mathrm{Soc}^j_{G_1} H_1^0(\lambda)) = \mathrm{Soc}^j_G H^0(\lambda)$, the length of the socle series of $H^0(\lambda)$ is no greater than the length of the G_1-socle series of $H_1^0(\lambda)$. Hence, it will suffice to show that the socle levels of $L(y_0 \cdot \lambda)$ in $H^0(\lambda)$ and of $L_1(y_0 \cdot \lambda)$ in $H_1^0(\lambda)$ are equal. By the first paragraph of 3.1, we need only show that

$$[H_1^0(\lambda): L_1(\mu^0 + \rho w \cdot \mu^1)] = 0 \quad \text{for } w \neq 1,$$

for which it suffices to show that $[H_1^0(\lambda): L_1(\mu^0 + ps_\beta \cdot \mu^1)] = 0$ for $\beta \in S$, since

$$[H_1^0(\lambda): L_1(\mu^0 + pw \cdot \mu^1)] \leq [H_1^0(\lambda): L_1(\mu^0 + ps_\beta \cdot \mu^1)]$$

when $l(ws_\beta) = l(w) - 1$, by [7, Proposition 2.8]. In the terms of **[6, 7]**, by Ye's theorem ([**11**, Corollary 3.5] and [7, Corollary 2.7]), the linkage class $\mathrm{WL} \cdot \lambda$ equals the set of highest weights of composition factors of $H_1^0(\lambda)$. To show that $\mu^0 + ps_\beta \cdot \mu^1 \notin \mathrm{WL} \cdot \lambda$, we have that $\mu^0 + ps_\beta \cdot \mu^1 = \mu - (\beta^v, \mu^1 + \rho)p\beta \in \mathrm{WL} \cdot \lambda$ if and only if $\lambda + (\beta^v, \mu^1 + \rho)p\beta \in \mathrm{WL}^{-1} \cdot \mu$ if and only if

$$y_0 \cdot (\lambda + (\beta^v, \mu^1 + \rho)p\beta) = \mu + (\beta^v, \mu^1 + \rho)pw_0(\beta) \in \mathrm{WL}^{-1} \cdot \mu,$$

by the symmetry of $\mathrm{WL}^{-1} \cdot \mu$ about v. But $w_0(\beta)$ is a negative root and $(\beta^v, \mu^1 + \rho) > 0$, by the dominance of μ; hence, $\mu + (\beta^v, \mu^1 + \rho)pw_0(\beta) \notin \mathrm{SL}^{-1} \cdot \mu$, which excludes that element from $\mathrm{WL}^{-1} \cdot \mu$, also.

3.2. In this section, we show that G_nB-structure of the G_n-socle layers of $H_n^0(\lambda)$ accounts for the loss of composition factors under the induction from G_nB to G-modules.

For an element $\sum_{\nu \in X(T)} m_\nu L_n(\nu)$ of the Grothendieck group of G_nB, denote the coefficient m_μ of $L_n(\mu)$ by $[\sum_{\nu \in X(T)} m_\nu L_n(\nu): L_n(\mu)]$. For an element $\sum_{\nu \in X(T)_+} m_\nu L(\nu)$ of the Grothendieck group of G, denote m_μ by $[\sum_{\nu \in X(T)_+} m_\nu L(\nu): L(\mu)]$.

Let λ and μ be dominant weights.

PROPOSITION. (a) *If* $M_j = \mathrm{Soc}^j_{G_n} H^0_n(\lambda)/\mathrm{Soc}^{j-1}_{G_n} H^0_n(\lambda)$ *is the highest socle layer of* $H^0_n(\lambda)$ *that contains any* G_nB*-composition factor from the orbit* $\{L_n(\mu^0 + p^n w \cdot \mu^1)\}_{w\in W}$, *then*

$$\begin{aligned}[H^0(\lambda): L(\mu)] &= [\mathrm{Soc}^j_G H^0(\lambda): L(\mu)] = [H^0(G/G_nB, \mathrm{Soc}^j_{G_n} H^0_n(\lambda)): L(\mu)] \\ &= [\chi(\mathrm{Soc}^j_{G_n} H^0_n(\lambda)): L(\mu)],\end{aligned}$$

and

$$[H^k(G/G_nB, \mathrm{Soc}^j_{G_n} H^0_n(\lambda)): L(\mu)] = 0 \quad \textit{for } k > 0.$$

