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ENLARGEMENTS OF QUANTUM LOGICS

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Let K be a quantum logic whose state space is nonvoid. Let B be a Boolean algebra and let C be a compact convex subset of a locally convex topological linear space. Then K can be enlarged to a logic L such that the centre of L equals B and the state space of L equals C. (The result remains valid when we replace the word "logic" with "orthomodular lattice".)

1. Introduction and preliminaries. An orthomodular partially ordered set L ("a quantum logic") naturally induces one algebraic and one measure-theoretic structure: The centre, $C(L)$, which is the set of all absolutely compatible elements in L, and the state space, $\mathcal{S}(L)$, which is the set of all "states" (= probability measures) on L. When we investigate the interplay of $C(L)$ and $\mathcal{S}(L)$, a natural question arises whether one can construct logics with given centres, state spaces and preassigned sublogics. (This question seems to be of certain significance also from the physical point of view if one wants to clarify the dependence, resp. independence, of the centre and the state space in the model. It should be noticed that, for instance, in the von Neumann algebra model, the state space determines the centre—see [**1**], [**6**], [**13**].)

Since centres of logics are exactly Boolean algebras (see e.g. [**5**]) and since state spaces of logics are (up to an affine homeomorphism) exactly compact convex subsets of locally convex topological linear spaces (see [**11**]), our question reads as follows: Given a Boolean algebra B and a compact convex set C of LCTLS, can every logic be enlarged to a logic L with $C(L) = B$ and $\mathcal{S}(L) = C$? In this paper we answer the latter question in the affirmative (obviously under "conditio sine que non" that the state space of the initial logic is nonvoid). We need rather advanced construction techniques with logics in some places—apart from applying key results and methods of the papers [**3**], [**4**], [**9**], [**10**], [**11**] we have to develop a fairly nontrivial "pasting" technique. This effort is needed mainly for establishing the following interesting lemma: Every logic K with $\mathcal{S}(K) \neq \emptyset$ can be enlarged to a logic L with card $\mathscr{S}(L) = 1$. It turns out that this very special case of our problem becomes in fact equivalent to the general case.

Let us start the investigation with reviewing basic definitions and facts.

DEFINITION 1.1. A (quantum) *logic* is a partially ordered set L with a least and a greatest element, 0, 1, together with an operation $x \to x'$ (an orthocomplementation) mapping L to L, such that the following conditions are satisfied for any $a, b \in L$ (the symbols $\vee$, $\wedge$ mean the lattice operations induced by $\leq$):

(i) $(a')' = a$,
(ii) $a \leq b$ implies $b' \leq a'$,
(iii) $a \vee a' = 1$,
(iv) if $a \leq b'$ then $a \vee b$ exists in L,
(v) if $a \leq b$ then $b = a \vee (a' \wedge b)$ (the orthomodular law).

Typical example of logics is a Boolean algebra or the lattice of all projections in a von Neumann algebra. Naturally, a logic need not be either distributive or a lattice. In what follows, let us reserve the letter L for logics.

DEFINITION 1.2. Let a, b be in L. We call a, b *orthogonal* (abbr. $a \perp b$) if $a \leq b'$, and we call a, b *compatible* if there are mutually orthogonal elements $x, y, z \in L$ such that $a = x \vee z$, $b = y \vee z$. The *centre* $C(L)$ of L is the set of all absolutely compatible elements ($C(L) = \{c \in L \mid c$ is compatible to any $d \in L\}$).

PROPOSITION 1.3 (*see* [5]). *The centre $C(L)$ of a logic L is a Boolean algebra (with the operations $\vee$, $\wedge$, $'$ inherited from L). Further, we have $C(L) = L$ if and only if L is a Boolean algebra. (Thus, obviously, every Boolean algebra is a centre of a logic.)*

DEFINITION 1.4. Let K, L be logics and let $f: K \to L$ be a mapping. Then f is called a (logic) *morphism* if the following conditions hold true:

(i) $f(0) = 0$,
(ii) $f(a') = f(a)'$ for any $a \in K$,
(iii) $f(a \vee b) = f(a) \vee f(b)$ whenever $a, b \in K$ and $a \perp b$.

When f is bijective and both f and f^{-1} are morphisms, then f is said to be an *isomorphism*. When $f: K \to f(K)$ is an isomorphism then f is called an *embedding*. In this case we call K a *sublogic* of L and L an *enlargement* of K. (Since a sublogic K of L is intrinsically identical with $f(K)$, we shall sometimes identify K with $f(K) \subset L$.)

