PACIFIC JOURNAL OF MATHEMATICS
Vol. 136, No. 2, 1989

CANONICAL ISOMETRY ON WEIGHTED BERGMAN SPACES

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We consider the spectral properties of a unitary operator called canonical isometry in the space of all holomorphic functions which are square integrable with respect to some measure on a domain in $\mathbf{C}^N$. Particularly, the correspondence of its eigenfunctions and fixed points of holomorphic automorphism is investigated.

1. Introduction. In the study of biholomorphic mappings the Hilbert space methods are very useful and fruitful. It was discovered by S. Bergman [**1, 2**] that in this area an important role is played by the space $L^2H(D)$, consisting of all functions which are holomorphic and square integrable with respect to Lebesgue measure on a domain $D \subset \mathbf{C}^N$, with scalar product

$$(1) \qquad \langle f, g\rangle = \int_D f(z)\overline{g(z)}\,dm(z), \qquad f, g \in L^2H(D).$$

Every biholomorphic automorphism $\varphi: D \to D$ induces a unitary operator, called canonical isometry $U_\varphi: L^2H(D) \to L^2H(D)$ given by the formula

$$(2) \qquad (U_\varphi f)(z) = f(\varphi(z))\frac{\partial\varphi}{\partial z}.$$

Here $\partial\varphi/\partial z$ denotes the complex Jacobian of φ. By $\mathrm{Aut}(D)$ we denote a group of biholomorphic automorphisms of a domain D.

In [**10**] the following spectral property of U_φ was established:

THEOREM 1. *Assume that $D \subset \mathbf{C}^N$ is a domain for which $L^2H(D) \neq \{0\}$. If $\varphi \in \mathrm{Aut}(D)$ has a fixed point, then there exists a linearly dense orthonormal system of eigenfunctions of U_φ.*

We are inspired by paper [**6**] with studying the more general situation, which has connections with theory of group representation [**8**], mathematical physics (see [**6**]) and ergodic theory (see [**3**]).

Let u be a measure on a domain $D \subset \mathbf{C}^N$, absolutely continuous and having strictly positive, continuous Radon-Nikodym derivative with respect to the Lebesgue measure. Consider the Hilbert space

$L^2H(D, u)$, consisting of all holomorphic functions on D which are square integrable with respect to u, with scalar product

$$\langle f, g\rangle_u = \int_D f(z)\overline{g(z)}\, du(z), \qquad f, g \in L^2H(D, u). \tag{3}$$

Consider a biholomorphic automorphism φ of D, such that the mapping $\psi \colon D \to \mathbf{C}$, in the following expression

$$u(\varphi(D')) = \int_{D'} |\psi|^2 du \quad \text{for all domains } D' \subset D \tag{4}$$

is nonzero and holomorphic. In fact the set of such automorphisms is a subgroup in $\mathrm{Aut}(D)$, with natural group law: $\varphi_1 \circ \varphi_2$ and φ^{-1} correspond to $\psi_1(\varphi_2)\psi_2$ and $1/\psi$ respectively. In the literature (see **[6]**, p. 21) it is called a subgroup of maps which leave u invariant modulo holomorphic change of gauge and denoted by $G(u)$.

In this context we obtain a unitary representation of $G(u)$ given by $V_\varphi \colon L^2H(D, u) \to L^2H(D, u)$, $\varphi \in G(u)$, defined as follows

$$(V_\varphi f)(z) = f(\varphi(z))\psi(z) \tag{5}$$

EXAMPLE. Case of Fock space—F_α^2. Let u_α, $\alpha > 0$, be a measure on $\mathbf{C}^N$, for which

$$\left(\frac{\alpha}{\pi}\right)^N e^{-\alpha|z|^2}$$

is the Radon-Nikodym derivative with respect to Lebesgue measure. $F_\alpha^2 := L^2H(\mathbf{C}^N, u_\alpha)$. Let G be a group of translations in $\mathbf{C}^N$. If $\varphi \in G$, $\varphi(z) = z - w$, then

$$\psi(z) = \left(\frac{\alpha}{\pi}\right)^{N/2} e^{\alpha\langle z, \bar{w}\rangle - \alpha|w|^2/2}. \tag{6}$$

The purpose of the present paper is to prove the following results:

THEOREM 2. *Assume that $L^2H(D, u) \neq \{0\}$, $D \subset \mathbf{C}^N$. If $\varphi \in G(u)$ has a fixed point, then there exists a linearly dense, orthogonal system of eigenfunctions of V_φ.*

