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FINITE DIMENSIONAL REPRESENTATION OF CLASSICAL CROSSED-PRODUCT ALGEBRAS

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The paper describes the structure of finite dimensional representations of B_T, the crossed-product algebra of a classical dynamical system $(\alpha_T, \mathbb{Z}, C(X))$ where T is a homeomorphism on a compact space X. The results are used to describe the topology of $\mathrm{Prim}_n(B_T)$ and to partially classify the hyperbolic crossed-product algebras over the torus. One of the main results is that the number of orbits of any fixed length with respect to T is an invariant of B_T. A consequence of that is that the entropy of T is an invariant of B_T, for T a hyperbolic automorphism on the m-torus.

Introduction. The purpose of this paper is to study finite dimensional representations of classical crossed-product algebras. The results are used to describe the primitive ideal space of these algebras and partially classify them. The first two sections deal primarily with finite dimensional representations of B_T, the crossed-product algebra B_T of a classical dynamical system of the form $(\alpha_T, \mathbb{Z}, C(X))$ where T is a homeomorphism on a compact space X. In §1 we study the general form of an irreducible n-dimensional representation of B_T. We show how to adjoin an orbit of length n to each such representation. The idea of adjoining an orbit to each finite dimensional representation is then further explored in §2. We show that the number of connected components in $\mathrm{Prim}_n(B_T)$ is equal to the number of orbits of length n with respect to T. A consequence of this result is that the entropy of T, for T a hyperbolic automorphism on $\mathbf{T}^m$, is an invariant of B_T. In §3 we investigate the classification of the B_T's corresponding to automorphisms on the 2-torus.

Preliminaries. For any integer n we define $E_n \colon B_T \to C(X)$ to be the (continuous) transformation that takes C in B_T to its nth "Fourier" coefficient f_n, see [**1**] for details. Symbolically, we write each C in B_T as $\sum f_n U^n$ where $f_n = E_n(C)$. Let $(\hat{\alpha}, \mathbf{T}, B_T)$ be the C^*-dynamical system defined by the dual action $\hat{\alpha}_\lambda(C) = \sum \lambda^n U^n$, [**2**]. It is known that the Fejer sums of the function $\lambda \to \hat{\alpha}_\lambda(C)$ converge uniformly to

$\hat{\alpha}_\lambda(C)$, see [3] for an elementary proof. In other words,

$$\sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) f_k U^k \lambda^k \to \hat{\alpha}_\lambda(C)$$

uniformly in λ, and in particular for $\lambda = 1$,

$$\sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) f_k U^k \to C.$$

1. Finite dimensional representations of B_T.

NOTATION. Let Y be a subset of X. Then by J_Y we denote the closed ideal in B_T generated by $\{f \in C(X); f|_Y = 0\}$.

LEMMA (1.1). *If Y is an invariant set then*

$$J_Y = \left\{\sum f_n U^n \in B_T; f_n|_Y = 0\right\}.$$

Here $\sum f_n U^n$ stands for the element C in B_T whose $E_n(C)$ is equal to f_n.

Proof. Show $\{\dots\} \subseteq J_Y$. Let $C = \sum f_n U^n$ be in B_T such that $f_n|_Y = 0$ for all n. Since the Fejer sums of C converge to C, as was mentioned in the preliminaries, it follows that C is in J_Y. Conversely, show that $J_Y \subseteq \{\dots\}$. Note that the collection $I = \{\sum_{\text{finite}} f_n U^n; f_n|_Y = 0\ \forall n\}$ is an ideal, not closed, in $K(\mathbb{Z}, C(X))$. *Reason*: If $f|_Y = 0$ then $(\alpha_T)^n(f) = f(T^{-n}(\cdot))$ is zero on Y since Y is invariant and therefore I is closed under multiplication. It is clearly closed under addition and scalar multiplication. Since $K(\mathbb{Z}, C(X))$ is dense in B_T it follows at once that the closure of I is an ideal of B_T. Therefore, the closure of I is exactly J_Y. Let $C = \sum f_n U^n$ be in J_Y and let $\{C_k = \sum f_n^k U^n\}$ in I be such that $C_k \to C$. From the continuity of E_n it follows that $f_n^k \to f_n$ for all n whence f_n is 0 on Y for all n. □