(b) *If the composition factors of* $H^0_n(\lambda)$ *from the orbit*

$$\{L_n(\mu^0 + p^n w \cdot \mu^1)\}_{w\in W}$$

occur only in the j*th socle layer, then*

$$\begin{aligned}[H^0(\lambda): L(\mu)] &= [\mathrm{Soc}^j_G H^0(\lambda)/\mathrm{Soc}^{j-1}_G H^0(\lambda): L(\mu)] \\ &= [H^0(G/G_nB, M_j): L(\mu)] = [\chi(M_j): L(\mu)],\end{aligned}$$

and

$$[H^k(G/G_nB, M_j): L(\mu)] = 0 \quad \textit{for } k > 0.$$

Proof. (a). By the hypothesis,

$$[H^k(G/G_nB, H^0_n(\lambda)/\mathrm{Soc}^j_{G_n} H^0_n(\lambda)): L(\mu)] = 0 \quad \text{for } k \geq 0.$$

Hence,

$$[H^k(G/G_nB, H^0_n(\lambda)): L(\mu)] = [H^k(G/G_nB, \mathrm{Soc}^j_{G_n} H^0_n(\lambda)): L(\mu)]$$

for $k \geq 0$, by the long exact sequence for $H^0(G/G_nB, -)$ associated to $0 \to \mathrm{Soc}^j_{G_n} H^0_n(\lambda) \to H^0_n(\lambda) \to H^0_n(\lambda)/\mathrm{Soc}^j_{G_n} H^0_n(\lambda) \to 0$. Since $H^k(G/G_nB, H^0_n(\lambda)) = 0$ for $k > 0$, by Kempf's vanishing theorem, the statements in (a) hold.

(b) follows from (a) and from the fact that

$$[H^k(G/G_nB, \mathrm{Soc}^{j-1}_{G_n} H^0_n(\lambda)): L(\mu)] = 0 \quad \text{for } k \geq 0,$$

by the hypothesis.

Let λ and μ be dominant weights and let

$$M = \mathrm{Soc}^j_{G_n} H^0_n(\lambda)/\mathrm{Soc}^{j-1}_{G_n} H^0_n(\lambda).$$

Let $M_{[\mu]}$ and $M^1_{[\mu]}$ be the submodule and subquotient of M defined in 1.2 (iii).

THEOREM. *If the composition factors of $H_n^0(\lambda)$ from the orbit $\{L_n(\mu^0 + p^n w \cdot \mu^1)\}_{w \in W}$ occur only in the single G_n-socle layer M, then* $[\mathrm{Soc}_G^j H^0(\lambda)/\mathrm{Soc}_G^{j-1} H^0(\lambda) \colon L(\mu)] = [\mathrm{Soc}_{G_n B} M^1_{[\mu]} \colon L_n(\mu)]$.

Proof. The theorem combines Proposition (b) above with the Proposition of 1.2.

REMARKS. (1) Under the hypothesis of the theorem, if $[H^0(\lambda) \colon L(\mu)] < [H_n^0(\lambda) \colon L_n(\mu)]$, then M and $M^1_{[\mu]}$ are not G_nB-semisimple.

(2) By Lemma 2 of §2, $H^0(G/G_nB, M)$ is G-semisimple, even though M may be not G_nB-semisimple. So, the effect of the G_nB-structure of the G_n-semisimple module M on the value of $H^0(G/G_nB, -)$ at M shows up, not in the structure of $H^0(G/G_nB, M)$, but in the loss of multiplicity in passing from M to $H^0(G/G_nB, M)$.