THEOREM 3. *Let D be a bounded simply connected domain in $\mathbf{C}^N$. Assume that D is complete and has nonpositive sectional curvature with respect to the Bergman metric. If V_φ has any eigenfunction then $\varphi \in G(u)$ has a fixed point.*

2. Evaluation functional and reproducing kernel. An evaluation functional (see **[13, 14]**)

$$x_z^* \colon L^2H(D, u) \to \mathbf{C}, \quad x_z^*(f) = f(z), \quad z \in D,$$

is linear and continuous. It implies that $L^2H(D,u)$ has a reproducing kernel. Let $x_z \in L^2H(D,u)$ be an element which represents x_z^* in terms of scalar product. The reproducing kernel is given by

$$K_u(w,z) = \langle x_z, x_2\rangle_u, \qquad w, z \in D. \tag{7}$$

LEMMA 1. *For arbitrary orthonormal bases $\{e_n\}$, $n = 1, 2, \ldots$, in $L^2H(D,u)$*

$$K_u(w,z) = \sum_{n=1}^{\infty} e_n(w)\overline{e_n(z)}.$$

This sum converges absolutely.

Since V_φ is a unitary operator, from Lemma 1 we obtain a transformation rule for reproducing kernel.

LEMMA 2. *Let $\varphi \in G(u)$ and $\psi\colon D \to \mathbf{C}$ be a holomorphic function related to φ as in* (4). *Then*

$$K_u(w,z) = K_u(\varphi(w),\varphi(z))\psi(w)\overline{\psi(z)}. \tag{8}$$

Proof. Let $\{e_n\}$, $n = 1, 2, \ldots$, be an orthonormal base in $L^2H(D,u)$. Since $V_\varphi(e_n)$ is also an orthonormal base, we have

$$\begin{aligned} K_u(w,z) &= \sum_{n=1}^{\infty} V_\varphi(e_n(w))\overline{V_\varphi(e_n(z))} \\ &= \sum_{n=1}^{\infty} e_n(\varphi(w))\psi(w)\overline{e_n(\varphi(z))\psi(z)} \\ &= \psi(w)\overline{\psi(z)}K_u(\varphi(w),\varphi(z)). \end{aligned}$$

□

Formula (8) can be written as

$$\begin{aligned} x_z &= (V_\varphi x_{\varphi(z)})\overline{\psi(z)}, \\ x_z(w) &= K_u(w,z) = K_u(\varphi(w),\varphi(z))\psi(w)\overline{\psi(z)} \\ &= \langle x_{\varphi(z)}, x_{\varphi(w)}\rangle_u \psi(w)\overline{\psi(z)} = x_{\varphi(z)}(\varphi(w))\psi(w)\overline{\psi(z)} \\ &= (V_\varphi x_{\varphi(z)})(w)\overline{\psi(z)}. \end{aligned} \tag{9}$$

Let us consider an operator $V_\varphi^*\colon L^2H(D,u)^* \to L^2H(D,u)^*$ adjoint to V_φ.

By definition for every $f \in L^2H(D,u)$

$$\begin{aligned} (V_\varphi^* x_z^*)(f) &= x_z^*(V_\varphi f) = (V_\varphi f)(z) = f(\varphi(z))\psi(z) \\ &= \psi(z)x_{\varphi(z)}^* f. \end{aligned}$$

Hence, we obtain the following formula for x_z^*

(10) $$V_\varphi^* x_z^* = \psi(z) x_{\varphi(z)}^*.$$

3. The proof of Theorem 2. The mapping

$$z \to x_z^* \in L^2H(D,u)^*, \qquad z \in D,$$

is holomorphic, and

$$f(x_z^*) = x_z^*(f) = f(z) \quad \text{for all } f \in L^2H(D,u).$$

Hence, for every $t \in D$, x_z^* can be developed for z close to t as a series (see **[12]**, Def. 3.30 and exercise 26)

(11) $$x_z^* = \sum_{k=(k_1,k_2,\dots,k_N)} g_k^*(z_1-t_1)^{k_1}(z_2-t_2)^{k_2}\cdots(z_N-t_N)^{k_N}$$

for some $g_k^* \in L^2H(D,u)^*$.

Denote by H_s^*, $s = 0,1,2,\dots$, the subspace of $L^2H(D,u)^*$ generated by elements g_k^*, $|k| = k_1 + k_2 + \dots + k_N \le s$.