We need some characterization of the J_Y's which is invariant under algebra isomorphism. This will be done by means of finite dimensional irreducible representations of B_T. The treatment of a general n-dimensional irreducible representation of B_T will be tailored after the 1-dimensional case which is described in what follows. Let $\rho\colon B_T \to \mathbb{C}$ be a non-degenerate representation. We know, [2], that $\rho = \pi \times W$ for some covariant representation $(\pi, W, \mathbb{C})$ of our dynamical system $(\alpha_T, \mathbb{Z}, C(X))$. Now, since π restricted to $C(X)$ is a representation of $C(X)$ on $\mathbb{C}$ it is known to be of the form $\pi(f) = f(x_0)$ for some x_0 in

X. Also, since W is unitary it is given by powers of some λ of absolute value 1. The covariant condition implies that $\pi(\alpha_1(f)) = W\pi(f)W^{-1}$ for all f in $C(X)$. As a result $T^{-1}x_0 = x_0$ whence x_0 is a fixed point.

Conversely, given any λ of absolute value 1 and x_0 a fixed point we can construct a covariant representation $(\pi, W, \mathbb{C})$ by defining $\pi(f) = f(x_0)$ for all f in $C(X)$ and $W(n) = \lambda^n$ for all n in $\mathbb{Z}$. We denote the dependence of ρ on x_0 and λ by $\rho_{x_0,\lambda}$. To summarize, the $\rho_{x_0,\lambda}$'s describe all the irreducible 1-dimensional representations of B_T.

We now turn to a general irreducible n-dimensional representation of B_T. First we describe some such representations and then we show that those are the only ones up to equivalence of representations. Let Y be the orbit of some periodic point of period n. Fix some $\lambda = \{\lambda_y\}_{y\in Y}$ where $|\lambda_y| = 1$ for all y in Y. As in the 1-dimensional case we will show that corresponding to Y and λ there is an n-dimensional representation $\rho_{Y,\lambda}$ of B_T. The representation $\rho_{Y,\lambda}$ will be constructed via a covariant representation $(\pi, W, l^2(Y))$ of our dynamical system. Let $\{e_y\}_{y\in Y}$ be the natural basis in $l^2(Y)$. Then for all f in $C(X)$, we define $\pi(f)$ as follows. For all y in Y, $\pi(f)e_y = f(y)e_y$. The unitary representation W is defined via the unitary W, with some abuse of notation, as follows. For all y in Y, $We_y = \lambda_y e_{Ty}$. Note that with respect to the basis $\{e_y\}$ the unitary W is the product of the unitaries W_0 and D, where W_0 is the unitary taking e_y to e_{Ty} and D is the diagonal unitary having λ_y's on the diagonal.

We check that the covariant condition is satisfied. Let n be an arbitrary integer. Then,

$$\pi(\alpha_n(f))e_y = \pi(f(T^{-n}(\cdot)))e_y = f(T^{-n}y)e_y.$$

On the other hand, $W^{-n}e_y = \mu e_{T^{-n}y}$ for some μ of absolute value 1. Therefore,

$$\begin{aligned} W^n\pi(f)W^{-n}e_y &= W^n\pi(f)(\mu e_{T^{-n}y}) = W^n\mu f(T^{-n}y)e_{T^{-n}y} \\ &= (\mu f(T^{-n}y))(W^n e_{T^{-n}y}) = (\mu f(T^{-n}y))(\mu^{-1}e_y) \\ &= f(T^{-n}y)e_y. \end{aligned}$$

Finally, we need to show that $\pi \times W$ is irreducible. Since the algebra $M_n(\mathbb{C})$ is simple it is sufficient to show that $\pi \times W$ contains all the elementary matrices in $M_n(\mathbb{C})$. Since Y is a finite orbit T acts on it transitively. Therefore, each elementary matrix in $M_n(\mathbb{C})$ will be equal to $\pi(f)W^m$ for appropriate f and m.

