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ISOMETRIC DEFORMATION OF SURFACES IN R^3 PRESERVING THE MEAN CURVATURE FUNCTION

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The purpose of this paper is to classify surfaces in Euclidean 3-space with constant Gaussian curvature which admit non-trivial one-parameter families of isometric immersions preserving the mean curvature function. It is shown that the Gaussian curvature must be zero and, if the mean curvature is not constant, then such isometric immersions are some deformations of the cylinder over a logarithmic spiral.

1. Introduction. The study of isometric deformations of surfaces in a Euclidean 3-space R^3 preserving the mean curvature has a longstanding history since the work by O. Bonnet [**1**]. He proved that a surface of constant mean curvature can be isometrically deformed preserving the mean curvature.

Recently S. S. Chern [**3**] has studied such a deformation for surfaces of non-constant mean curvature and he gave an interesting criterion for its existence.

On the other hand, W. Scherrer [**7**] and R. A. Tribuzy [**9**] have found another necessary and sufficient condition for existence of such deformations.

The purpose of this paper is twofold; first to give a simple and unified treatment of Scherrer's and Tribuzy's result; second, to classify surfaces with constant Gaussian curvature which admit non-trivial isometric deformations preserving the mean curvature function, as an application of our result. This generalizes a recent result by Roussos [**6**].

By a non-trivial family of surfaces we mean surfaces which do not differ by rigid motions. We suppose that surfaces in this paper do not contain umbilic points.

Our theorems are local in nature, because a theorem of Lawson and Tribuzy [**4**] says that if the mean curvature of a compact surface in R^3 is not constant, then there exist at most two geometrically distinct isometric immersions of the surface with the same mean curvature.

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2. A surface theory. In this section we shall develop some local theory for any surface in R^3 which will be applied in the next section to the study of isometric deformations of surfaces. As a result, we can give a unified treatment of the above mentioned necessary and sufficient conditions of Scherrer [7] and Tribuzy [9].

We consider a piece of oriented surface M in R^3 which does not contain any umbilic point. Over M there is a well-defined field of orthonormal frames $x\ e_1,\ e_2,\ e_3$ such that $x \in M$, e_1, e_2 are unit tangent vectors at x and e_3 is the unit normal vector field at $x \in M$. We then have

$$\begin{aligned} (1) \qquad & dx = w_1 e_1 + w_2 e_2, \\ & de_1 = w_{12}e_2 + w_{13}e_3, \quad de_2 = w_{21}e_1 + w_{23}e_3, \\ & de_3 = -w_{13}e_1 - w_{23}e_2, \end{aligned}$$

where the w's are 1-forms on M, w_{12} is the connection form, w_{13} and w_{23} define the second fundamental form of M in R^3. They satisfy the structure equations:

$$\begin{aligned} (2) \qquad & dw_1 = w_2 \wedge w_{21}, \qquad dw_2 = w_1 \wedge w_{12}, \\ & dw_{12} = -Kw_1 \wedge w_2, \\ & dw_{13} = w_{12} \wedge w_{23}, \qquad dw_{23} = w_{21} \wedge w_{13}. \end{aligned}$$

Let H be the mean curvature function of M. For fixed e_1, e_2 and e_3, there exist some functions x, y such that we can write $w_{13} = (H+x)w_1 + yw_2$, $w_{23} = yw_1 + (H-x)w_2$. Since the Gaussian curvature K is written by $K = (H+x)(H-x) - y^2$, we have that $H^2 - K = x^2 + y^2$, which is positive by our assumption. Therefore, we can write, following Švec [8], for some α

$$\begin{aligned} (3) \qquad & w_{12} = \left(H + \sqrt{H^2 - K}\cos\alpha\right) w_1 + \sqrt{H^2-K}\sin\alpha w_2, \\ & w_{23} = \sqrt{H^2-K}\sin\alpha w_1 + \left(H - \sqrt{H^2-K}\cos\alpha\right) w_2. \end{aligned}$$

