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A REMARK ON SPINOR NORMS OF LOCAL INTEGRAL ROTATIONS I

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The spinor norms of the integral rotations on the modular quadratic forms over a local field, which could not be expressed in the convenient closed forms in [1], are expressed in a convenient closed form.

The spinor norms of integral rotations on a modular quadratic form over a local field were determined in [1], but there remained one case to solve. In the present paper, we will solve this problem. Familiarity of [1] and [2] is assumed, and we also adopt the notations of [1] and [2]. Thus, F denotes a dyadic local field of characteristic 0, $\mathscr{O}$ the ring of integers in F, $\mathscr{B} = \pi\mathscr{O}$ the maximal ideal of $\mathscr{O}$, $\mathscr{U}$ the group of units in $\mathscr{O}$, $\mathscr{D}(\cdot)$ the quadratic defect function, V a regular quadratic space of dimension 2 over F, L a unimodular lattice of determinant d on V, a the norm generator, $O^+(V)$ the group of rotations on V, $O^+(L)$ the corresponding subgroup of units of L, and $\theta(\cdot)$ the spinor norm function.

Write $L \cong A(a, -\delta a^{-1})$, adapted to a basis $\{x, y\}$, where $\mathscr{D}(1+\delta) = \delta\mathscr{O}$ and $-\delta a^{-1}$ belongs to $\mathfrak{w}L$. Put $\operatorname{ord}(a) = \nu$, $\operatorname{ord}(2) = e$, and $\mu = e - \nu$. We have the following proposition.

PROPOSITION. *If $e + [\mu/2] \geq \operatorname{ord}(\delta a^{-1}) > e$, then*

$$\theta(O^+(L)) = (1 + \mathscr{B}^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]})\dot{F}^2 \cap Q(\langle 1, d\rangle)\dot{F}^2$$

where $Q(\langle 1, d\rangle) = \{a \cdot c | c \in Q(\dot{V})\}$.

Proof. Take any symmetry S_z in $O(L)$ where z is a maximal anisotropic vector of L. Put $z = s \cdot x + t \cdot y$ where $s, t \in \mathscr{O}$ and one of them must be a unit. Since $\operatorname{ord}(Q(z)) = \operatorname{ord}(s^2 a + 2st - t^2 \cdot \delta \cdot a^{-1}) \leq e$, we obtain $0 \leq \operatorname{ord}(s) \leq [\mu/2]$ and $Q(z) = s^2 a \cdot (1 + 2s^{-1}ta^{-1} - (s^{-1}t)^2(\delta a^{-1})a^{-1})$.

If s is a unit,

$$\begin{aligned}\operatorname{ord}(2s^{-1}ta^{-1} - (s^{-1}t)^2(\delta a^{-1})a^{-1}) &= \operatorname{ord}(2s^{-1}ta^{-1}) \geq e - \nu \\ &= (\operatorname{ord}(\delta) + 2\mu - 2e - 2[\mu/2]) + (e + [\mu/2] + \nu - \operatorname{ord}(\delta)) + [\mu/2] \\ &\geq \operatorname{ord}(\delta) + 2\mu - 2e - 2[\mu/2].\end{aligned}$$

If t is a unit,

$$\begin{aligned}&\operatorname{ord}(2s^{-1}ta^{-1}-(s^{-1}t)^2(\delta a^{-1})a^{-1})\\&\qquad\geq\min(\mu-\operatorname{ord}(s),\operatorname{ord}(\delta)-2\nu-2\operatorname{ord}(s)).\end{aligned}$$

When $\operatorname{ord}(s)\geq\operatorname{ord}(\delta)+\mu-2e$,

$$\begin{aligned}&\min(\mu-\operatorname{ord}(s),\operatorname{ord}(\delta)-2\nu-2\operatorname{ord}(s))\\&\quad=\operatorname{ord}(\delta)-2\nu-2\operatorname{ord}(s)\\&\quad\geq\operatorname{ord}(\delta)-2\nu-2[\mu/2]\\&\quad=\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2].\end{aligned}$$

When $\operatorname{ord}(s)<\operatorname{ord}(\delta)+\mu-2e$,

$$\begin{aligned}&\min(\mu-\operatorname{ord}(s),\operatorname{ord}(\delta)-2\nu-2\operatorname{ord}(s))\\&\quad=\mu-\operatorname{ord}(s)>2e-\operatorname{ord}(\delta)\\&\quad=(\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2])+2(e+[\mu/2]+\nu-\operatorname{ord}(\delta))\\&\quad\geq\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2].\end{aligned}$$

By the theorem of [3], we obtain

$$\theta(O^+(L))\subseteq(1+\mathscr{B}^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]})\dot{F}^2.$$

It is obvious that V is anisotropic in this case, so $\theta(O^+(V))=Q<1$, $d>\dot{F}^2$. Hence, $\theta(O^+(L))\subseteq(1+\mathscr{B}^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]})\dot{F}\cap Q(\langle 1,d\rangle)\dot{F}^2$.

Take any $a\cdot h\dot{F}^2\subseteq(1+\mathscr{B}^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]})\dot{F}^2\cap Q(\langle 1,d\rangle)\dot{F}^2$ where $h\in Q(\dot{V})$, so there exists z in $\dot{V}$ such that $h=Q(z)$. Without loss of generality, we can assume that $z=s\cdot x+t\cdot y$ where $s,t\in\mathscr{O}$ and one of them must be a unit.