Next, we show that any n-dimensional representations of B_T must have, up to equivalence of representations, the form $\rho_{Y,\lambda}$ for some Y, λ

as described above. Let ρ be any irreducible representation of B_T on some n-dimensional vector space $\mathbb{C}^n$. Then, $\rho = \pi \times W$ for some covariant representation $(\pi, W, \mathbb{C}^n)$ of B_T. Since π reduced to $C(X)$ is a representation of that algebra, it is known that with respect to some orthonormal basis in $\mathbb{C}^n$, π is given by $f \to \text{diagonal}(f(y_0), \ldots, f(y_{n-1}))$. We index this basis by the y_i's so that $\{e_i\}$, $0 \leq i \leq n-1$, is the new basis. We may assume that the representation of π is with respect to this basis. Let Y be the collection $\{y_0, \ldots, y_{n-1}\}$. Note that for the time being we do not know that the y_i's are all distinct.

First, we show that Y is invariant. Since $(\pi, W, \mathbb{C}^n)$ is a covariant representation then for all f in $C(X)$, $\pi(\alpha_1(f)) = W\pi(f)W^{-1}$. If Y was not invariant under T then there would exist y in Y such that $T^{-1}y$ is not in Y. Choose f in $C(X)$ such that f is 0 on Y but is 1 on $T^{-1}y$. In that case $W\pi(f)W^{-1} = 0$ but $\pi(\alpha_1(f)) \neq 0$—contradiction.

Next, we show that Y is an orbit. Note that a priori we do not know that the y_i's are all distinct so that we also have to show that there is no duplication among the y_i's. Let Y_1 be the orbit of some arbitrary element y in Y. Let $\{i_j\}$ be a subsequence of $\{i\}$ such that the y_{i_j}'s are distinct and their union is Y_1. Also, let H_{Y_1} be the linear subspace of $\mathbb{C}^n$ generated by $\{e_{i_j}\}$ and let f in $C(X)$ be such that f is 1 on Y_1. The definition of π implies that $\pi(f)$ is the orthogonal projection P onto H_{Y_1} and moreover since Y_1 is invariant $\pi(\alpha_n(f)) = \pi(f)$. Therefore the covariant condition implies now that P commutes with W^j for all j whence H_{Y_1} is a reducing subspace for W. Since it is also a reducing subspace for $\pi(C(X))$ it follows that it is a reducing subspace for $(\pi \times W)(B_T)$ and as a result $H_{Y_1} = \mathbb{C}^n$. We may conclude that $Y = Y_1$ or in other words Y is an orbit and there is no duplication among the y_i's.

We summarize the previous discussion in the following

PROPOSITION (1.2). *The $\rho_{Y,\lambda}$'s describe, up to equivalence of representations, all the irreducible n-dimensional representations of B_T.*

In the next proposition we find a necessary and sufficient condition for two representations of the form $\rho_{Y,\lambda}$, Y is fixed, to be equivalent. Note that the previous discussion let us identify the representation space with $l^2(Y)$.

PROPOSITION (1.3). *Let $\rho_{Y,\lambda}$ and $\rho_{Y,\mu}$, where $\lambda = \{\lambda_y\}$ and $\mu = \{\mu_y\}$, be irreducible n-dimensional representations of B_T. Then, $\rho_{Y,\lambda}$ is equivalent to $\rho_{Y,\mu}$ if and only if $\prod_{y\in Y} \lambda_y = \prod_{y \in Y} \mu_y$.*

Proof. First assume that $\rho_{Y,\lambda}$ is equivalent to $\rho_{Y,\mu}$. Let U be the unitary in B_T induced by T and let $\{e_y\}$ be the natural basis in $l^2(Y)$, the representation space. Since $T^n y = y$, the definition of $\rho_{Y,\lambda}$ implies that

$$\rho_{Y,\lambda}(U^n)e_y = \lambda_y \lambda_{Ty} \cdots \lambda_{T^{n-1}y} e_y = \left(\prod_{y \in Y} \lambda_y\right) e_y.$$