3.3. *Illustrations.* (1) Let G be a group of type A_2 or B_2. Let λ be a dominant p-regular weight such that the highest weights of the composition factors of $H_1^0(\lambda)$ have 1-parts in $\bigcup_{w \in W} w \cdot \overline{C}_0$. Suppose that there is a weight δ such that $\lambda + p\delta$ is a generic weight [3, §2.1] in the lowest p^2-alcove. Then

$$[\mathrm{Soc}_{G_1}^j H_1^0(\lambda + p\delta) \colon L_1(\mu)] = [\mathrm{Soc}_G^j H^0(\lambda + p\delta) \colon L(\mu)]$$

for each weight μ [4]. For A_2 and B_2, those values are computed in [6, Figure 2A, 2B], and the hypothesis of 3.2 Proposition (b) is satisfied for each μ. Hence, $\mathrm{Soc}_G^j H^0(\lambda)$ can be computed from $\mathrm{Soc}_{G_1}^j H_1^0(\lambda)$ by cancellation. For example, see Figure 1 for the group of type B_2. In Figure 1(a), λ lies in the alcove labelled 1, and the numbers in the alcoves give the socle layers of the composition factors of $H_1^0(\lambda)$ whose highest weights lie in those alcoves. Figure 1(b) gives the socle layers of the composition factors of $H^0(\lambda)$, computed by cancellation from Figure 1(a) according to 3.2 Proposition (b).

(2) Let G be the group of type G_2. Let α and β be the simple roots orthogonal to the hyperplanes $H_{\alpha,0}$ and $H_{\beta,0}$ in Figure 2. Assume that λ is a generic weight in the lowest p^2-alcove. In Figure 2 (Figure 3 of [8]), the numbers give the Jantzen filtration layers of the composition factors of $H^0(\lambda)$, normalized so that $L(\lambda)$ is on the first layer. We assume that the Jantzen and the socle filtrations of $H^0(\lambda)$ coincide. Then the figure shows also the socle layers of the composition factors of $H_1^0(\lambda)$, where we can, in that infinitesimal setting, move λ from a generic position to the corresponding p-translated position in the 1-box $V_{-\rho}$ with lower vertex $-\rho$, without disturbing the socle layers and multiplicities of the p-translated composition factors.

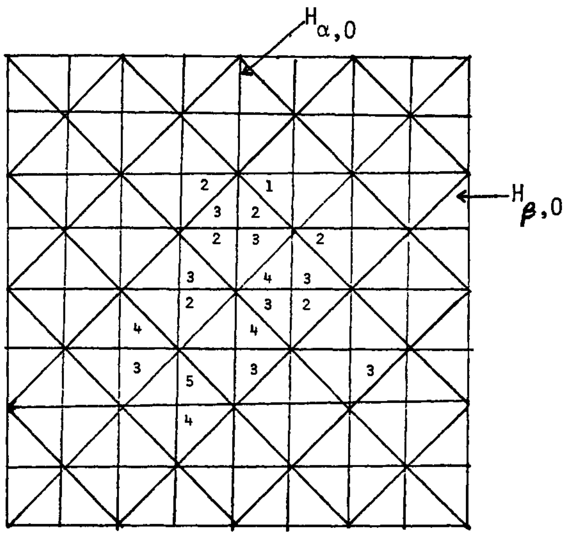

FIGURE 1(a)

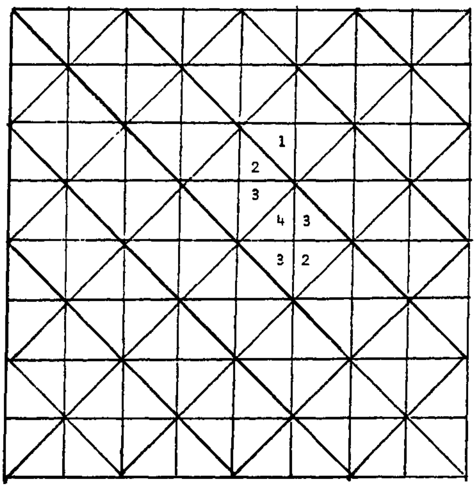

FIGURE 1(b)

Let μ be the weight in the fundamental alcove C_0 which is linked to λ. In Figure 2, we see that the composition factors of $H_1^0(\lambda)$ from the orbit $\{L_1(\mu^0 + pw \cdot \mu^1)\}_{w \in W}$ occur only in the fourth socle layer $\mathrm{Soc}^4_{G_1} H_1^0(\lambda)/\mathrm{Soc}^3_{G_1} H_1^0(\lambda)$. There are four such factors, each of multiplicity one, distinguished by heavy boundary lines in the figure. By the remark of 1.3, those factors belong to $M^1_{[\mu]}$. We will show that the $G_1 B$-module $M^1_{[\mu]}$ has the following structure, all factors of multiplicity 1.