Substituting (11) and the developments of ψ and $\varphi = (\varphi_1,\dots,\varphi_N)$ in a power series about fixed point of φ into (10), we can see (similarly as in **[10]**) that for every $s = 0,1,2,\dots$,

(12) $$V_\varphi^* H_s^* = H_s^*.$$

Consider the mapping $P\colon L^2H(D,u)^* \to L^2H(D,u)$ defined as follows: $P(g^*) = g$, where g represents g^* in terms of scalar product. P maps H_s^* onto the subspace $H_s \subset L^2H(D,u)$, generated by $g_k = P(g_k^*)$, $|k| \le s$.

It turns out that for every $s = 0,1,2,\dots$, **[10]**.

(13) $$V_\varphi H_s = H_s.$$

Denote by $\hat{H}_s$, $s = 1,2,\dots$, the orthogonal complement of H_{s-1} in H_s. For every $s = 1,2,\dots$,

(14) $$V_\varphi \hat{H}_s = \hat{H}_s.$$

Every $\hat{H}_s$ has finite dimension and $L^2H(D,u)$ is an orthogonal sum of $\hat{H}_s$.

There exists the orthogonal system E_s, $s = 1,2,\dots$, of eigenfunctions of $V_\varphi\colon \hat{H}_s \to \hat{H}_s$. If $\hat{H}_s = \{0\}$, set $E_s = \varnothing$. Define

$$E_0 = \begin{cases} \left\{ \dfrac{g_0}{\|g_0\|} \right\} & \text{if } g_0 \ne 0, \\ \varnothing & \text{otherwise.} \end{cases}$$

Finally $E = \bigcup_{s=0}^{\infty} E_s$ composes the complete orthonormal system in $L^2H(D,u)$.

4. The proof of Theorem 3. Let now D be a bounded domain in $\mathbf{C}^N$.

Denote by K_D the Bergman function of D **[1, 13, 14]** (K_D is a reproducing kernel in $L^2H(D)$).

A riemannian structure on D is given by the Bergman metric tensor (see **[1, 7, 14]**)

$$T(z) = \sum_{i,j=1}^{N} \left(\frac{\partial^2 \log K_D(z,z)}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j + \frac{\partial^2 \log K_D(z,z)}{\partial \bar{z}_i \partial z_j} d\bar{z}_i \otimes dz_j \right). \tag{15}$$

The Bergman and Euclidean distances generate the same topology.

The orbit of $\varphi \in \mathrm{Aut}(D)$ in $z \in D$ we will call the set

$$\mathrm{Orb}_\varphi(z) := \{w \in D\colon w = \varphi^n(z),\ n = 0, \pm 1, \pm 2, \dots\}. \tag{16}$$

We shall need the following classical result:

THEOREM 4 (HOPF-RINOV). *Let M be a connected riemannian manifold. The following conditions are equivalent:*
1°. *M is complete metric space;*
2°. *every closed and bounded subset of M is compact.*

The above theorem and the fact that biholomorphic mappings are isometries of the Bergman metric yield:

LEMMA 3. *Let $D \subset \mathbf{C}^N$ be a domain complete with respect to the Bergman metric. For every $\gamma \in \mathrm{Aut}(D)$ and $z, w \in D$:*

$$\mathrm{Orb}_\gamma(z) \subset\subset D \quad \textit{if and only if} \quad \mathrm{Orb}_\gamma(w) \subset\subset D.$$

The fundamental observation is contained in the following:

LEMMA 4. *Let $D \subset\subset C^N$ be a domain complete with respect to the Bergman metric. For every $\varphi \in G(u)$ if there exists $z \in D$, such that $\mathrm{Orb}_\varphi(z)$ is not relatively compact in D, then V_φ has no eigenfunction.*

Proof. Suppose, that for some $l \in \mathbf{C}$ and $0 \neq f \in L^2H(D,u)$,

$$V_\varphi f = lf \quad \text{or equivalently} \quad f(\varphi(z))\psi(z) = lf(z), \quad z \in D.$$

Since $|l| = 1$, we have $|f(\varphi(z))|^2|\psi(z)|^2 = |f(z)|^2$. Let $B(z^0, 1) \subset D$ be a ball with center z^0 and radius 1.

