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LOCALIZATION OF A CERTAIN SUBGROUP OF SELF-HOMOTOPY EQUIVALENCES

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Let X be a simple, finite C.W. complex. The group $\mathscr{E}_\#(X)$ is known to be nilpotent. In this paper, we give a proof of the naturality of localization on this group, $\mathscr{E}_\#(X)_{(P)} = \mathscr{E}_\#(X_{(P)})$. The result is then applied to study the group structures of $\mathscr{E}_\#(X)$ of rational Hopf spaces and some Lie groups.

Introduction. Let X be a pointed topological space. We use the notation $\mathscr{E}(X)$ to denote the group of based self-homotopy equivalences of X. (For this group there are other notations, for example, $\mathrm{AUT}^\circ(X)$ in [**2**].) Throughout the paper our spaces X will be connected of finite type and either finite dimensional or Postnikov pieces, namely, spaces with finite number of non-trivial homotopy groups. Then we denote by $\mathscr{E}_\#^m(X)$ the subgroup of $\mathscr{E}(X)$ which is the kernel of the obvious map (cf. [**1**], [**16**]):

$$\mathscr{E}(X) \to \prod_{j\le m} \operatorname{Aut}\pi_j(X).$$

We simply denote $\mathscr{E}_\#(X)$ when $m = \dim X$, where

$$\dim X = \max\{i|\pi_i(X) \neq 0\}$$

if X is a Postnikov piece.

E. Dror and A. Zabrodsky have proved that $\mathscr{E}_\#(X)$ is a nilpotent group for an arbitrary finite dimensional C.W. complex or a Postnikov piece ([**2**], Theorem A). If $m \geq \dim X$, $\mathscr{E}_\#^m(X)$ is a subgroup of $\mathscr{E}_\#(X)$ and thus also nilpotent. Hence these groups can be localized in a natural way. For example, the reader may consult the book [**5**] which provides basic matters on the theory of localization of nilpotent groups (and spaces).

In this paper, our main result is the following.

THEOREM 0.1. *Let X be a simple C.W. complex and P be an arbitrary collection of prime numbers. Assume that $m \geq \dim X$. Then the* *natural map*:

$$l_{\#}: \mathscr{E}_{\#}^{m}(X) \to \mathscr{E}_{\#}^{m}(X_{(P)})$$

is the P-localization map, where $X_{(P)}$ is the localization at P.

In other words $\mathscr{E}_{\#}^{m}(X)_{(P)} = \mathscr{E}_{\#}^{m}(X_{(P)})$. When a space is P-equivalent to the simpler space, this theorem enables us to determine $\mathscr{E}_{\#}(X)$ effectively. For example, for 0-regular spaces we obtain the following result which is concerned with the classical result of **[1]** (Theorem 5.4).

THEOREM 0.2. *Let X be a simple finite rational H-space with* $\beta_{2n_i-1} - \mathrm{rank}_Q(\pi_{2n_i-1}(X) \otimes Q) \leq 1$ *for* $i \leq k$. *Then* $\mathscr{E}_{\#}(X)/torsion = Z \oplus \cdots \oplus Z$, *the free abelian group of*

$$\mathrm{rank} = \sum_{i=1}^{k} \mathrm{rank}_Q(\pi_{2n_i-1}(X) \otimes Q) \cdot (\beta_{2n_i-1} - \mathrm{rank}_Q(\pi_{2n_i-1}(X) \otimes Q)),$$

where β_j is the jth Betti number and $H^*(X, Q) = E(x_1, \ldots, x_k)$ (the exterior algebra) with $\deg x_i = 2n_i - 1$.

In many cases $\mathscr{E}_{\#}(X)$ is an abelian group. We give the following non-abelian example.

EXAMPLE (Example 3.1). $\mathscr{E}_{\#}(\mathrm{SO}(6))$ and $\mathscr{E}_{\#}(\mathrm{SU}(4))$ are not abelian.

This paper is organized as follows. In the first section we prove our main theorem (Theorem 0.1). In the second section we prove Theorem 0.2. In the final section three we will show the above example.