What follows is that $\rho_{Y,\lambda}(U^n) = (\prod_{y\in Y} \lambda_y)I$. Hence, $U^n - (\prod_{y\in Y} \lambda_y)I$ is in $\ker(\rho_{Y,\lambda})$. Since we assumed that $\rho_{Y,\lambda}$ is equivalent to $\rho_{Y,\mu}$ it follows that $U^n - (\prod_{y\in Y} \lambda_y)I$ is also in $\ker(\rho_{Y,\mu})$. But the above calculation also shows that $\rho_{Y,\mu}(U^n) = (\prod_{y\in Y} \mu_y)I$ whence the first half of the proposition follows. Conversely, assume that λ and μ satisfy $\prod_{y\in Y} \lambda_y = \prod_{y\in Y} \mu_y$. We need to find a unitary W in $B(l^2(Y))$ such that $W\rho_{Y,\lambda}W^{-1} = \rho_{Y,\mu}$. Fix some y in Y. We then define W in the following way. We let $We_{T^i y} = \alpha_{T^i y} e_{T^i y}$, for $0 \le i \le n-1$, where $\alpha_y = 1$ and for $1 \le i \le n-1$,

$$\alpha_{T^i y} = \prod_{j=0}^{i-1} \mu_{T^j y} \prod_{j=0}^{i-1} \lambda_{T^j y}^{-1}.$$

First note that if f is in $C(X)$ then $W\rho_{Y,\lambda}(f)W^{-1} = \rho_{Y,\mu}(f)$. Therefore, since B_T is generated by U and $C(X)$ it follows that in order to show that $W\rho_{Y,\lambda}W^{-1} = \rho_{Y,\lambda}$ it is enough now to prove that for $0 \le i \le n-1$

$$W\rho_{Y,\lambda}(U)W^{-1}e_{T^i y} = \rho_{Y,\mu}(U)e_{T^i y} = \mu_{T^i y} e_{T^{i+1} y}.$$

Check the case $i = 0$:

$$W\rho_{Y,\lambda}(U)W^{-1}e_y = W\rho_{Y,\lambda}(U)e_y = W\lambda_y e_{Ty} = \lambda_y \mu_y \lambda_y^{-1} e_{Ty} = \mu_y e_{Ty}.$$

Check the case $0 < i < n-1$:

$$W\rho_{Y,\lambda}(U)W^{-1}e_{T^i y} = W\rho_{Y,\lambda}(U)\left(\prod_{j=0}^{i-1} \mu_{T^j y}^{-1} \prod_{j=0}^{i-1} \lambda_{T^j y}\right) e_{T^i y}$$

$$= W(\lambda_{T^i y})\left(\prod_{j=0}^{i-1} \mu_{T^j y}^{-1} \prod_{j=0}^{i-1} \lambda_{T^j y}\right) e_{T^{i+1} y}$$

$$= \left(\prod_{j=0}^{i} \mu_{T^j y} \prod_{j=0}^{i} \lambda_{T^j y}^{-1}\right)(\lambda_{T^i y})\left(\prod_{j=0}^{i-1} \mu_{T^j y}^{-1} \prod_{j=0}^{i-1} \lambda_{T^j y}\right) e_{T^{i+1} y}$$

$$= \mu_{T^i y} e_{T^{i+1} y}.$$

Check the case $i = n - 1$:

$$\begin{aligned} W\rho_{Y,\lambda}(U)W^{-1}e_{T^{n-1}y} &= W\rho_{Y,\lambda}(U)\left(\prod_{j=0}^{n-2}\mu_{T^jy}^{-1}\prod_{j=0}^{n-2}\lambda_{T^jy}\right)e_{T^{n-1}y} \\ &= W(\lambda_{T^{n-1}y})\left(\prod_{j=0}^{n-2}\mu_{T^jy}^{-1}\prod_{j=0}^{n-2}\lambda_{T^jy}\right)e_y \\ &= (\lambda_{T^{n-1}y})\left(\prod_{j=0}^{n-2}\mu_{T^jy}^{-1}\prod_{j=0}^{n-2}\lambda_{T^jy}\right)e_y \\ &= \left(\prod_{j=0}^{n-2}\mu_{T^jy}^{-1}\prod_{j=0}^{n-1}\lambda_{T^jy}\right)e_y = \mu_{T^{n-1}y}e_y. \end{aligned}$$

The last equality follows from the hypothesis that $\prod_{y\in Y}\lambda_y = \prod_{y\in Y}\mu_y$. □

2. The structure of $\mathrm{Prim}_n(B_T)$. In this section we use the description of irreducible representations of B_T to study the structure of $\mathrm{Prim}_n(B_T)$. The number of connected components of $\mathrm{Prim}_n(B_T)$ is proven to be equal to the number of orbits of length n.