Let us now introduce measure-theoretic notions we shall need in the sequel.

DEFINITION 1.5. A *state* on a logic L is a mapping $s: L \to \langle 0, 1\rangle$ such that

(i) $s(1) = 1$,
(ii) if $a, b \in L$ and $a \perp b$ then $s(a \vee b) = s(a) + s(b)$.

Let us denote by $\mathscr{S}(L)$ the set of all states on L (called "state space"). The set $\mathscr{S}(L)$ is naturally endowed with a topological and convex structure (as a subset of $\langle 0, 1\rangle^L$). When understood this way, $\mathscr{S}(L)$ is obviously a compact convex set. In fact, we have the following result.

PROPOSITION 1.6 (*see* **[11]**). *State spaces are (up to affine homeomorphisms) exactly compact convex subsets in locally convex topological linear spaces.*

In the constructions which follow later we shall frequently use the following corollary of Proposition 1.6. There are logics S, R with $\operatorname{card}\mathscr{S}(S) = 0$ (i.e., $\mathscr{S}(S) = \emptyset$) and $\operatorname{card}\mathscr{S}(R) = 1$ (i.e., $\mathscr{S}(R)$ is a singleton). We call the former logics *stateless* and the latter logics *rigid*. (Obviously, a stateless logic cannot be Boolean. There are non-Boolean rigid logics—for instance, it suffices to put $R = S \times \{0, 1\}$ for a stateless logic S, see **[10]**).

Let us finally recall a simple construction. Observe first that the set $\langle 0, a\rangle = \{x \in L | x \leq a\}$ (resp. $\langle b, 1\rangle = \{x \in L | x \geq b\}$) becomes a logic when considered with the ordering and the orthocomplementation naturally inherited from L (see e.g. **[7]**, **[12]**). Using this fact, we can easily prove the following proposition.

PROPOSITION 1.7. (*A straightforward generalization of* **[3]**, *Theorem* 3.4; *we can also obtain a proof of this proposition as a by-product of the final part of the proof of Theorem* 2.2 *in the next paragraph*) : *Let K, L be logics. Suppose that, for an element $a \in K \cap L$, we have $K \cap L = \langle 0, a\rangle \cup \langle a', 1\rangle$, where $\langle 0, a\rangle$ (considered as an interval in K) is isomorphic to $\langle 0, a\rangle$ (considered as an interval in L). Then $M = K \cup L$ becomes a logic (called the (a, a')-pasting of K and L) if endowed with the partial ordering $\leq$ such that $a \leq b$ in M if and only if $a \leq b$ in K or in L, and with the orthocomplementation $'$ inherited from K and L. Moreover, both K and L become sublogics of M.*

Particularly, if $a = 0$ we call the latter construction the *(0, 1)-pasting* of K and L. We have the following result which will be frequently used.

PROPOSITION 1.8. *Let L be a logic. Let S (resp. R) be a stateless (resp. rigid) logic. Then the following statements hold true:*

(i) *If M denotes the $(0, 1)$-pasting of L and S then M is stateless. Moreover, if* $\operatorname{card} L > 2$ *then $C(M) = \{0, 1\}$;*
(ii) *If M denotes the $(0, 1)$-pasting of L and R then $\mathcal{S}(M)$ and $\mathcal{S}(L)$ are affinely homeomorphic.*
Moreover, if $\operatorname{card} L > 2$ *and* $\operatorname{card} R > 2$ *then $C(M) = \{0, 1\}$.*

The proof of Proposition 1.8 is elementary.

2. Results. The first result says that we solve our problem as soon as we can embed arbitrary logics into rigid logics. (As always in what follows, the sign "=" means "Boolean isomorphic" when applied for two Boolean algebras, and "affinely homeomorphic" when applied for two state spaces.)

THEOREM 2.1. *Let R, H be logics and let B be a Boolean algebra. Let us suppose that R be rigid. Then R can be embedded into a logic L such that $C(L) = B$ and $\mathcal{S}(L) = \mathcal{S}(H)$.*

Proof. Denote by M the $(0, 1)$-pasting of R and H. Denote further by P the $(0, 1)$-pasting of M and a stateless logic S. Then $\mathcal{S}(M) = \mathcal{S}(H)$ and $\mathcal{S}(P) = \emptyset$. We may suppose that $C(M) = C(P) = \{0, 1\}$.