1. Proof of the main theorem. Let X_n be an nth Postnikov stage of X. Then there exist natural homomorphism $J_X^n: \mathscr{E}(X_n) \to \mathscr{E}(X_{n-1})$ and its restriction, $J_X^n: \mathscr{E}_{\#}(X_n) \to \mathscr{E}_{\#}(X_{n-1})$ which is denoted by the same symbol. When $m \geq \dim X$, $\mathscr{E}_{\#}^{m}(X) = \mathscr{E}_{\#}(X_m)$ and we can prove our theorem by induction on the Postnikov decomposition of X. The following exact sequence is due to Y. Nomura **[10]** (cf. **[6]**, **[13]** and **[16]**).

$$0 \to I(1_{X_n}) \to H^n(X_{n-1}; \pi_n(X)) \xrightarrow{\Delta} \mathscr{E}_{\#}(X_n) \xrightarrow{J_X^n} \mathscr{E}_{\#}(X_{n-1}).$$

The localization map $l: X \to X_{(P)}$ can be restricted to the Postnikov stages and the following diagram is commutative.

$$\begin{array}{ccccccc}
0 \to I(1_{X_n}) & \to & H^n(X_{n-1}; \pi_n(X)) & \xrightarrow{\Delta} & \mathscr{E}_{\#}(X_n) & \to & \mathrm{Im}\, J_X^n \to 1 \\
\downarrow l & & \downarrow l & & \downarrow l_{n\#} & & \downarrow l_{n-1\#} \\
0 \to I(1_{X_{n(P)}}) & \to & H^n(X_{n-1(P)}; \pi_n(X_{(P)})) & \xrightarrow{\Delta} & \mathscr{E}_{\#}(X_{n(P)}) & \to & \mathrm{Im}\, J_{X_{(P)}}^n \to 1.
\end{array}$$

In the above diagram,

$$l\colon H^n(X_{n-1};\pi_n(X)) = [X_{n-1}, K(\pi_n(X), n)] \to [X_{n-1}, K(\pi_n(X_{(P)}), n)]$$
$$= [X_{n-1(P)}, K(\pi_n(X_{(P)}), n)] = H^n(X_{n-1(P)};\pi_n(X_{(P)}))$$

obviously P-localizes. Since we can show by induction that $\mathscr{E}_\#(X_{n(P)})$ is a P-local group for any n, it suffices to show that the restriction of $l_{n-1\#}$ to $\operatorname{Im} J_X^n$ and the restriction of l to $I(1_{X_n})$ are both localization maps (Theorem 3.2 Ch. I, **[5]**).

(I) $l_{n-1\#}\colon \operatorname{Im} J_X^n \to \operatorname{Im} J_{X_{(P)}}^n$ P-localizes.

First we recall that $\operatorname{Im} J_X^n = \{f \in \mathscr{E}_\#(X_{n-1}) | f^* k^{n+1} = k^{n+1}\}$, where k^{n+1} is the k-invariant of X (cf. Theorem 2.9 **[10]**). This group can be identified with the isotropy subgroup at k^{n+1} with respect to the action of $\mathscr{E}_\#(X_{n-1})$ on the cohomology group $H^{n+1}(X_{n-1};\pi_n(X))$.

Secondly we assert that this action is nilpotent (for the nilpotent action see §4 of **[5]**). Let us consider the fibrations

$$K(\pi_m(X), m) \to X_m \to X_{m-1}, \qquad m \leq n-1.$$

For $m = 1, X_1 = K(\pi_1(X), 1)$ and $\mathscr{E}_\#(X_{n-1})$ acts trivially on the cohomology. There exists the $\mathscr{E}_\#(X_{n-1})$-module spectral sequence converging to $H^*(X_m)$.

$$E_2^{p,q} = H^p(X_{m-1}; H^q(\pi_m(X), m)) \Rightarrow H^*(X_m).$$

If we assume that $\mathscr{E}_\#(X_{n-1})$ acts on $H^*(X_{m-1})$ nilpotently, then so does $H^*(X_m)$. Thus $\mathscr{E}_\#(X_{n-1})$ acts on $H^*(X_{n-1})$ nilpotently. By the universal coefficient theorem, we obtain the assertion.

Let Q be a nilpotent group acting on a group N nilpotently. Then the localizaton Q_P acts on N_P compatibly in the sense of §1 of **[3]**.

THEOREM. (*P. Hilton* **[3]**, *Theorem* 1.1) $Q(a)_P = Q_P(ea)$, *where* $e\colon N \to N_P$ *localizes and* $Q(a)$ *denotes the isotropy subgroup of* Q *at* $a \in N$.