Now we shall remark that α in (3) depends on the frame e_1, e_2 and e_3. (Therefore the covariant derivatives of α in [8, formula (29)] shall be defined in a modified fashion.) Let $\tilde{e}_1$, $\tilde{e}_2$ be another unit tangent frame and denote $(\tilde{e}_1 + i\tilde{e}_2) = \exp(i\theta)(e_1 + ie_2)$. A simple calculation shows that $\tilde{\alpha} = \alpha + 2\theta$, $\tilde{w}_{12} = w_{12} - d\theta$ (mod 2π), where we are denoting the quantities pertaining to the frame fields $x\ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ by the same symbols with "$\sim$". It follows that $d\alpha + 2w_{12}$ is a globally defined 1-form on M and we denote it by

$$D\alpha := d\alpha + 2w_{12} = \alpha_1 w_1 + \alpha_2 w_2, \tag{4}$$

where α_1 and α_2 are coefficients of the 1-form $D\alpha$.

For any tensor field α_i of $(0, 1)$-type, we define its covariant derivatives $\alpha_{i,j}$ as follows:

$$D\alpha_i := d\alpha_i + \sum \alpha_s w_{si} = \sum \alpha_{i,j} w_j, \qquad 1 \le i \le 2. \tag{5}$$

It is easily verified that $\alpha_{1,1} + \alpha_{2,2}$ is independent of the choice of e_i in case of the α_i's defined by (4) and we write $\Delta\alpha = \alpha_{1,1} + \alpha_{2,2}$.

Exterior differentiation of (3) gives, using (2) and (4),

$$\begin{aligned}\sqrt{H^2 - K} D\alpha = {} & \cos\alpha(H_1 w_2 + H_2 w_1) - \sin\alpha(H_1 w_1 - H_2 w_2) \\ & + \left(\sqrt{H^2 - K}\right)_2 w_1 - \left(\sqrt{H^2 - K}\right)_1 w_2,\end{aligned} \tag{6}$$

where H_i and $(\sqrt{H^2 - K})_i$, $i = 1, 2$, are exterior derivatives of the scalar functions H and $\sqrt{H^2 - K}$, respectively.

We introduce the 1-forms

$$\beta_1 = \frac{H_1 w_1 - H_2 w_2}{\sqrt{H^2 - K}}, \qquad \beta_2 = \frac{H_2 w_1 + H_1 w_2}{\sqrt{H^2 - K}}. \tag{7}$$

By using the $*$-operator of Hodge, such that $*w_1 = w_2$ and $*w_2 = -w_1$, the formula (6) can be written

$$D\alpha = -\sin\alpha \cdot \beta_1 + \cos\alpha \cdot \beta_2 - *d\log\sqrt{H^2 - K}, \tag{6$'$}$$

which is one of the fundamental formulas in this paper.

In order to obtain the exterior derivative of (6)$'$, we first calculate:

$$\begin{aligned} d\beta_1 = \frac{1}{\sqrt{H^2 - K}} \Big[\Big\{ & \left(\log\sqrt{H^2 - K}\right)_1 H_2 \\ & + \left(\log\sqrt{H^2 - K}\right)_2 H_1 - 2H_{12} \Big\} w_1 \wedge w_2 \\ & - 2\sqrt{H^2 - K}\beta_2 \wedge w_{12} \Big], \\ d\beta_2 = \frac{1}{\sqrt{H^2 - K}} \Big[\Big\{ & -\left(\log\sqrt{H^2 - K}\right)_1 H_1 \\ & + \left(\log\sqrt{H^2 - K}\right)_2 H_2 + H_{11} - H_{22} \Big\} w_1 \wedge w_2 \\ & + 2\sqrt{H^2 - K}\beta_1 \wedge w_{12} \Big], \end{aligned} \tag{8}$$

where H_{ij}'s are covariant derivatives of H_i above defined in (5).

Using (4) and (8), the exterior differentiation of (6)′ gives the condition:

$$(9) \qquad -2A\sin\alpha + B\cos\alpha + P = 0,$$

where we set

$$(10) \quad \begin{aligned} A &= H_{12}\sqrt{H^2-K} - H_2\left(\sqrt{H^2-K}\right)_1 - H_1\left(\sqrt{H^2-K}\right)_2, \\ B &= (H_{22}-H_{11})\sqrt{H^2-K} + 2H_1\left(\sqrt{H^2-K}\right)_1 \\ &\quad - 2H_2\left(\sqrt{H^2-K}\right)_2, \\ P &= (H^2-K)\left(\Delta\log\sqrt{H^2-K} - 2K\right) - |\operatorname{grad} H|^2, \end{aligned}$$

and Δ is the Laplacian for the induced metric of M.