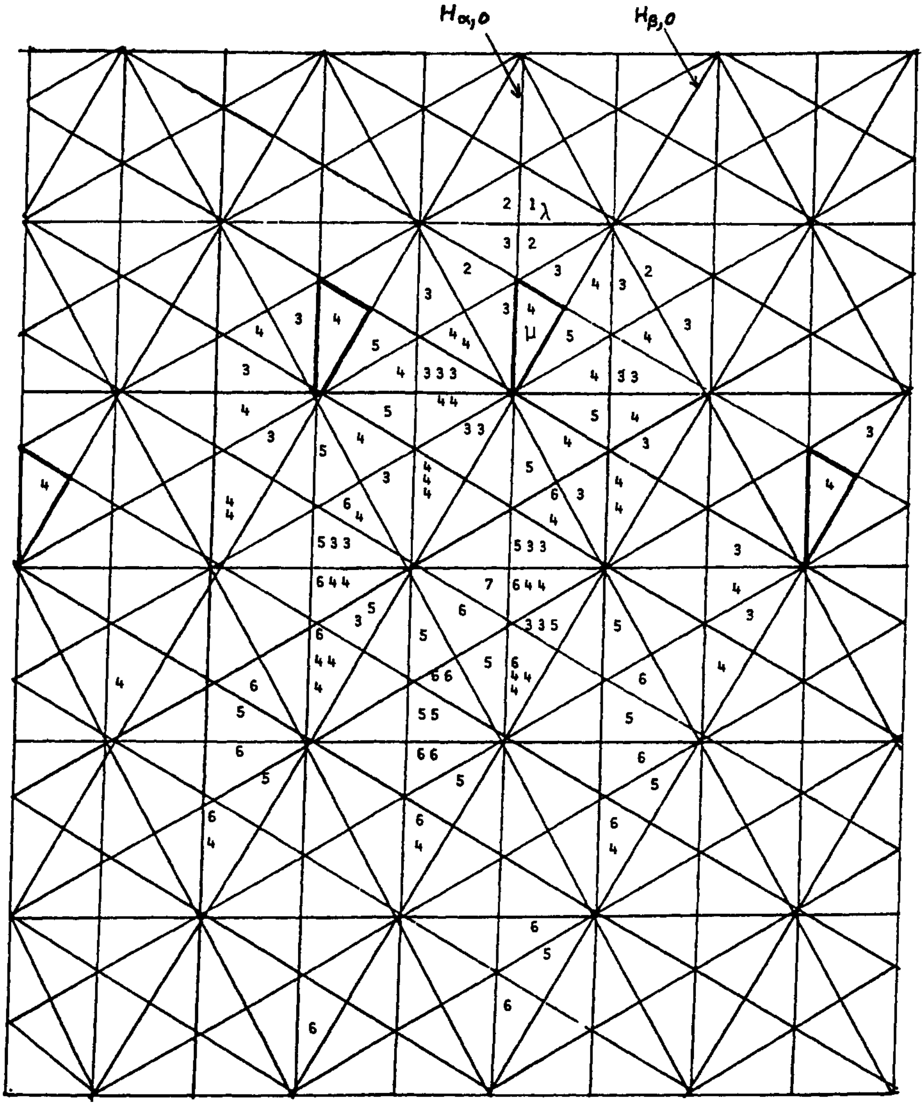

FIGURE 2

$$L_1(\mu)$$

$$L_1(\mu^0 + ps_\alpha \cdot \mu^1) = L_1(\mu - p\alpha) \qquad L_1(\mu - p\beta) = L_1(\mu^0 + ps_\beta \cdot \mu^1)$$

