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# THE ISOMORPHISM PROBLEM FOR ORTHODOX SEMIGROUPS

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**The author's structure theorem for orthodox semigroups produced an orthodox semigroup $\mathscr{H}(E, T, \psi)$ from a band $E$, an inverse semigroup $T$ and a morphism $\psi$ between two inverse semigroups, namely $T$ and $W_E/\gamma$, an inverse semigroup constructed from $E$. Here, we solve the isomorphism problem: when are two such orthodox semigroups isomorphic? This leads to a way of producing all orthodox semigroups, up to isomorphism, with prescribed band $E$ and maximum inverse semigroup morphic image $T$.**

**1. Preliminaries.** A semigroup $S$ is called *regular* (in the sense of von Neumann for rings) if for each $a \in S$ there exists $x \in S$ such that $axa = a$; and $S$ is called an *inverse semigroup* if for each $a \in S$ there is a unique $x \in S$ such that $axa = a$ and $xax = x$. A *band* is a semigroup in which each element is idempotent, and an *orthodox semigroup* is a regular semigroup in which the idempotents form a subsemigroup (that is, a band).

We follow the notation and conventions of Howie [**4**].

*Result* 1 [**3**, *Theorem* 5]. *The maximum congruence contained in Green's relation* $\mathscr{H}$ *on any regular semigroup* $S$, $\mu = \mu(S)$ *say, is given by* $\mu = \{(a,b) \in \mathscr{H}$: *for some* [*for each pair of*] $\mathscr{H}$*-related inverses* $a'$ *of* $a$ *and* $b'$ *of* $b$, $a'ea = b'eb$ *for each idempotent* $e \leq aa'\}$.

A regular semigroup $S$ is called *fundamental* if $\mu$ is the identity relation on $S$. For each band $E$, the semigroup $W_E$ is fundamental, orthodox, has its band isomorphic to $E$, and contains, for each orthodox semigroup $S$ with band $E$, a copy of $S/\mu$ as a subsemigroup: see the author [**1**] (or [**3**] with $E = \langle E \rangle$ and $W_E = T_{\langle E \rangle}$) or Howie [**4**, §VI.2].

Now take any inverse semigroup $T$, and, if such exist, any idempotent-separating morphism $\psi: T \to W_E/\gamma$ whose range contains the semilattice of all idempotents of $W_E/\gamma$, where $\gamma$ denotes the least inverse semigroup congruence on $W_E$. A semigroup $\mathscr{H}(E, T, \psi)$ (see

$S(E, T, \psi)$ in the author [**2**], or see Howie [**4**, §VI.4]) is defined by

$$\mathscr{H}(E, T, \psi) = \{(w, t) \in W_E \times T : w\gamma^\natural = t\psi\};$$

that is, $\mathscr{H}(E, T, \psi)$ occurs in the pullback diagram

$$\begin{array}{ccc} \mathscr{H}(E, T, \psi) & \xrightarrow{p_2} & T \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \psi} \\ W_E & \xrightarrow[\gamma^\natural]{} & W_E/\gamma \end{array} .$$

Here, $p_1$ and $p_2$ are projections.

The semigroup $\mathscr{H}(E, T, \psi)$ is orthodox, has band isomorphic to $E$, and has its maximum inverse semigroup morphic image isomorphic to $T$; conversely every such semigroup is obtained in this way (the author [**2**], or Howie [**4**, §VI.4]).

## 2. The isomorphism problem.

LEMMA 1. *Take any two morphisms $\varphi$, $\psi$ from a regular semigroup $T$ to a regular semigroup $S$ such that the range of each of $\varphi$ and $\psi$ contains the set $E(S)$ of all the idempotents of $S$. If $\varphi|E(T) = \psi|E(T)$ then $(t\varphi, t\psi) \in \mu$, for all $t \in T$; in particular, if also $S$ is fundamental, then $\varphi = \psi$.*