Let us assume that B is a Boolean algebra of subsets of a set X. Choose a point $z \in X$. Let L be the set of all functions $f: X \to P$ with the following three properties:

(i) the range of f is finite,
(ii) the set $f^{-1}(b) = \{x \in X \mid f(x) = b\}$ belongs to B for any $b \in P$,
(iii) $f(z) \in M$.

Let us endow the set L with the usual "pointwise" partial ordering and orthocomplementation inherited from P. (We put $f = g'$, resp. $f \leq g$, if $f(x) = g(x)'$, resp. $f(x) \leq g(x)$, for any $x \in X$.) A routine verification gives that L is a logic. Obviously, the mapping $h: R \to L$ such that $(h(k))(x) = k$ ($k \in R$, $x \in X$) is an embedding. It remains to be shown that $C(L) = B$ and $\mathcal{S}(L) = \mathcal{S}(H)$.

To show that $C(L) = B$, let us observe that a function $f \in L$ belongs to $C(L)$ if and only if $f(x) \in \{0, 1\}$ for any $x \in X$. Indeed, if it is not the case, we have $f(x) = b \in M - \{0, 1\}$. Since $C(M) = \{0, 1\}$, there is an element $c \in M$ which is not compatible with b. Take a function $g: X \to L$ such that $g(x) = c$ for $f(x) = b$, $g(x) = f(x)$ otherwise. Then $g \in L$ and g is not compatible to f. We see that $C(L)$ are exactly the "characteristic functions" of the subsets of X which belong to B. Thus, $C(L) = B$.

Let us now consider $\mathscr{S}(L)$. Let us first notice that if Y is a subset of X such that $Y \in B$ and $z \notin Y$, then the set L_Y of all constant functions $h: Y \to P$ forms a stateless logic (isomorphic to P). Suppose now that $s \in \mathscr{S}(L)$. By the foregoing property, we easily see that if $f \in L$ such that $f(z) = 0$ then $s(f) = 0$. Thus, for any $f \in L$, the value of $s(f)$ depends only on $f(z)$. It follows that the mapping $\alpha: \mathscr{S}(M) \to \mathscr{S}(L)$ defined by the equality $\alpha(t)(f) = t(f(z))$ $(t \in \mathscr{S}(M))$ is an affine homeomorphism and therefore $\mathscr{S}(M) = \mathscr{S}(L)$. The proof of Theorem 2.1 is complete.

The question now remains whether an arbitrary logic admits an embedding into a rigid logic. A positive answer to this question is given in the following theorem. It is obvious that the combination of Theorems 2.1, 2.2 establishes our main result.

THEOREM 2.2. *Let K be a logic and let s belong to $\mathscr{S}(K)$. Then there is a rigid logic R and an embedding $e: K \to R$ such that, for the (only) state $\tilde{s} \in \mathscr{S}(R)$, we have $\tilde{s}e = s$.*

Proof. We shall need a few lemmas. Since the arguments are fairly technical in some places, the reader acquainted with the papers **[3]**, **[10]**, **[11]** will probably find himself in a more convenient situation.

LEMMA 2.3. *Suppose that H_3 is a three-dimensional Hilbert space and suppose that $L(H_3)$ is the logic of all projections in H_3. Suppose further that a is an atom in $L(H_3)$ (i.e., we suppose that a is a projection on a one-dimensional subspace). Then there is precisely one state $s \in \mathscr{S}(L(H_3))$ such that $s(a) = 1$ and, moreover, for any real number $r \in \langle 0, 1 \rangle$, we can find an atom $a_r \in L(H_3)$ such that $s(a_r) = r$.*

Proof. Lemma 2.3 immediately follows from the description of states on $L(H_3)$ given by the Gleason theorem (see **[2]**).

LEMMA 2.4. *Let a real number $r \in \langle 0, 1 \rangle$ be given. Then we can construct a rigid logic R_r such that, for an atom $a_r \in R_r$ and the (only) state $s \in \mathscr{S}(R_r)$, we have $s(a_r) = r$.*

Proof. For $r = 1$, the rigid logic R_1 (with the appropriate atom a_1) was constructed in **[11]**, Lemma 2. If $r \neq 1$, then we can choose an atom $a \in L(H_3)$ and identify a with a_1 ($a_1 \in R_1$) and, dually, a' with a'_1. We also identify the zeros and units of $L(H_3)$ and R_1. Then the (a, a')-pasting of $L(H_3)$ and R_1 can be taken for R_r (Lemma 2.3).