$$L_1(\mu - p\beta - p\alpha)$$

$$L_1(\mu - p\beta - 2p\alpha)$$

$$L_1(\mu - p\beta - 3p\alpha)$$

$$L_1(\mu - p\beta - 4p\alpha) = L_1(\mu^0 + ps_\alpha s_\beta \cdot \mu^1).$$

By the definition of $M^1_{[\mu]}$, all factors of $\mathrm{Soc}_{G_1B}M^1_{[\mu]}$ come from the orbit $\{L_1(\mu^0 + pw \cdot \mu^1)\}_{w\in W}$. We show that $\mathrm{Soc}_{G_1B}M^1_{[\mu]}$ is the irreducible module $L_1(\mu^0 + ps_\alpha s_\beta \cdot \mu^1)$.

(i) $L_1(\mu)$ is not a factor of the socle of $M^1_{[\mu]}$. In fact, since $[H^0(\lambda): L(\mu)]$ is zero by cancellation, $[\mathrm{Soc}_{G_1B}M^1_{[\mu]}: L_1(\mu)]$ is 0 by the Theorem of 3.2.

(ii) $L_1(\mu^0 + ps_\beta \cdot \mu^1)$ is not a factor of the socle of $M^1_{[\mu]}$. Here, we need to assume that there is a dominant weight δ such that $(\alpha^v, \delta) = 0$, and that the highest weights of composition factors of $H^0_1(\lambda + p\delta) = H^0_1(\lambda)\otimes p\delta$ now lie in the $1\cup s_\alpha$-chambers, but still have their 1-parts in $\bigcup_{w\in W} w \cdot \overline{C}_0$. The orbit $\{L_1(\mu^0 + pw \cdot (s_\beta \cdot \mu^1 + \delta))\}_{w\in W}$ contains just two composition factors of $H^0_1(\lambda + p\delta)$, namely, $L_1(\mu^0 + p(s_\beta \cdot \mu^1 + \delta))$ and $L_1(\mu^0 + ps_\alpha \cdot (s_\beta \cdot \mu^1 + \delta)) = L_1(\mu^0 + p(s_\alpha s_\beta \cdot \mu^1 + \delta))$. By the theorem of 3.2, we conclude that $L_1(\mu^0 + p(s_\beta \cdot \mu^1 + \delta))$ is not a factor of the socle of $(M \otimes p\delta)^1_{[\mu^0 + p(s_\beta\cdot\mu^1 + \delta)]}$. Since there is a canonical surjection $M^1_{[\mu]} \to (M \otimes p\delta)^1_{[\mu^0 + p(s_\beta\cdot\mu^1+\delta)]} \otimes (-p\delta)$,

$$L_1(\mu^0 + p(s_\beta \cdot \mu^1 + \delta)) \otimes (-p\delta) = L_1(\mu^0 + ps_\beta \cdot \mu^1)$$

is not a factor of $\mathrm{Soc}_{G_1B}M^1_{[\mu]}$.

(iii) $L_1(\mu^0 + ps_\alpha \cdot \mu^1)$ is not a factor of the socle of $M^1_{[\mu]}$. Let $M^2_{[\mu]}$ be the maximal quotient of $M^1_{[\mu]}$ whose socle contains only factors from $\{L_1(\mu), L_1(\mu^0 + ps_\beta \cdot \mu^1)\}$. An argument modelled after that of (ii), shows that $L_1(\mu)$ is not a factor of the socle of $M^2_{[\mu]}$. Hence, if $L_1(\mu^0 + ps_\alpha \cdot \mu^1)$ were a factor of $\mathrm{Soc}_{G_1B}M^1_{[\mu]}$, then we could form the quotient $M^1_{[\mu]}/L_1(\mu^0 + ps_\alpha \cdot \mu^1)$, where $L_1(\mu)$ would be not a factor of $\mathrm{Soc}_{G_1B}(M^1_{[\mu]}/L_1(\mu^0 + ps_\alpha \cdot \mu^1))$. Consider the exact sequence $0 \to N \to M^1_{[\mu]}/L_1(\mu^0 + ps_\alpha \cdot \mu^1) \xrightarrow{f} M^2_{[\mu]} \to 0$, where N denotes the kernel of the canonical map f. Since $[N: L_1(\mu)] = 0$, $[H^0(G/G_1B, N): L(\mu)] = 0$; by the Proposition of 1.2, applied to $M = M^2_{[\mu]}$, we have $[H^0(G/G_1B, M^2_{[\mu]}): L(\mu)] = 0$. Hence,