If $[\mu/2]<\operatorname{ord}(s)\leq\mu-e+(\operatorname{ord}(\delta)-1)/2$, then t is a unit. Since

$$\begin{aligned}\operatorname{ord}((st^{-1})\cdot(a\delta^{-1})\cdot 2)&=\operatorname{ord}(s)-\operatorname{ord}(\delta a^{-1})+e\\&\geq\operatorname{ord}(s)-[\mu/2]>0\end{aligned}$$

we know that $(s\cdot t^{-1})\cdot(a\delta^{-1})\cdot 2-1$ is a unit. Let $\operatorname{ord}(s)=m$, so

$$\begin{aligned}a\cdot h&=a(s^2a+2st-t^2(\delta a^{-1}))\\&=(as)^2\cdot(1+(s^{-1}t)^2(\delta a^{-1})\cdot(a^{-1})\cdot((st^{-1})\cdot(a\delta^{-1})\cdot 2-1))\end{aligned}$$

in $(1+\mathscr{B}^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]})\dot{F}^2$. Notice

$$\operatorname{ord}((s^{-1}t)^2(\delta a^{-1})\cdot(a^{-1}))=-2m+\operatorname{ord}(\delta)-2\nu.$$

We obtain the equation

$$1+w\pi^{\operatorname{ord}(\delta)-2m-2\nu}=f^2(1+r\pi^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]})\tag{1}$$

where $w \in \mathscr{U}$, $r \in \mathscr{O}$, $f \in \dot{F}$. Since

$$\operatorname{ord}(\delta) + 2\mu - 2e - 2[\mu/2] > \operatorname{ord}(\delta) - 2\nu - 2m \geq 1,$$

we can assume $f = 1 + q\pi^k$ where $q \in \mathscr{U}$, $k \geq 1$. So the following equation is yielded from (1).

$$(2)\quad w\pi^{\operatorname{ord}(\delta)-2\nu-2m} - r\pi^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]} - 2rq\pi^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]+k} - rq^2\pi^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]+2k} - 2q\pi^k = q^2\pi^{2k}.$$

Since

$$\begin{aligned}\operatorname{ord}(\delta) - 2\nu - 2m &\leq e + [\mu/2] + \nu - 2\nu - 2[\mu/2] \\ &= e - \nu - [\mu/2] \leq e < e + k = \operatorname{ord}(2q\pi^k)\end{aligned}$$

and $\operatorname{ord}(\delta) - 2\nu - 2m$ is odd, consider the orders of the elements at both sides of (2), a contradiction is derived.

If $\operatorname{ord}(s) \geq \mu - e + (\operatorname{ord}(\delta) + 1)/2$, then t is a unit again. Since

$$\operatorname{ord}(a^2(st^{-1})^2 \cdot \delta^{-1}) = 2\nu + 2\operatorname{ord}(s) - \operatorname{ord}(\delta) \geq 1$$

and

$$\begin{aligned}\operatorname{ord}(2(st^{-1})a\delta^{-1}) &= e + \operatorname{ord}(s) + \nu - \operatorname{ord}(\delta) \\ &\geq e + (1 - \operatorname{ord}(\delta))/2 \geq e + (1 - e - [\mu/2] - \nu)/2 \\ &= (e - [\mu/2] - \nu)/2 + 1/2 > 0\end{aligned}$$

we know $(1 - a^2(st^{-1})^2\delta^{-1} - 2(st^{-1})a\delta^{-1}) \in \mathscr{U}$. Notice

$$\begin{aligned}a \cdot h &= a(s^2a + 2st - t^2(\delta a^{-1})) \\ &= \delta(-t^2)(1 - a^2(st^{-1})^2\delta^{-1} - 2(st^{-1})a\delta^{-1})\end{aligned}$$

in $(1 + \mathscr{B}^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]})\dot{F}^2$, so we obtain the equation

$$(3)\qquad \delta \cdot \eta = \zeta \cdot f^2$$

where $\eta \in \mathscr{U}: f \in \dot{F}$, $\zeta \in (1 + \mathscr{B}^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]}) \subseteq \mathscr{U}$. Since $\operatorname{ord}(\delta)$ is odd, consider the orders of the elements at both sides of (3), a contradiction is derived.

Now the only possibility is $0 \leq \operatorname{ord}(s) \leq [\mu/2]$, so

$$\operatorname{ord}(Q(z)) = \operatorname{ord}(s^2a + 2st - t^2(\delta a^{-1})) \leq e$$

and z is a maximal vector of L, thus $Sz \in O(L)$, and $Sx \cdot Sz \in O^+(L)$. Notice

$$\theta(Sx \cdot Sz) = a \cdot h\dot{F}^2;$$

hence, $\theta(O^+(L)) = (1 + \mathscr{B}^{\operatorname{ord}(\delta)+2\mu-2e-2[\mu/2]})\dot{F}^2 \cap Q(\langle 1, d\rangle)\dot{F}^2$. □

Combining the above proposition with the results obtained in **[1]**, we conclude that the spinor norms of integral rotations on a modular quadratic form over a local field are determined completely and all the results are expressed in the conventional closed forms.

References

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