LEMMA 2.5. *Let K be a logic and let s belong to $\mathscr{S}(K)$. Choose an element $a \in K$. Then there is a logic L_a such that the following conditions are satisfied (for a logic L the symbol $\langle 0, x\rangle_L$, resp. $\langle x', 1\rangle_L$, means the interval $\langle 0, x\rangle$, resp. $\langle x', 1\rangle$, considered in L):*

(i) *$K \cap L_a = \langle 0, a\rangle_K \cup \langle a', 1\rangle_K = \langle 0, a\rangle_{L_a} \cup \langle a', 1\rangle_{L_a}$ and the partial ordering and orthocomplementation in K and in L_a coincide on $K \cap L_a$;*

(ii) *for any $t \in \mathscr{S}(L_a)$ we have $t(a) = s(a)$,*

(iii) *if $t_1, t_2 \in \mathscr{S}(L_a)$ and $t_1(k) = t_2(k)$ for any $k \in K \cap L_a$, then $t_1 = t_2$;*

(iv) *there exists a state $\bar{s} \in \mathscr{S}(L_a)$ such that $\bar{s}(k) = s(k)$ for any $k \in K \cap L_a$.*

Proof. Put $r = s(a)$. By Lemma 2.4 we can find a rigid logic R_r and an atom $a_r \in R_r$ such that $t(a_r) = r$ for the state $t \in \mathscr{S}(R_r)$. In the logic R_r we first fill the interval $\langle 0, a\rangle_K$ in $\langle 0, a_r\rangle_{R_r}$ and identify a with a_r. We obtain a set $\tilde{R}_r$. Further, for all $b \in R_r$ with $b \perp a_r$ and all $c \in \langle 0, a\rangle_K$, we add the elements $b \vee c$ to $\tilde{R}_r$ and identify $b \vee a$ with $b \vee a_r$. We obtain a set L_a. We endow L_a with the partial ordering equal to the transitive closure of the orderings in R_r and $\langle 0, a\rangle_K$ and with the orthocomplementation defined as follows: If $b \in L_a - R_r$, then we can write $b = c \vee d$, where $c \in \langle 0, a'_r\rangle_{R_r}$ and $d \in \langle 0, a\rangle_K$, and define $b' = (c' \wedge a'_r) \vee (d' \wedge a)$, and if $b \in R_r$, then b' is defined as the orthocomplement of b in R_r. It can be easily checked that L_a becomes a logic. (The above construction is quite intuitive and thus we allow ourselves to leave the details to the reader. A thorough investigation of the construction together with further generalizations can be found in **[8]**.)

The intersection $K \cap L_a$ consists of all elements of $\langle 0, a\rangle_K$ and their complements. Every state $t \in \mathscr{S}(K)$ restricted to $\langle 0, a\rangle_K$ has an extension to a state on L_a if and only if $t(a) = s(a)$. We infer that L_a possesses all the properties required in Lemma 2.5.

LEMMA 2.6. *Let K be a logic. Let us call a subset S of K overlapping if $a \leq b'$ for no pair $a, b \in S$. Let us denote by $\mathscr{S}$ the collection of all* *overlapping subsets of K and let us order the set $\mathcal{S}$ by inclusion. Then there is a maximal element T in $\mathcal{S}$ and, moreover, for any $a \in K$ we have $a \in T$ or $a' \in T$.*

Proof. The existence of a maximal set $T \in \mathcal{S}$ follows immediately from Zorn's lemma. Suppose that $\{b, b'\} \cap T = \emptyset$ for some $b \in K$. Since T is maximal, there exist $c, d \in T$ such that $c \leq b'$ and $d \leq (b')' = b$. It follows that $d \leq b \leq c'$ and therefore $d \leq c'$—a contradiction. This completes the proof of Lemma 2.6.

Let us now return to the proof of Theorem 2.2. Let us first take a maximal overlapping subset of K, some T, and put $T' = \{a \in K | a = b' \text{ for some } b \in T\}$. For each $a \in T'$ take a logic L_a from Lemma 2.5. We may (and shall) assume that $(L_a - K) \cap (L_b - K) = \emptyset$ for $a \neq b$. Let us put $\mathcal{R} = \{K\} \cup \{L_a | a \in T'\}$. Let $R = \bigcup_{P \in \mathcal{R}} P$. Define $\leq, '$ in R as follows: We have $c \leq d$ (resp. $c = d'$) if and only if there is $P \in \mathcal{R}$ such that $c \leq d$ in P (resp. $c = d'$ in P). We claim now that R is a logic with the properties required in Theorem 2.2. We first show that R is indeed a logic. This will be verified in the following four statements.