Let ρ be a finite dimensional irreducible representation of B_T.

NOTATION. We denote by ρ_λ the composition $\rho\cdot\hat{\alpha}_\lambda$ where $\hat{\alpha}$ is the dual action.

LEMMA (2.1). *For any λ in $\mathbf{T}$, $\mu = \{\mu_y\}$ and Y a finite invariant set of T,*

$$(\rho_{Y,\mu})_\lambda = \rho_{Y,\lambda\mu}.$$

Proof. For any f in $C(X)$, $(\rho_{Y,\mu})_\lambda(f) = \rho_{Y,\lambda\mu}(f)$; therefore we only need to check that $(\rho_{Y,\mu})_\lambda(U) = \rho_{Y,\lambda\mu}(U)$. Let $\{e_y\}$ be the natural orthonormal basis in $l^2(Y)$. Then for any y in Y,

$$(\rho_{Y,\mu})_\lambda(U)e_y = \rho_{Y,\mu}(\lambda U)e_y = \lambda\rho_{Y,\mu}(U)e_y = \lambda\mu_y e_{Ty} = \rho_{Y,\lambda\mu}(U). \quad \square$$

PROPOSITION (2.2). *Let $\rho = \rho_{Y,\lambda}$ be an n-dimensional irreducible representation of B_T. Then,*

$$J_Y = \bigcap_{\lambda\in\mathbf{T}}\ker(\rho_\lambda).$$

Proof. Assume that $\rho = \pi \times W$. Let $C = \sum f_n U^n$ be in J_Y. By Lemma (1.1) the f_n's are 0 on Y and hence the $\pi(f_n)$'s are all 0. We noted in the preliminaries that

$$\sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) f_k U^k \lambda^k \to \hat{\alpha}_\lambda(C)$$

uniformly in λ. As a result,

$$\sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) \pi(f_k) W^k \lambda^k \to \rho \cdot \hat{\alpha}_\lambda(C) = \rho_\lambda(C)$$

for all λ in $\mathbf{T}$ and therefore C is in $\bigcap_\lambda \ker(\rho_\lambda)$.

Conversely, let $C = \sum f_n U^n$ be in $\bigcap_\lambda \ker(\rho_\lambda)$. By Lemma (1.1) we need to show that f_n is 0 on Y for all n. Let $\{C_k\} \subseteq K(\mathbb{Z}, C(X))$ be such that $C_k \to C$. Since $\rho_\lambda(C_k) \to \rho_\lambda(C)$ uniformly in λ it follows by our hypothesis that $\rho_\lambda(C_k) \to 0$ uniformly. Therefore for all ξ, η in $l^2(Y)$, $(\rho_\lambda(C_k)\xi, \eta) \to 0$ uniformly in λ. Let $\xi = e_y$ and $\eta = e_{y'}$. Assume that for all k, $\sum a_n^k \lambda^n$ is the Fourier expansion of $\lambda \to (\rho_\lambda(C_k)e_y, e_{y'})$. Then, $a_n^k \to 0$ for all n. Let $C_k = \sum f_n^k U^n$ for all k. Then, $a_n^k = (\pi(f_n^k)W^n e_y, e_{y'})$. Since $f_n^k \to f_n$ for all n, it follows that $(\pi(f_n)W^n e_y, e_{y'}) = 0$. But $W^n e_y = \delta e_{T^n y}$, for some δ of absolute value 1. Therefore what we have shown is that for all n and for all y, y' in Y, $(e_{T^n y}, \overline{f_n(y')} e_{y'}) = 0$. In particular if we pick $y = T^{-n} y'$ we get that $f_n(y') = 0$. Since n in y' are arbitrary it follows that f_n is 0 on Y for all n. □

Let $\{Y_i\}_{i \in I}$ be the set of all orbits of length n with respect to T. Assume that I is finite.