The formula (9) holds at non-umbilic points of any surface in R^3. We shall remark that this is essentially the same as the formula (3) in §2 of Scherrer [7] and the formula (39) of Švec [**8**].

Now we shall give another important formula obtained from (6)′. Applying the $*$-operator to (6)′, we get

$$\alpha_1 w_2 - \alpha_2 w_1 = -\sin\alpha\cdot\beta_2 - \cos\alpha\cdot\beta_1 + d\log\sqrt{H^2-K}.$$

Exterior differentiation gives, using (4), (6) and (8),

$$(11) \qquad (H^2-K)\Delta\alpha = 2A\cos\alpha + B\sin\alpha.$$

It follows from (9) and (11) that, for a given surface M in R^3, the conditions $A = B = 0$ are equivalent to the conditions $P = \Delta\alpha = 0$. This will be used later as

$$(12) \qquad A = B = 0 \Leftrightarrow P = \Delta\alpha = 0.$$

3. Deformation of surfaces with constant Gaussian curvature. We apply the local theory for a surface in R^3 developed in the previous section to the study of a one-parameter family of isometric surfaces having the same mean curvature at corresponding points. Such a family is called an isometric deformation of surfaces preserving the mean curvature function.

Here ∇ denotes the covariant differentiation of the induced metric and $Z = (e_1 - ie_2)/2$.

THEOREM 1. *Let M be a piece of an oriented surface in R^3 such that it has no umbilic points. Then, M admits a non-trivial isometric* *deformation preserving the mean curvature function if and only if one of the following conditions holds*:

$$\nabla\left(\frac{\nabla H}{H^2-K}\right)(Z,Z)=0; \tag{13}$$

$$(H^2-K)\left(\Delta\log\sqrt{H^2-K}-2K\right)-|\operatorname{grad}H|^2=0 \quad \textit{and} \quad \Delta\alpha=0. \tag{14}$$

Proof. In order to construct some non-trivial isometric deformation of M preserving the mean curvature function, it is necessary and sufficient to find a family of α's such that, for each α, the forms (3) satisfy the last equation of (2). Now we consider (6)′ as a total differential equation for unknown functions α. The complete integrability condition is given by (9). If the condition $A=B=0$ or $P=\Delta\alpha=0$ holds for the M, then, by (12), (6)′ is completely integrable.

Conversely, if M admits a non-trivial isometric deformation preserving the mean curvature function, then (6)′ is completely integrable, and so (9) and (11) hold for α's. By differentiation of (9) twice with respect to the direction of the deformation, we get $2A\cos\alpha+B\sin\alpha=-2A\sin\alpha+B\cos\alpha=0$. Coupling these formulas with (9) and (11), we get $P=\Delta\alpha=0$ on M, which is equivalent to $A=B=0$.

Finally, (13) is simply the complex representation of $A=B=0$: in fact, we have

$$4(H^2-K)^{3/2}\nabla\left(\frac{\nabla H}{H^2-K}\right)(Z,Z)=-(B+2iA). \tag{15}$$

This proves Theorem 1.

REMARK 1. Equation (13) is another representation of Scherrer's condition [7, p. 81] in complex notation. On the other hand, a complex representation of (14) has been obtained in a different way by Tribuzy [9, Proposition 3.2]. Our calculation shows that these two conditions are closely related and only the exterior differentiations of (6)′ and the dual of (6)′ in the sense of $*$-operation are essential.

REMARK 2. By [3], we know that on a surface with non-constant mean curvature, the condition (13) or (14) implies that the metric $d\hat{s}^2:=\beta_1^2+\beta_2^2$ has constant Gaussian curvature equal to -1. It is clear that this is also proved by our method: we denote by β_{12} and $\hat{K}$

the connection form of the new metric $d\hat{s}^2$ and its Gaussian curvature, respectively. Some calculation shows that

$$\beta_{12} = -\sin\alpha \cdot \beta_1 + \cos\alpha \cdot \beta_2 - d\alpha. \tag{16}$$

Exterior differentiation of this and the equation (6)′ give $\hat{K} \equiv -1$.