*Proof*. Take any $t \in T$ and any inverse $t'$ of $t$ in $T$. Of course, in $S$, $t'\varphi$ and $t'\psi$ are inverses of $t\varphi$ and $t\psi$ respectively and $(t'\varphi)(t\varphi) = (t't)\varphi = (t't)\psi = (t'\psi)(t\psi)$. Likewise $(t\varphi)(t'\varphi) = (t\psi)(t'\psi)$, so $(t\varphi)\mathscr{H}(t\psi)$ and $(t'\varphi)\mathscr{H}(t'\psi)$. Take any idempotent $e$ of $S$ such that $e \leq (tt')\varphi$ and any $x \in T$ such that $x\varphi = e$: then $(tt'xtt')\varphi = [(tt')\varphi]e[(tt')\varphi] = e$, so $e \in \text{range}(\varphi|tt'Ttt')$. Now $tt'Ttt'$ is a regular semigroup, so by Lallement's Lemma [**4**, Lemma II.4.7] there is an idempotent $f \in tt'Ttt'$ such that $f\varphi = e$. Since $t'ft$ is idempotent, we have

$$\begin{aligned}(t'\varphi)e(t\varphi) &= (t'\varphi)(f\varphi)(t\varphi) = (t'ft)\varphi = (t'ft)\psi \\ &= (t'\psi)(f\psi)(t\psi) = (t'\psi)e(t\psi).\end{aligned}$$

Thus $(t\varphi, t\psi) \in \mu$, as required, completing the proof.

Take any isomorphism $\alpha : E \to E'$ from a band $E$ to a band $E'$. Consider $W_E$ and $W_{E'}$ and, as usual, identify $E$ and $E'$ with the bands of

$W_E$ and $W_{E'}$ respectively. Since $W_{E'}$ is constructed from $E'$ precisely as $W_E$ is constructed from $E$, there is an isomorphism from $W_E$ to $W_{E'}$ extending $\alpha$, say $\alpha^*$ (in fact, there is a unique such isomorphism, by Lemma 1). Denote by $\gamma$ and $\gamma'$ the least inverse semigroup congruences on $W_E$ and $W_{E'}$ respectively: then the map $\alpha^{**}: W_E/\gamma \to W_{E'}/\gamma'$, given by $w\gamma\alpha^{**} = w\alpha^*\gamma'$, for all $w \in W_E$, is an isomorphism such that $\gamma^\natural\alpha^{**} = \alpha^*\gamma'^\natural$, and is the unique such isomorphism. Summarizing, we have that the following diagram commutes, and $\alpha^*, \alpha^{**}$ are the unique morphisms making the diagram commute.

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E' \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \subseteq} \\ W_E & \xrightarrow{\alpha^*} & W_{E'} \\ \downarrow{\scriptstyle \gamma^\natural} & & \downarrow{\scriptstyle \gamma'^\natural} \\ W_E/\gamma & \xrightarrow{\alpha^{**}} & W_{E'}/\gamma' \end{array}$$

THEOREM 2. *Take any bands $E$, $E'$, inverse semigroups $T$, $T'$ and idempotent-separating morphisms $\psi: T \to W_E/\gamma$ and $\psi': T' \to W_{E'}/\gamma'$ whose ranges contain the idempotents of $W_E/\gamma$ and $W_{E'}/\gamma'$ respectively. Then $\mathscr{H}(E, T, \psi)$ is isomorphic to $\mathscr{H}(E', T', \psi')$ if and only if there exist isomorphisms $\alpha: E \to E'$ and $\beta: T \to T'$ such that the following diagram commutes*

$$\begin{array}{ccc} T & \xrightarrow{\beta} & T' \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi'} \;; \\ W_E/\gamma & \xrightarrow{\alpha^{**}} & W_{E'}/\gamma' \end{array}$$

*that is, such that $\psi' = \beta^{-1}\psi\alpha^{**}$.*

*Proof*. (a) *if* statement. Suppose such $\alpha$, $\beta$ exist. Informally we could say that $E'$, $T'$, $\psi'$ are a renaming of $E$, $T$, $\psi$ respectively, obtained by renaming each $e \in E$ by $e\alpha$ and each $t \in T$ by $t\beta$, and so $\mathscr{H}(E', T', \psi')$ is isomorphic to $\mathscr{H}(E, T, \psi)$. More formally, we consider the isomorphism $(\alpha^*, \beta): W_E \times T \to W_{E'} \times T'$ given by $(w, t)(\alpha^*, \beta) = (w\alpha^*, t\beta)$ for all $(w, t) \in W_E \times T$, and we show that $\mathscr{H}(E, T, \psi)(\alpha^*, \beta) = \mathscr{H}(E', T', \psi')$.