By the hypothesis of induction, $l_{n-1\#}\colon \mathscr{E}_\#(X_{n-1}) \to \mathscr{E}_\#(X_{n-1(P)})$ localizes and by the naturality of $l_{n-1\#}$ it is compatible with the given action of $\mathscr{E}_\#(X_{n-1})$ on the cohomology. Thus there is a commutative diagram:

$$\begin{array}{ccc} \mathscr{E}_\#(X_{n-1}) & \to & \operatorname{Aut} H^{n+1}(X_{n-1};\pi_n(X)) \\ \downarrow l_{n-1\#} & & \downarrow \\ \mathscr{E}_\#(X_{n-1(P)}) & \to & \operatorname{Aut} H^{n+1}(X_{n-1(P)};\pi_n(X_P)). \end{array}$$

Finally put $Q = \mathscr{E}_{\#}(X_{n-1}), N = H^{n+1}(X_{n-1};\pi_n(X))$ and $a = k^{n+1}$. Then (I) is derived from the above Hilton's Theorem.

(II) $l: I(1_{X_n}) \to I(1_{X_{n(P)}})$ P-localizes.
$I(1_{X_n})$, the kernel of the homomorphism Δ, is a subgroup of the group $H^n(X_{n-1};\pi_n(X))$ and has the following form ([**10**]):

$$I(1_{X_n}) = \{x \in H^n(X_{n-1};\pi_n(X)) | \mu(x \times 1_{X_n})d = 1_{X_n}\},$$

where μ denotes the action of $K(\pi_n(X), n)$ on X_n, d is a diagonal map. Hence $I(1_{X_n})$ can be regarded as an isotropy subgroup at 1_{X_n} with respect to the action of $H^n(X_{n-1};\pi_n(X))$ on $[X_n, X_n]$. In this case we cannot apply the argument like above because $[X_n, X_n]$ is not generally a group. But we can use the argument which is dual to that of the proof of Theorem 2.5, Ch. II, [**5**]. We have a fibration:

$$X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{k^{n+1}} K(\pi_n(X), n+1)\ (= K).$$

This gives rise to a fibration:

$$F(X_n, X_n) \xrightarrow{p_{n*}} F(X_n, X_{n-1}) \xrightarrow{k_*^{n+1}} F(X_n, K(\pi_n(X), n+1)),$$

where $F(\ ,\ )$ denotes the function space. Then we obtain the following commutative diagram

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
\pi_1(F(X_n, X_{n-1}), p_n) & \xrightarrow{l_*} & \pi_1(F(X_n, X_{n-1(P)}), lp_n) \\
\downarrow\Psi & & \downarrow\Psi' \\
\pi_1(F(X_n, K), 0) & \xrightarrow{l_*} & \pi_1(F(X_n, K_{(P)}), 0) \\
\downarrow & & \downarrow \\
\pi_0(F(X_n, X_n), 1_{X_n}) & \xrightarrow{l_*} & \pi_0(F(X_n, X_{n(P)}), l) \\
\downarrow & & \downarrow
\end{array}$$

It is well known that $\operatorname{Im}\Psi$ ($\operatorname{Im}\Psi'$) coincides with the isotropy subgroup of the $\pi_1(F(X_n, K), 0) = H^n(X_n, K(\pi_n(X), n)$ action ($\pi_1(F(X_n, K_{(P)}), 0) = H^n(X_n, K(\pi_n(X_{(P)}), n))$ action) on $\pi_0(F(X_n, X_n), 1_{X_n})$ ($\pi_0(F(X_n, X_{n(P)}), l)$) at 1_{X_n} (l). We should note that these actions can be regarded as to be induced from the actions of $K(\pi_n(X), n)$ ($K(\pi_n(X_{(P)}), n)$) on X_n ($X_{n(P)}$). Thus if we restrict these actions to $H^n(X_{n-1};\pi_n(X))$ (or $H^n(X_{n-1};\pi_n(X_{(P)}))$), we get actions mentioned earlier. Let $i_N: X_n^N \to X_n$ be the inclusion of N-skeleton of X_n, with N sufficiently large. The space $F(X_n, X_{n-1}) = F(\varinjlim_j X_n^{N+j}, X_{n-1})$ is homotopy equivalent to $\varprojlim_j F(X_n^{N+j}, X_{n-1})$.