As an application of Theorem 1, we can classify surfaces with constant Gaussian curvature which admit an isometric deformation preserving the mean curvature function.

THEOREM 2. *Let M be a piece of an oriented surface in R^3 without umbilic points such that the Gaussian curvature K is constant on M. If M admits a non-trivial isometric deformation preserving the mean curvature function, then K must be zero.*

Proof. If M is a minimal surface, then K is zero by a theorem of Pinl [5]. In case of $H \neq 0$, we consider a tensor field of $(0, 1)$-type defined by $f_i = H_i/H^2 - K$, $i = 1, 2$. Since K is constant, we have $f_{i,j} = \{(H^2-K)H_{i,j} - 2HH_iH_j\}/(H^2-K)^2$. By conditions $A = B = 0$, there exists some scalar function λ with $f_{i,j} = \lambda\delta_{ij}$. By taking the trace of these equations, we have

$$2\lambda = \sum f_{i,i} = \frac{(H^2 - K)\Delta H - 2H|\operatorname{grad} H|^2}{(H^2 - K)^2}.$$

On the other hand the condition $P = 0$ is equivalent to

$$(H^2 - K)\Delta H - 2H|\operatorname{grad} H|^2 = \frac{2K(H^2 - K)^2}{H}.$$

These formulas give that $\lambda = K/H$, and so we have

$$Hf_{i,j} = K\delta_{ij}, \qquad 1 \leq i, j \leq 2. \tag{17}$$

Since K is constant, we have, from (17),

$$H_k f_{i,j} - H_j f_{i,k} + H(f_{i,j,k} - f_{i,k,j}) = 0. \tag{18}$$

Now, we need the Ricci identities of the tensor field f_i on a two dimensional Riemannian manifold:

$$f_{1,2,1} - f_{1,1,2} = Kf_2, \quad f_{2,1,2} - f_{2,2,1} = Kf_1. \tag{19}$$

By (17), (18) and (19), we have $KH_i = 0$, $i = 1, 2$. If $H_i = 0$, $i = 1, 2$, then H is constant and $f_{i,j}$'s vanish identically. Therefore K must be zero, finishing the proof.

COROLLARY. *Non-umbilical surfaces with non-zero constant Gaussian curvature do not admit any non-trivial isometric deformation preserving the mean curvature function.*

4. Examples of deformations. Theorem 2 shows that flat surfaces having the property (13) should be studied. Let M be a piece of Euclidean 2-plane with the standard flat metric $ds^2 = du^2 + dv^2$. We consider an isometric immersion $X(u,v)\colon M \to R^3$ such that it has no umbilic points which implies non-vanishing of the mean curvature H and satisfies the condition (13).

We put $w_1 = du$ and $w_2 = dv$, and so we have $w_{12} = 0$. The condition (13), then, is equivalent to

$$HH_{uv} - 2H_uH_v = 0, \tag{20}$$

$$H(H_{vv} - H_{uu}) + 2(H_u^2 - H_v^2) = 0. \tag{21}$$

We shall determine w_{13} and w_{23} of X in (3). If H is constant, then X is a piece of plane in R^3 or a circular cylinder within isometries of (u,v)-domain and R^3. We may assume $H \neq 0$. Then

(22)

$$X_t(u,v) = \left(\frac{1}{2H}\cos(2H\tilde{u}), \frac{1}{2H}\sin(2H\tilde{u}), \tilde{v}\right), \qquad t \in (-\infty, \infty),$$

where we set $\tilde{u} = \cos t \cdot u - \sin t \cdot v$ and $\tilde{v} = \sin t \cdot u + \cos t \cdot v$, is the non-trivial isometric deformation preserving the mean curvature of the circular cylinder. This is a special case of the Bonnet's Theorem **[1]** and also see J. A. Wolf **[10]**.

From now on, we assume that H is positive and not constant. The general solution of (20) is $H(u,v) = 1/(\phi(u) + \psi(v))$, where ϕ and ψ are any functions. Considering (21), we get

$$H(u,v) = \frac{1}{a(u^2+v^2) + bu + cv + d},$$

where a, b, c and d are some real numbers. We shall study, separately, two cases according to the value of a. At first we assume $a = 0$. By taking some orthogonal transformation and parallel translation of the standard coordinates of R^2, we may assume that

$$H(u,v) = \frac{b}{u}, \tag{23}$$

where b is a non-zero real number.