Take any $(w, t) \in \mathscr{H}(E, T, \psi)$: then $w\gamma^\natural = t\psi$, and so

$$t\beta\psi' = t\beta\beta^{-1}\psi\alpha^{**} = t\psi\alpha^{**} = w\gamma^\natural\alpha^{**} = w\alpha^*\gamma'^\natural,$$

so $(w, t)(\alpha^*, \beta) = (w\alpha^*, t\beta) \in \mathscr{H}(E', T', \psi')$ and hence $\mathscr{H}(E, T, \psi)(\alpha^*, \beta) \subseteq \mathscr{H}(E', T', \psi')$.

From symmetry, we deduce that

$$\mathscr{H}(E', T', \psi')(\alpha^*, \beta)^{-1} = \mathscr{H}(E', T', \psi')(\alpha^{*-1}, \beta^{-1}) \subseteq \mathscr{H}(E, T, \psi),$$

whence $\mathscr{H}(E, T, \psi)(\alpha^*, \beta) = \mathscr{H}(E', T', \psi')$, as required.

(b) *only if* statement. Informally, we could say that, for any orthodox semigroup $S$ with band $E$ and least inverse semigroup congruence $\mathscr{Y}$, there is a unique morphism $\psi$ making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{(\rho,\lambda)} & W_E \\ {\scriptstyle \mathscr{Y}^\natural}\downarrow & & \downarrow{\scriptstyle \gamma^\natural} \\ S/\mathscr{Y} & \xrightarrow{\psi} & W_E/\gamma \end{array} .$$

Hence $E$, $S/\mathscr{Y}$, $\psi$ are all determined to within isomorphisms (or renamings) by $S$. Formally, we proceed as follows.

Take any isomorphism $\theta: S \to S'$, where $S = \mathscr{H}(E, T, \psi)$ and $S' = \mathscr{H}(E', T', \psi')$. Put $\theta|E = \alpha$, an isomorphism of $E$ upon $E'$, by Lallement's Lemma [**4**, Lemma II.4.7]. Let $\mathscr{Y}$ and $\mathscr{Y}'$ denote the least inverse semigroup congruences on $S$ and $S'$ respectively. Clearly there is a unique isomorphism $\beta: S/\mathscr{Y} \to S'/\mathscr{Y}'$ making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{\theta} & S' \\ {\scriptstyle \mathscr{Y}^\natural}\downarrow & & \downarrow{\scriptstyle \mathscr{Y}'^\natural} \\ S/\mathscr{Y} & \xrightarrow{\beta} & S'/\mathscr{Y}' \end{array} .$$

Now $T \cong S/\mathscr{Y}$ and $T' \cong S'/\mathscr{Y}'$ ([**2**, Theorem 1] or [**4**, Theorem VI.4.6]), so we assume without loss of generality that $T = S/\mathscr{Y}$ and $T' = S'/\mathscr{Y}'$; it remains to show that $\psi' = \beta^{-1}\psi\alpha^{**}$.

We shall see that Diagram 1 commutes ($p_1$, $p_2$, $p_1'$, $p_2'$ are projections).

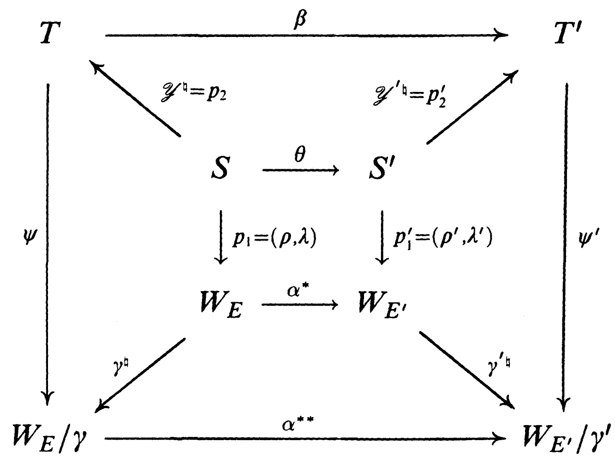
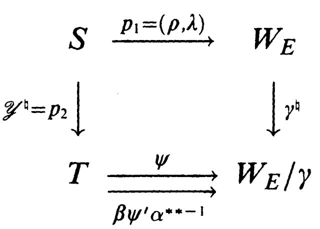



DIAGRAM 1

We have seen already that each of the four outer faces is a commuting diagram: we consider the central face. Now $\theta p_1'$ and $p_1\alpha^*$ are morphisms which agree on $E$ (with $\alpha = \theta|E$), and which map $E$ (isomorphically) onto $E'$, the band of $W_{E'}$. Hence, by Lemma 1, $\theta p_1' = p_1\alpha^*$; that is, the central face commutes.