$$[H^0(M^1_{[\mu]}/L_1(\mu^0 + ps_\alpha \cdot \mu^1)): L(\mu)] = 0.$$

That leads to a contradiction as follows. Consider the exact sequence

$$\begin{aligned} 0 &\to H^0(G/G_1B, L_1(\mu^0 + ps_\alpha \cdot \mu^1)) \\ &\to H^0(G/G_1B, M^1_{[\mu]}) \to H^0(G/G_1B, M^1_{[\mu]}/L_1(\mu^0 + ps_\alpha \cdot \mu^1)) \\ &\to H^1(G/G_1B, L_1(\mu^0 + ps_\alpha \cdot \mu^1)) \to H^1(G/G_1B, M^1_{[\mu]}). \end{aligned}$$

Since $[\mathrm{Soc}_{G_1B} M^1_{[\mu]} : L_1(\mu)] = 0$, we have $[H^0(G/G_1B, M^1_{[\mu]}) : L(\mu)] = 0$ by the Proposition of 1.2. By Proposition (b) of 3.2, we have $[H^1(G/G_1B, M^1_{[\mu]}) : L(\mu)] = 0$. Hence,

$$\begin{aligned}&[H^0(G/G_1B, M^1_{[\mu]}/L_1(\mu^0 + ps_\alpha \cdot \mu^1)) : L(\mu)]\\ &\quad = [H^1(G/G_1B, L_1(\mu^0 + ps_\alpha \cdot \mu^1)) : L(\mu)] = 1.\end{aligned}$$

By (i)–(iii), we have $\mathrm{Soc}_{G_1B} M^1_{[\mu]} = L_1(\mu^0 + ps_\alpha s_\beta \cdot \mu^1)$. Hence, by the fact [2, Corollary 2.4] that

$$\mathrm{Ext}^1_B(\nu, \eta) = \begin{cases} k & \text{if } \eta = \nu - p^j\beta,\ j \geq 0,\ \beta \in S, \\ 0 & \text{otherwise,} \end{cases}$$

the factors displayed for $M^1_{[\mu]}$ between $L_1(\mu^0 + ps_\alpha s_\beta \cdot \mu^1)$ and $L_1(\mu)$ must occur as shown, and with multiplicity one only.

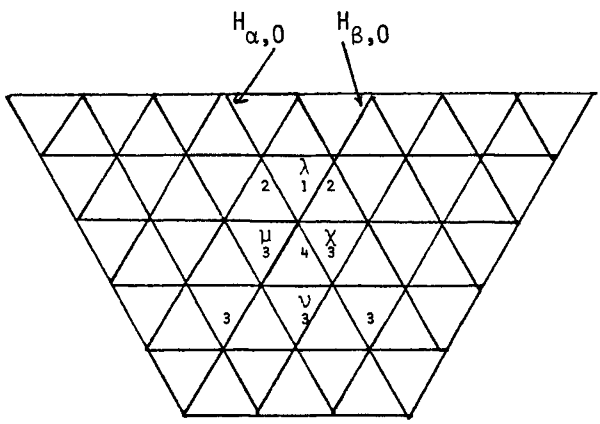

FIGURE 3

(3) Let $G = \mathrm{SL}_3$. Let λ and μ be weights positioned as shown in Figure 3. The numbers there give the socle layers of the composition factors of $H^0_1(\lambda)$. The third socle layer M equals $M_{[\mu]} \oplus M_{[\chi]} \oplus M_{[\nu]}$, in the terms of 1.2. By the theorem, $M_{[\mu]}$ and $M_{[\chi]}$ are non-split G_1B-extensions and $\mathrm{Soc}_{G_1B} M = L_1(\mu - p\beta) + L_1(\chi - p\alpha) + L_1(\nu)$. See [9] for a complete presentation of the structure of $H^0_1(\lambda)$ for SL_3.