By (6) and (23), we have

$$\frac{\partial \alpha}{\partial u} = \frac{\sin\alpha}{u} \quad \text{and} \quad \frac{\partial \alpha}{\partial v} = \frac{1 - \cos\alpha}{u}.$$

This is integrable and any solutions are

$$\tan\frac{\alpha}{2} = \frac{ut}{1-vt}, \tag{24}$$

where t is any real number. By (3), (23) and (24), we get a 1-parameter family of isometric immersions X_t preserving the mean curvature function. The square norm of the $D\alpha$ in case of (24) is given by

$$|D\alpha|^2 = \frac{4t^2}{u^2t^2 + (1-vt)^2}. \tag{25}$$

Therefore, any X_t is not congruent with $X_{t'}$, for $t \neq t'$, $t, t' \geq 0$. The total differential equation (1) of X_t can be written easily. By a similarity transformation of X, we may assume $b = 1/2$. In particular, putting $t = 0$, we get

$$X_0(u,v) = \left(\int \cos\log u \cdot du, \int \sin\log u \cdot du, v\right).$$

We put $\xi = \log u$ and $\tau = v$. Then this gives a *cylinder over a logarithmic spiral*:

$$X_0(\xi,\tau) = \left(\frac{1}{\sqrt{2}}e^{\xi}\cos\left(\xi - \frac{\pi}{4}\right), \frac{1}{\sqrt{2}}e^{\xi}\sin\left(\xi - \frac{\pi}{4}\right), \tau\right). \tag{26}$$

Moreover, we know that the curvature and the torsion of the curve $X_t(u, v_0)$, for a fixed v_0, are $(1-v_0t)^2/u(u^2t^2 + (1-v_0t)^2)$ and $-t(1-v_0t)/(u^2t^2 + (1-v_0t)^2)$, respectively. For a fixed $u_0 > 0$, the curvature and the torsion of the curve $X_t(u_0, v)$ are given by $u_0t^2/(u_0t^2 + (1-vt)^2)$ and $-t(1-vt)/(u_0^2t^2 + (1-vt)^2)$, respectively. By the Theorem 3 of Roussos **[6]**, X_∞ is a generalized flat cone. In fact, if we assume $t = \infty$ in (24), then we have

$$\sin\alpha = -2uv/(u^2+v^2) \quad \text{and} \quad \cos\alpha = -(u^2-v^2)/(u^2+v^2).$$

Put $u = r\cos\theta$ and $v = r\sin\theta$ in the total differential equation (1) of X_∞. It is easily verified that $X_\infty(r\cos\theta, r\sin\theta) = rE_1(\theta)$, where $E_1(\theta) = \cos\theta \cdot e_1 + \sin\theta \cdot e_2$ is a unit vector in R^3 and the geodesic curvature of $E_1(\theta)$ as a curve on the unit sphere is $1/\cos\theta$.

By those observations, one can imagine the isometric deformation X_t of X_0 preserving the mean curvature function. Conversely, it is easily proved that the flat cylinder (26) satisfies the condition (13).

Now we shall prove that the constant a must be zero. If a is not zero, then, by taking some orthogonal transformation and parallel translation of the standard coordinates of R^2, we may assume that

$H(u,v) = 1/a(u^2+v^2)+d$. By using the polar coordinates, (6) can be written, in this case,

$$\frac{\partial\alpha}{\partial r} = \frac{-2ar}{ar^2+d}\sin(2\theta-\alpha),$$
$$\frac{\partial\alpha}{\partial\theta} = \frac{2ar^2}{ar^2+d}(1-\cos(2\theta-\alpha)).$$

But this system is not integrable. Thus we have proved the following.

THEOREM 3. *Let M be a flat surface in R^3 without umbilic points and H the mean curvature function of M. If H is not constant and satisfies the differential equation* (13), *then there exists a non-trivial isometric deformation M_t preserving the mean curvature function such that $M = M_t$ for some t, M_0 is a cylinder over a logarithmic spiral and M_∞ is a generalized cone.*

REMARK 3. Simple calculation proves that there is no flat circular cone that admits a non-trivial isometric deformation preserving the mean curvature function. Hence, M_t is not a flat circular cone for any t $(\leq\infty)$.

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