Consideration of the external face leads us to the following diagram.




The commuting of the five internal faces of Diagram 1 gives us that $p_1\gamma^\natural = p_2\beta\psi'\alpha^{**-1}$. But the mapping $s\mathscr{Y} \mapsto (\rho_S, \lambda_S)\gamma$ (for all $s \in S$), namely $\psi$, is the unique morphism from $T$ to $W_E/\gamma$ making this diagram commute, and hence $\psi = \beta\psi'\alpha^{**-1}$ (that is, the external face commutes) and so $\psi' = \beta^{-1}\psi\alpha^{**}$ as required.

**3. Orthodox semigroups, up to isomorphism.** Consider the following problem: given a band $E$ and an inverse semigroup $T$, find, up to isomorphism, the orthodox semigroups with band $E$ and with maximum inverse semigroup morphic image isomorphic to $T$.

The author's structure theorem ([**2**, Theorem 1] or [**4**, Theorem VI.4.6]) and Theorem 2 above together immediately yield a solution as follows.

Denote by $\text{Aut}(S)$ the group of automorphisms of any semigroup $S$. From Lemma 1, for any $\varphi \in \text{Aut}(W_E)$, we see that $\varphi = (\varphi|E)^*$, so we have that $\text{Aut}(E) \cong \text{Aut}(W_E)$ under the map $\alpha \mapsto \alpha^*$ for each $\alpha \in \text{Aut}(E)$. The map $\text{Aut}(W_E) \to \text{Aut}(W_E/\gamma), \alpha^* \mapsto \alpha^{**}$ (for each $\alpha \in \text{Aut}(E)$), is a morphism; we denote its image by $[\text{Aut}(E)]^{**}$; then $[\text{Aut}(E)]^{**} = \{\alpha^{**} : \alpha \in \text{Aut}(E)\}$.

Denote by $M$ the set of idempotent-separating morphisms from $T$ into $W_E/\gamma$ whose ranges each contain the idempotents of $W_E/\gamma$. By [**2**, Corollary 1] or [**4**, Theorem VI.4.6], there exists an orthodox semigroup with band $E$ and with maximum inverse semigroup morphic image isomorphic to $T$, if and only if $M$ is nonempty. Assume henceforth that $M$ is nonempty. Define an action on $M$ by the group $\text{Aut}(T) \times [\text{Aut}(E)]^{**}$ as follows:

$$\psi(\beta, \alpha^{**}) = \beta^{-1}\psi\alpha^{**},$$

for all $\psi \in M, \beta \in \text{Aut}(T), \alpha \in \text{Aut}(E)$.

The orbits of $M$ under $\text{Aut}(T) \times [\text{Aut}(E)]^{**}$ are the sets

$$\psi(\text{Aut}(T) \times [\text{Aut}(E)]^{**}) = \{\beta^{-1}\psi\alpha^{**} : \beta \in \text{Aut}(T), \alpha \in \text{Aut}(E)\},$$

for each $\psi \in M$ (thus these sets partition $M$). By Theorem 2, we have, for all $\psi, \psi' \in M$, $\mathscr{H}(E, T, \psi) \cong \mathscr{H}(E, T, \psi')$ if and only if $\psi$ and $\psi'$ are in the same orbit. Thus, if $\{\psi_i : i \in I\}$ is a transversal of the set of orbits (that is, a selection of precisely one morphism from each orbit) then $\mathscr{H}(E, T, \psi_i)$, $i \in I$, is a list of all the orthodox semigroups with band $E$ and maximum inverse semigroup morphic image isomorphic to $T$, and the semigroups are pairwise nonisomorphic.

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