This gives rise to a homotopy equivalence of $(F(X_n, X_{n-1}), p_n)$ and $\varprojlim_j (F(X_n^{N+j}, X_{n-1}), p_n i_{N+j})$. We have a cofibration

$$V \to X_n^{N+j} \to X_n^{N+j+1},$$

where V is a wedge of $N+j$-spheres, giving rise to a fibration

$$(F(V, X_{n-1}), 0) \leftarrow (F(X_n^{N+j}, X_{n-1}), p_n i_{N+j}) \leftarrow (F(X_n^{N+j+1}, X_{n-1}), p_n i_{N+j+1}).$$

Since N is sufficiently large, $(F(V, X_{n-1}), 0)$ is weakly contractible, thus the total space and fibre of this fibration are weakly homotopy equivalent. As a result,

$$\begin{aligned}(F(X_n, X_{n-1}), p_n) &= \varprojlim (F(X_n^{N+j}, X_{n-1}), p_n i_{N+j}) \\ &\simeq (F(X_n^N, X_{n-1}), p_n i_N).\end{aligned}$$

Thus these are nilpotent spaces by Theorem 2.5, Ch. II, **[5]** and moreover the upper l_* in the above diagram localizes (Theorem 3.11, Ch. II, **[5]**) and so does the middle l_*. Therefore, $l_*\colon \operatorname{Im}\Psi \to \operatorname{Im}\Psi'$ localizes. We have the following.

$$\begin{aligned}I(1_{X_n})_{(P)} &= (\operatorname{Im}\Psi \cap H^n(X_{n-1}; \pi_n(X))_{(P)} \\ &= (\operatorname{Im}\Psi)_{(P)} \cap H^n(X_{n-1}; \pi_n(X)_{(P)}), \quad \text{by Theorem 1.2, [4]}, \\ &= \operatorname{Im}\Psi' \cap H^n(X_{n-1(P)}; \pi_n(X_{(P)})) \\ &= I(1_{X_{n(P)}}).\end{aligned}$$

This completes the proof of (II).

2. Proof of Theorem 0.2. In this section we prove Theorem 0.2. Again we use the induction on the Postnikov decomposition. First we introduce the following proposition.

PROPOSITION 2.1. *Let* $X = K(Q, 2n_k - 1) \times \cdots \times K(Q, 2n_1 - 1)$, $1 \le n_1 \le \cdots \le n_k$ *with* $\beta_{2n_i-1} - \operatorname{rank}_Q(\pi_{2n_i-1}(X)) \le 1$ *for* $i \le k$. *Then* $\mathscr{E}_\#(X) = Q \oplus \cdots \oplus Q$, *the direct sum of rationals of the rank (over* Q*)* $= \sum_{l=1}^k \operatorname{rank}_Q(\pi_{2n_l-1}(X)) \cdot (\beta_{2n_l-1} - \operatorname{rank}_Q(\pi_{2n_l-1}(X)))$.

Proof. On the first Postnikov stage, $\mathscr{E}_\#(X_{2n_1-1}) = 1$. Assume that $\mathscr{E}_\#(X_{2n_{l-1}-1})$ is an abelian group. By Theorem 2.10 of **[11]**,

$$\mathscr{E}_\#(X_{2n_l-1}) = H^{2n_l-1}(X_{2n_{l-1}-1}; \pi_{2n_l-1}(X)) \underset{T}{\times} E_\#(X_{2n_l-1}),$$

where $\underset{T}{\times}$ means the semidirect product. Moreover the action of f on ω ($f \in \mathscr{E}_\#(X_{2n_{l-1}-1})$, $\omega \in H^{2n_l-1}(X_{2n_{l-1}-1}; \pi_{2n_l-1}(X))$) is given by $f \cdot \omega = (f)^{*-1}\omega$. Each element ω can be written as the sum,

$$\sum a_{n_{i(1)}} \cup \cdots \cup a_{n_{i(m)}},$$

where $a_{n_{i(j)}}$ is the fundamental class of $H^{2n_{i(j)}}(K(Q, 2n_{i(j)} - 1); Q)$, $1 \leq i(j) \leq k$, $\cup$ means the cup product. Obviously, $(f)^{*-1}$ maps ω into ω identically. It follows that $f \cdot \omega = \omega$. What we have just proved is that $\mathscr{E}_\#(X_{2n_l-1})$ is actually the direct product of $\mathscr{E}_\#(X_{2n_{l-1}-1})$ and the cohomology group. The rank (over Q) of $\mathscr{E}_\#(X)$ can be computed as follows (cf. the proof of Theorem 5.4 **[1]**). Let ρ_l stand for the dimension of $\mathscr{E}_\#(X_{n_l-1})$.