NOTATION. Let $F_{Y_i} = \{R \in \mathrm{Prim}_\pi(B_T); R \supseteq J_{Y_i}\}$.

By definition, F_{Y_i} is closed in $\mathrm{Prim}_n(B_T)$. Also by Proposition (2.2) each R in $\mathrm{Prim}_n(B_T)$ is in one of the F_{Y_i}'s. Since the Y_i's are mutually exclusive it follows that the F_{Y_i}'s are too. Consequently the F_{Y_i}'s are open and closed in $\mathrm{Prim}_n(B_T)$.

Finally, we show that if $\{\ker(\rho)\} \in F_{Y_i}$, then the connected component of $\{\ker(\rho)\}$ includes F_{Y_i}. Fix ρ such that $\{\ker(\rho)\} \in F_{Y_i}$. Now, the function $\lambda \to \{\ker(\rho_\lambda)\}$ is continuous with respect to the Jacobson topology on $\mathrm{Prim}_n(B_T)$. *Reason*: $\rho_\lambda = \rho \cdot \hat{\alpha}_\lambda$ and $\hat{\alpha}_\lambda$ is continuous with respect to the pointwise topology. Therefore, $\lambda \to \{\ker(\rho_\lambda)\}$ is a continuous function from $\mathbf{T}$ to $\mathrm{Prim}_n(B_T)$.

Conclusion. The connected component of $\{\ker(\rho)\}$ includes the set

$$\left\{R \in \operatorname{Prim}_n(B_T); R \supseteq \bigcap_\lambda \ker(\rho_\lambda)\right\}.$$

But by Proposition (2.2), $\bigcap_\lambda \ker(\rho_\lambda) = J_{Y_i}$ and therefore the connected component of $\{\ker(\rho)\}$ includes F_{Y_i}. Since the F_{Y_i}'s are open and closed it follows that the connected component of $\{\ker(\rho)\}$ is exactly F_{Y_i}.

We summarize the above discussion in the following theorem.

NOTATION. For any homeomorphism T we denote by $O(T)$ the set of all finite orbits of T.

THEOREM (2.3). *Let Θ: $B_T \to B_S$ be an isomorphism. Let Y be a finite orbit with respect to T. Then, $\Theta(J_Y) = J_Z$ for some Z a finite orbit with respect to S having the same cardinality as Y. The correspondence $Y \to Z$ defines a set theoretic isomorphism Θ' between $O(T)$ and $O(S)$. Moreover, $(\Theta')^{-1} = (\Theta^{-1})'$. Note that T and S may act on different spaces.*

Proof. We know that the map $\operatorname{Prim}_n(\Theta)$: $\operatorname{Prim}_n(B_T) \to \operatorname{Prim}_n(B_S)$, defined by $\{\ker(\rho)\} \to \{\ker(\rho \cdot \Theta^{-1})\}$ is a homeomorphism. Therefore, the image of F_Y under $\operatorname{Prim}(\Theta)$ must be equal to some F_Z where Z is a finite orbit of S having the same cardinality as Y. Now, $\Theta(J_Y) = J_Z$ because $\Theta(\ker(\rho)) = \ker(\rho \cdot \Theta^{-1})$ and $\bigcap_{\{R; R \in \operatorname{Prim}_n(B_T), R \supseteq J_Y\}} R = J_Y$. Finally, Θ' is a set theoretic isomorphism because $\operatorname{Prim}(\Theta)$ is a homeomorphism. □

THEOREM (2.4). *Let ρ be an irreducible n-dimensional representation of B_T. Assume that T has finitely many orbits of length n. Then the connected component of $\{\ker(\rho)\}$ in $\operatorname{Prim}_n(B_T)$ is equal to*

$$\{\ker(\rho_\lambda); 0 \leq \arg(\lambda) < 2\pi/n\}.$$

The number of connected components in $\operatorname{Prim}_n(B_T)$ is equal to the number of orbits of length n.