Statement 1. If $P, Q \in \mathcal{R}$ and $P \neq Q$ then $P \cap Q$ is a subset of K which is closed under the formation of the least upper bounds (in K) of orthogonal elements. Moreover, with any $x \in P \cap Q$ the set $P \cap Q$ contains either the entire set $\langle 0, x \rangle_K$ or the entire set $\langle x, 1 \rangle_K$.

Indeed, we have

$$\begin{aligned} L_a \cap L_b &= (L_a \cap K) \cap (L_b \cap K) \\ &= (\langle 0, a \rangle_K \cup \langle a', 1 \rangle_K) \cap (\langle 0, b \rangle_K \cup \langle b', 1 \rangle_K). \end{aligned}$$

Since both a, b belong to T', we have ensured that a' is not orthogonal to b' and therefore $\langle 0, a \rangle_K \cap \langle b', 1 \rangle_K = \emptyset = \langle 0, b \rangle_K \cap \langle a', 1 \rangle_K$. We obtain

$$L_a \cap L_b = (\langle 0, a \rangle_K \cap \langle 0, b \rangle_K) \cup (\langle a', 1 \rangle_K \cap \langle b', 1 \rangle_K).$$

Also, $L_a \cap K = \langle 0, a \rangle_K \cup \langle a', 1 \rangle_K$. Thus, using this expression for $P \cap Q$, we immediately have that $P \cap Q$ possesses the required properties.

Statement 2. The relation $\leq$ is transitive on R.

Indeed, if $b \leq c$ in P and $c \leq d$ in Q for $P, Q \in \mathcal{R}$, then $c \in P \cap Q \subset K$. The set $P \cap Q$ contains either the entire $\langle 0, c \rangle_K$ or the entire $\langle c, 1 \rangle_K$. Hence either b or d belongs to $P \cap Q$. Thus either P or Q contains all the elements b, c, d and therefore we have obtained $b \leq d$ as desired.

Statement 3. Let V be a subset of R such that for any pair $x, y \in V$ there is a logic belonging to the collection $\mathscr{R}$ which contains both x and y. Then the entire V is contained in a logic belonging to $\mathscr{R}$.

Indeed, if V is not contained in K, then there is an element x such that $x \in V - K$. Then there is exactly one logic $P \in \mathscr{R}$ containing x and therefore we have $V \subset P$.

Statement 4. The set R is closed under the formation of the least upper bounds of orthogonal pairs.

Indeed, let $b, c \in R$ be orthogonal in R. Then there is a logic $P \in \mathscr{R}$ which contains the triple $\{b, c, b \vee c\}$, where the l.u.b. is taken in P. If d is an upper bound of b, c in R then b, c, d are contained in a logic $Q \in \mathscr{R}$ and therefore $b \vee c$ (in P) belongs to $P \cap Q$. Hence $b \vee c$ (in P) is not greater than d (with respect to the ordering in Q) and we have $b \vee c$ (in P) $= b \vee c$ (in R).

It is obvious that the foregoing statements verify that R is a logic. Moreover, the natural inclusion mapping $e: K \to R$ embeds K into R. Finally, let $\tilde{s} \in \mathscr{S}(R)$. Since $T \cup T' = K$, we see that $\tilde{s}e = s$. By the construction, every state on L_a was completely determined by its values on $\langle 0, a\rangle$ (Lemma 2.5). Therefore R admits only one state and the proof of Theorem 2.2 is complete.

Let us now state our main result. In view of the characterization of the state spaces of logics **[11]**, we can formulate it as follows. (Moreover, if we take the trouble in verifying the lattice stability of the constructions—and the authors did—we can also add the lattice version.)

THEOREM 2.3. *Suppose that we are given a Boolean algebra B and a compact convex set C of LCTLS. Then every non-stateless logic K (resp. every non-stateless orthomodular lattice K) can be embedded in a logic L (resp. can be lattice-theoretically embedded in an orthomodular lattice L) such that $C(L) = B$ and $\mathscr{S}(L) = C$.*

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