$$\rho_l = \operatorname{rank}_Q(H^{2n_l-1}(X_{2n_{l-1}-1}; \pi_{2n_l-1}(X))) + \rho_{l-1}.$$

Therefore,

$$\begin{aligned}\operatorname{rank}_Q \mathscr{E}_\#(X) = \rho_k &= \sum_{l=2}^{k} \operatorname{rank}_Q(H^{2n_l-1}(X_{2n_{l-1}-1}; \pi_{2n_l-1}(X))) \\ &= \sum_{l=1}^{k} \operatorname{rank}_Q(\pi_{2n_l-1}(X)) \cdot (\beta_{2n_l-1} - \operatorname{rank}_Q(\pi_{2n_l-1}(X))).\end{aligned}$$

Proof of the theorem. By the above proposition, $\mathscr{E}_\#(X_0) =$ the direct sum of rationals. Thus $\mathscr{E}_\#(X)_{(0)} = \mathscr{E}_\#(X_{(0)})$ is also an abelian group. Recall that all torsion elements of a nilpotent subgroup form a normal subgroup. Then the injectivity of rationalization,

$$\mathscr{E}_\#(X)/\text{torsion} \to (\mathscr{E}_\#(X)/\text{torsion})_{(0)} = \mathscr{E}_\#(X)_{(0)},$$

implies that $\mathscr{E}_\#(X)/\text{torsion}$ is a free abelian group of the rank mentioned above (since we are assuming that X is of finite type, this group is finitely genrated).

3. Further application.

EXAMPLE 3.1 Let Π be the collection of all primes. We see that $\mathscr{E}_\#(\mathrm{SO}(6))_{(\Pi-2)} = \mathscr{E}_\#(\mathrm{SU}(4))_{(\Pi-2)}$. Let us denote this group by G. Then $G = G_{(3)} \times Z_5 \times Z_5$ and $G_{(3)}$ has the following presentation (cf. **[11]**).

$$G_{(3)} = \langle a, b, c \,|\, a^9, b^9, c^3, [a, b], [a, c], [b, c]a^{-3} \rangle.$$

Proof. $\mathrm{SO}(6) \simeq_p \mathrm{Spin}(6) = \mathrm{SU}(4)$ for an odd prime p, where $\simeq_p$ denotes the p-equivalence. Due to this equivalence a half part of the

Example 3.1 is obvious. To determine the group structure we recall the following theorem.

THEOREM (SIERADSKI, THEOREM 4, COROLLARY 8, [14]). *Let X and Y be homotopy associative H-spaces. If the homotopy set $[A\vee B, A\wedge B]$ is trivial, then there is a short exact sequence of multiplicative groups.*

$$(3.2)\qquad 1\to[X\wedge Y, X\times Y]\xrightarrow{q^{\#}+1}\mathscr{E}(X\times Y)\to \mathrm{GL}(2,\Lambda_{IJ})\to 1,$$

where $q\colon X\times Y\to X\wedge Y$ is the projection, for Λ_{IJ} see [14].

The group structure. Let us denote $\Pi-2$ by l. There is another l-equivalence: $\mathrm{SO}(6)\simeq_l \mathrm{SO}(5)\times S^5$. We first investigate the group $\mathscr{E}(\mathrm{SO}(5)_{(l)}\times S^5_{(l)})$. As it is well known, $\mathrm{SO}(5)_{(l)}$ is homotopy equivalent to $\mathrm{Sp}(2)_{(l)}$. $(\mathrm{SO}(5)\wedge S^5)_{(l)}\simeq(\mathrm{Sp}(2)\wedge S^5)_{(l)}\simeq(S^8\cup e^{12}\vee S^{15})_{(l)}$, the triviality of the attaching map of the top cell is due to Lemma 2.1, (ii), [7]. We use this cell structure to obtain (i) $[(\mathrm{SO}(5)\vee S^5)_{(l)}, (\mathrm{SO}(5)\wedge S^5)_{(l)}]=0$, (ii) $[\mathrm{SO}(5)_{(l)}, S^5_{(l)}]=0=[S^5_{(l)},\mathrm{SO}(5)_{(l)}]$. Therefore (3.2) ($X=\mathrm{SO}(5)_{(l)}, Y=S^5_{(l)}$), its restriction, has the following form.

$$(3.3)\quad 0\to[(S^8\cup e^{12})\vee S^{15}, \mathrm{Sp}(2)_{(l)}\times S^5_{(l)}]\to\mathscr{E}_{\#}(\mathrm{SO}(6)_{(l)})\to\mathscr{E}_{\#}^{(15)}(\mathrm{Sp}(2))_{(l)}\to 1.$$