We complete this illustration with a remark about the evaluation map $e: H^0(\lambda) \to H^0_1(\lambda)$, when λ is positioned so that all the composition factors of $H^0_1(\lambda)$ have dominant highest weights. From the fact that $H^0(G/G_1B, e(M)) = H^0(G/G_1B, M)$ for any G_1B-module M, in the case before us, we conclude that $H^0(G/G_1B, H_1^0(\lambda)/e(H^0(\lambda))) = (0)$ and that $e(H^0(\lambda)) = H_1^0(\lambda)$. Then, in the case of SL_3, $e\colon \mathrm{Soc}^j_G H^0(\lambda) \to \mathrm{Soc}^j_{G_1} H_1^0(\lambda)$ is surjective for each j. Let M_G be the G-semisimple module $\mathrm{Soc}^3_G H^0(\lambda)/\mathrm{Soc}^2_G H^0(\lambda)$ and let M_{G_1} be the G_1-semisimple G_1B-module $\mathrm{Soc}^3_{G_1} H_1^0(\lambda)/\mathrm{Soc}^2_{G_1} H_1^0(\lambda)$. $e\colon M_G \to M_{G_1}$ is surjective, and

$$\begin{aligned} e(M_G) &= e(H^0(G/G_1B, (M_{G_1})_{[\mu]})) \oplus e(H^0(G/G_1B, (M_{G_1})_{[\chi]})) \\ &\quad \oplus e(H^0(G/G_1B, (M_{G_1})_{[\nu]})) \\ &= (M_{G_1})_{[\mu]} \oplus (M_{G_1})_{[\chi]} \oplus (M_{G_1})_{[\nu]}, \quad \text{term-by-term.} \end{aligned}$$

In particular,

$$\begin{aligned} e(H^0(G/G_1B, (M_{G_1})_{[\mu]})) &= e(L(\mu) \oplus L(\mu - p\beta)) \\ &= e(L(\mu)) + e(L(\mu - p\beta)) = (M_{G_1})_{[\mu]} \end{aligned}$$

is the non-split G_1B-extension of $L_1(\mu)$ by $L_1(\mu - p\beta)$. Since $\mu - p\beta < \mu$, $e(L(\mu - p\beta))$ cannot contain the G_1B-composition factor $L_1(\mu)$. Hence, it must be that $e(L(\mu)) = (M_{G_1})_{[\mu]}$. Thus, the G_1B-extension of $L_1(\mu)$ by $L_1(\mu - p\beta)$ in $(M_{G_1})_{[\mu]}$ also occurs in M_G, but it occurs within $L(\mu)$, instead of between $L(\mu)$ and $L(\mu - p\beta)$.

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References

[1] H. H. Andersen, *Cohomology of line bundles on G/B*, Ann. Sci. École Norm. Sup., **12** (1979), 85–100.
[2] ———, *Extensions of modules for algebraic groups*, Amer. J. Math., **106** (1984), 489–504.
[3] ———, *On the generic structure of cohomology modules for semisimple algebraic groups*, TAMS, **295** (1986), 397–415.
[4] ———, *Jantzen's filtration of Weyl modules*, Math. Z., **194** (1987), 127–142.
[5] E. Cline, B. Parshall, L. Scott, W. van der Kallen, *Rational and generic cohomology*, Invent. Math., **39** (1977), 143–163.
[6] S. Doty and J. Sullivan, *On the structure of the higher cohomology modules of line bundles on G/B*, J. Algebra, **114** (1988), 286–332.
[7] ———, *Filtration patterns for representations of algebraic groups and their Frobenius kernels*, Math. Z., **195** (1987), 391–407.
[8] J. Humphreys, *Cohomology of line bundles on G/B for the exceptional group G_2*, J. Pure Appl. Algebra, **44** (1987), 227–239.
[9] R. Irving, *The structure of certain highest weight modules for SL_3*, J. Algebra, **99** (1986), 438–457.

[10] J. Jantzen, *Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne*, J. Reine Angew. Math., **317** (1980), 157–199.

[11] J.-C. Ye, *Filtrations of principal indecomposable modules of Frobenius kernels of reductive groups*, Math. Z., **189** (1985), 515–527.

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