Integrating by change of variables yields

$$\begin{aligned}\int_{B(z^0,1)} |f(z)|^2\, du(z) &= \int_{B(z^0,1)} |f(\varphi(z)|^2 |\psi(z)|^2\, du(z)\\ &= \int_{\varphi(B(z^0,1))} |f(z)|^2\, du(z).\end{aligned}$$

From Lemma 3, $\mathrm{Orb}_\varphi(z^0)$ is not relatively compact. Since the closure $B(z^0, 1)$ is compact, there exists $z^1 \in \mathrm{Orb}_\varphi(z^0)$ such that

$$B(z^0, 1) \cap B(z^1, 1) = \varnothing.$$

For the closure of $B(z^0, 1) \cup B(z^1, 1)$ is also compact, there exists $z^2 \in \mathrm{Orb}_\varphi(z^0)$, for which

$$(B(z^0, 1) \cup B(z^1, 1)) \cap B(z^2, 1) = \varnothing.$$

In this way we are able to construct an infinite sequence of balls B_n, $n = 1, 2, \ldots$, which do not intersect each other and

$$\int_{B_n} |f(z)|^2\, du(z) = \text{const} > 0 \quad \text{for all } n = 1, 2, \ldots.$$

But

$$\int_D |f(z)|^2\, du(z) > \sum_{n=1}^{\infty} \int_{B_n} |f(z)|^2\, du(z) = \infty,$$

contrary to the assumption. □

Let us now consider $\varphi \in G(u)$ with relatively compact orbits. Denote by Λ a group

$$\Lambda := \{\gamma \in \mathrm{Aut}(D) \colon \gamma = \varphi^n,\ n = 0, \pm 1, \pm 2, \ldots\}$$

and by $\overline{\Lambda}$ its closure in a topology of locally uniform convergence. Our aim is to show that $\overline{\Lambda}$ is a compact subgroup of $\mathrm{Aut}(D)$.

LEMMA 5. *Let $\xi \in \mathrm{Aut}(D)$ and for some $z^0 \in D$, $\mathrm{Orb}_\xi(z^0)$ be relatively compact. Then*

(a) $\xi^n(\mathrm{Orb}_\xi(z^0)) \subset \mathrm{Orb}_\xi(z^0)$, $n = 0, \pm 1, \pm 2, \ldots$, *and*

(b) *if γ_m, $m = 1, 2, \ldots$, is an arbitrary subsequence in Λ, locally uniformly convergent to γ, then $\gamma(\mathrm{Orb}_\xi(z^0)) \subset \overline{\mathrm{Orb}_\xi(z^0)}$.*

Proof. (a) It is obvious.

(b) Set $\gamma_m = \xi^k$. From continuity of γ_m and (a) we have

$$\gamma_m(z) \in \overline{\mathrm{Orb}_\xi(z^0)} \quad \text{for every } z \in \mathrm{Orb}_\xi(z^0).$$

Hence

$$\gamma(z) = \lim_{m\to\infty} \gamma_m(z) \in \overline{\mathrm{Orb}_\xi(z^0)}. \qquad \square$$

THEOREM 5 (H. CARTAN), [**11**]. *Let $D \subset \mathbf{C}^N$ be a bounded domain. For every sequence $(\gamma_n) \subset \mathrm{Aut}(D)$, $n = 1, 2, \ldots$, which is locally uniformly convergent to a mapping $\gamma\colon D \to \mathbf{C}^N$ the following conditions are equivalent*:

(a) $\gamma \in \mathrm{Aut}(D)$,

(b) $\gamma(D) \not\subset \partial D$ *(boundary of D)*

In view of Theorem 5, we see, that $\overline{\Lambda}$ is a group of automorphisms. Since D is bounded, the compactness of $\overline{\Lambda}$ follows from:

THEOREM 6 (MONTEL). *Every family of commonly bounded holomorphic mappings $\gamma_m\colon D \to \mathbf{C}^N$ is compact in the topology of locally uniform convergence.*

The topology of locally uniform convergence in $\mathrm{Aut}(D)$ is the same as topology induced by the Bergman metric. Hence, $\overline{\Lambda}$ is a compact group of isometries of D as a riemannian manifold. The statement of Theorem 3 is a consequence of:

THEOREM 7 (E. CARTAN). *Every compact group of isometries of simply connected, complete riemannian manifolds with nonpositive sectional curvature has a common fixed point.*

5. Remark. Very fine characterization of domain with nonpositive sectional curvature with respect to the Bergman metric is given in paper [**4**]. Namely, if $D \subset \mathbf{C}^N$ is a bounded, homogeneous domain, then the following conditions are equivalent:

(a) D has nonpositive sectional curvature with respect to the Bergman metric;

(b) D is symmetric domain.

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Received April 7, 1987.

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