The left-hand side term of this sequence is isomorphic to $Z_9\oplus Z_{45}$. Let us consider the 3-component. First we recall the result of [12]. Let λ be the map which is introduced in [12] (§1, 1.2):

$$\lambda\colon\pi_{10}(\mathrm{Sp}(2))\to\mathscr{E}(\mathrm{Sp}(2)).$$

Then a generator of $\mathscr{E}_{\#}(\mathrm{Sp}(2))_{(3)}$ $(=\mathscr{E}_{\#}(\mathrm{SO}(5))_{(3)})=Z_3$ can be represented by $\lambda(i\alpha_2)$, where $i\colon S^3\to\mathrm{Sp}(2)$ is the inclusion and $\alpha_2\in\pi_{10}(S^3)_{(3)}=Z_3$ is the generator [15]. Using the group structure of Sp(2), $\lambda(i\alpha_2)$ has the other description $1+i\alpha_2 p$ (Corollary 2.2, [8]), where p is the collapsing map to the top cell. The only nontrivial homotopy group $\pi_i(\mathrm{Sp}(2))_{(3)}$, $10<i\le 15$, is $\pi_{14}(\mathrm{Sp}(2))_{(3)}=Z_3$ and its generator is $i\alpha_3(3)$ [9], where $\alpha_3(3)\in\pi_{14}(S^3)_{(3)}=Z_3$, is the generator. Since $(1+i\alpha_2 p)(i\alpha_3(3))=i\alpha_3(3)$ it follows that $\mathscr{E}_{\#}^{15}(\mathrm{Sp}(2))_{(3)}=\mathscr{E}_{\#}(\mathrm{Sp}(2))_{(3)}$. Similarly, $\mathscr{E}_{\#}^{15}(\mathrm{Sp}(2))_{(5)}=\mathscr{E}_{\#}(\mathrm{Sp}(2))_{(5)}$.

A generator of the summand $Z_9=(Z_{45})_{(3)}$ on the left-hand side of (3.3) may be given by:

$$S^8\cup e^{12}\vee S^{15}\xrightarrow{r}S^8\cup e^{12}\xrightarrow{\tilde{\alpha}_1}S^5\xrightarrow{j}\mathrm{SO}(5)\times S^5,$$

where r is the retraction and $\tilde{\alpha}_1$ stands for the extension of $\alpha_1 \in \pi_8(S^5)_{(3)} = Z_3$ is the generator, j is the natural inclusion. The composition,

$$(3.4) \qquad (\lambda(-\alpha_2) \times 1)(1 + q^{\#}(j\tilde{\alpha}_1 r))(\lambda(\alpha_2) \times 1) \in \mathscr{E}(\mathrm{Sp}(2) \times S^5)$$

is homotopic to $(\lambda(-\alpha_2) \times 1)(\lambda(\alpha_2) \times 1 + (j\tilde{\alpha}_1 r)q(\lambda(\alpha_2) \times 1))$.

$(j\tilde{\alpha}_1 r)q(\lambda(\alpha_2) \times 1)$ can be easily seen to be $(j\tilde{\alpha}_1 r)(\lambda(\alpha_2) \wedge 1)$. By the definition of λ, $\lambda(\alpha_2) \wedge 1$ is homotopic to $1 + \Sigma^5(i\alpha_2 p)$. Hence $(j\tilde{\alpha}_1 r)(\lambda(\alpha_2) \wedge 1) = j\tilde{\alpha}_1 r + j\alpha_1\Sigma^5(\alpha_2 p)$. Hence (3.4) is equal to $1 + q^{\#}(j\tilde{\alpha}_1 r + j\alpha_1\Sigma^5(\alpha_2 p))$. At $\pi_{15}(S^5)$, $\alpha_1\alpha_2 = -3\beta_1(5)$ **[15]**, p. 180. The analogous argument permits us to show that the $(1 + q^{\#})$-image of the other Z_9-summand commutes with $\lambda(\alpha_2) \times 1$.

Put $c = \lambda(-\alpha_2) \times 1$, $b = 1 + q^{\#}(j(-\tilde{\alpha}_1(r))$, $a = 1 + q^{\#}j\beta_1(5)(\Sigma^5 p)$. These imply the assertion on the group structure of $G_{(3)}$. Since we see easily that the 5-components have no non-trivial extensions, we complete the proof.

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