Proof. The only part that was not proven is that the component of $\{\ker(\rho)\}$ in $\operatorname{Prim}_n(B_T)$ is equal to $\{\ker(\rho_\lambda); 0 \leq \arg(\lambda) < 2\pi/n\}$. By Proposition (1.2) we know that ρ is equivalent to some $\rho_{Y,\mu}$, where Y is an orbit of length n and $\mu = \{\mu_y\}$, and the discussion preceding Theorem (2.3) shows that the connected component of ρ is equal to

$F_Y = \{R \in \mathrm{Prim}_n(B_T); R \supseteq J_{Y,}\}$. Therefore, what is left to show is that for any $\nu = \{\nu_y\}$, $\ker(\rho_{Y,\nu})$ is equal to $\ker(\rho_\lambda)$ for some λ such that $0 \leq \arg(\lambda) < 2\pi/n$. By Proposition (1.3) the class of $\rho_{Y,\nu}$ is dependent only on $\prod_{y\in Y} \nu_y$ and by Lemma (2.1) $\rho_\lambda = (\rho_{Y,\mu})_\lambda = \rho_{Y,\lambda\mu}$. Therefore we are done because for $\{\lambda; 0 \leq \arg(\lambda) < 2\pi/n\}$ the range of the function $\lambda \to \prod_{y\in Y} \lambda\mu_y$ is $\mathbf{T}$. □

3. Partial classification of hyperbolic crossed-product algebras. We now specialize to the case $X = \mathbf{T}^m$ and T an automorphism on $\mathbf{T}^m$.

NOTATION. Denote by $N_p(T)$ the cardinality of the set $\{x \in X; T^p x = x\}$ and by $O_p(T)$ the cardinality of the set of all periodic points of period equal to p.

DEFINITION. An automorphism T is called hyperbolic if it has no eigenvalue of unit modulus.

THEOREM (3.1). *A partial classification of the B_T's. Let T and S be hyperbolic automorphisms on tori not necessarily of the same dimensions. If the algebra B_T is isomorphic to B_S, then for all $p \geq 1$, $N_p(T) = N_p(S)$. In particular, T and S must have the same entropy.*

Proof. If $\Theta: B_T \to B_S$ is an isomorphism then it induces a homeomorphism between $\mathrm{Prim}_n(B_T)$ and $\mathrm{Prim}_n(B_S)$ for $n \geq 1$. Since the number of connected components is a topological invariant it must be the same for $\mathrm{Prim}_n(B_T)$ and $\mathrm{Prim}_n(B_S)$. On the other hand we know that the number of connected components in $\mathrm{Prim}_n(B_T)$ is equal to the number of orbits of length n. Therefore, $O_n(S) = O_n(T)$. Note that $N_n(T)$ is not quite the number of periodic points of period n because it includes all points of period m for m which divides n. But $N_n(T)$ can be recovered from the $O_m(T)$'s simply because

$$N_n(T) = \sum_{\{m\geq 1; m|n\}} O_m(T).$$

Let $\sigma(T) = \{\lambda_1, \dots, \lambda_k\}$ be the spectrum of T and $\sigma(S) = \{\mu_1, \dots, \mu_l\}$ be the spectrum of S. Recall that $N_p(T) = |\det(T^p - I)|$, [4]. By the above discussion we know that if B_T is isomorphic to B_S then for all p, $|\det(T^p - I)| = |\det(S^p - I)|$. Now,

$$|\det(T^p - I)| = \prod_{i=1}^{k} |\lambda_i^p - 1|.$$

Fix some $\varepsilon > 0$. Note that we can make the following estimations. If $\lambda \in \sigma(T)$ and $|\lambda| > 1$ then for p sufficiently large,

$$(1-\varepsilon)|\lambda|^p \leq |\lambda^p - 1| \leq (1+\varepsilon)|\lambda|^p$$

and if $\lambda \in \sigma(T)$ and $|\lambda| < 1$ then for p sufficiently large,

$$(1-\varepsilon) \leq |\lambda^p - 1| \leq (1+\varepsilon).$$

Denote by Λ the quantity $\prod_{\{i;|\lambda_i|>1\}} |\lambda_i|$. By the above estimation,

$$(1-\varepsilon)^k \Lambda^p \leq N_p(T) \leq (1+\varepsilon)^k \Lambda^p.$$

Repeating the above calculation for S we get that for any fixed $\varepsilon' > 0$ and for p sufficiently large

$$(1-\varepsilon')^l M^p \leq N_p(S) \leq (1+\varepsilon')^l M^p$$

where M is analogous to Λ. Claim: Λ must be equal to M. Proof: Assume without loss of generality that $\Lambda < M$. Then for any positive $\varepsilon, \varepsilon'$

$$(1+\varepsilon)^k \Lambda^p < (1-\varepsilon')^l M^p$$

for p sufficiently large. The last inequality implies that $N_p(T) < N_p(S)$—contradiction. We have completed the proof since the entropy of an automorphism T is equal to $\log(\Lambda)$, [**4**]. □

What can be now deduced about the classification of the crossed-product algebras over the 2-torus. Note that if T is an automorphism on the 2-torus then the equation for its characteristic polynomial, regarded as a linear transformation on the plane, is

$$x^2 - \text{trace}(T)x + \det(T) = 0.$$

From this relation we deduce that if T and S have the same trace and determinant then they have the same eigenvalues and conversely.

In the last section we showed that the entropy of T is an invariant of B_T. Since the product of the eigenvalues of T is 1 in absolute value it follows that if $B_T \cong B_S$ then

$$\{|\lambda|; \lambda \in \sigma(T)\} = \{|\mu|; \mu \in \sigma(T)\}.$$

Let us make the following notations. Let $\delta = \det(T)$, $\delta' = \det(S)$, $\tau = \text{trace}(T)$ and $\tau' = \text{trace}(S)$. Since the eigenvalues of T and S are real we now have that

$$\frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2} = \pm \frac{\tau' \pm \sqrt{\tau'^2 - 4\delta'}}{2}.$$

Claim. The above equation for the eigenvalues implies that $|\tau| = |\tau'|$ and $\tau^2 - 4\delta = (\tau')^2 - 4\delta'$. Therefore also $\delta = \delta'$. Proof: Recall that the eigenvalues of hyperbolic automorphisms are irrational, [**5**]. In general, if k, l, m, n are integers and $m + \sqrt{n}$, $k + \sqrt{l}$ are irrational numbers satisfying $m + \sqrt{n} = k + \sqrt{l}$ then $m = k$ and $n = l$. Reason: $\sqrt{n} = (k-m)+\sqrt{l}$ and therefore $n = (k-m)^2+l+2(k-m)\sqrt{l}$. If $k \neq m$ then $\sqrt{l}$ is rational whence $k + \sqrt{l}$ is also rational—contradiction.

Can we furthermore deduce that $\operatorname{trace}(T) = \operatorname{trace}(S)$? From the last section we know that $|\det(T^n - I)| = |\det(S^n - I)|$ for all $n \geq 1$. Observe that

$$|\det(T - I)| = |\det(T) + 1 - \operatorname{trace}(T)|.$$

Therefore if $\det(T) = \det(S) = 1$ then $|2 - \tau| = |2 - \tau'|$. Since $|\tau| = |\tau'|$ it follows that $\tau = \tau'$.

We may summarize the above discussion in the following

COROLLARY (3.2). *Let T and S be hyperbolic automorphisms on the 2-torus. If $B_T \cong B_S$ then:*
(i) $\det(T) = \det(S)$,
(ii) $|\operatorname{trace}(T)| = |\operatorname{trace}(S)|$.
If $\det(T)$ *or* $\det(S)$ *is equal to* 1 *then*
(iii) $\operatorname{trace}(T) = \operatorname{trace}(S)$.

REMARKS. In the case $\det(T) = \det(S) = -1$ it is not true that $B_T \cong B_S$ implies that $\operatorname{trace}(T) = \operatorname{trace}(S)$. Example: Let T be a hyperbolic automorphism having determinant -1. Let $S = T^{-1}$. Note that $\operatorname{trace}(S) = -\operatorname{trace}(T)$ but $B_T \cong B_{T^{-1}} = B_S$.

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