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REGULARIZATION OF ACTIONS OF GROUPS AND GROUPOIDS ON MEASURED EQUIVALENCE RELATIONS

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Dedicated to H. A. Dye

The paper deals with the regularization problem for locally compact groups of non-strict automorphisms of measured equivalence relations. It is shown that by means of inessential reorganization of the equivalence relation and the group action one can make all the automorphisms of the given action to be strict with respect to the equivalence relation. A similar problem is solved for an action of a measure groupoid which leaves invariant mod 0 a measured equivalence relation.

1. The study of full groups introduced by H. A. Dye [**5**] as well as a consideration of outer conjugacy of subgroups of the full group normalizer (see the paper by A. Connes and W. Krieger [**4**] and the subsequent works [**1**], [**2**], [**3**], [**8**]) forces one to deal with maps and families of transformations that behave properly only almost everywhere. This as a rule does not cause any difficulties when the transformation groups in question are nonsingular and countable. One needs only to discard a null set in order to make the action of such a group regular. However, in the case of continuous transformation groups this method of regularization is inappropriate in general.

The first step in studying the regularization of actions of continuous groups was made by G. Mackey in [**10**], where the existence and uniqueness of a point realization for an action of a locally compact group G as automorphisms of a Boolean algebra were established. In order to form the point realization of a Boolean G-space, the author applies the properties of a universal G-space on which the G-action is regular. This approach got its complete basis later in works of A. Ramsay [**12**], [**14**]. Similar problems were considered by A. M. Vershik [**15**] with another technique used.

The present paper contains a solution of the regularization problem for groups of non-strict automorphisms of measured equivalence relations based on results of [**12**], [**14**] (§2). It is shown that by means of inessential reorganization of the equivalence relation and the group action one can make all the automorphisms of the group action to be strict automorphisms of the equivalence relation. This allows us, in particular, to formulate in the general case the notion of semidirect product of a measure groupoid by a locally compact group of non-strict automorphisms. A similar question arose in studying the outer conjugacy for actions of continuous groups [**8**, §2], where its solution was obtained for the case of discrete equivalence relations.

A similar problem is solved in §3 for action of a measure groupoid which leaves invariant mod 0 a measured equivalence relation. Another approach to this kind of problem is described in the paper by A. L. Fedorov [**6**].

2. All the necessary definitions concerning the measured equivalence relations can be found in [**11**].

Let (X, μ) be a Lebesgue probability space on which a nonsingular action of a locally compact separable (l.c.s.) group G by the automorphisms $\alpha(g)$, $g \in G$, is given in such a way that the map $(g, x) \mapsto \alpha(g)x$ is Borel ($x \in X$, $g \in G$). Assume also that every automorphism $\alpha(g)$ leaves invariant mod 0 the measured equivalence relation $(R, [\nu])$ on X, i.e. $\alpha(g)$ is a strict isomorphism of some inessential reductions (i.r.) of R.

It follows from [7, Theorem 6.4] that R is generated mod 0 by an action of a l.c.s. group H on X by the automorphisms $\beta(h)$, $h \in H$.

For every $g \in G$ consider the H-action β_g on X by the automorphisms $\beta_g(h) = \alpha(g)\beta(h)\alpha(g)^{-1}$. Denote also by $R_\beta(g)$ the associated Borel equivalence relation on X; in particular, $R_\beta(e) = R$. Clearly, every $\beta_g(h)$ is an (a.e.) inner automorphism of $R_\beta(e)$.

Recall that the family $B(g)$, $g \in G$, of subsets of X is called a Borel field of sets if the set $\{(x, g) \colon x \in B(g)\}$ is Borel in $X \times G$.

LEMMA 2.1. *There exist a Borel field of equivalence relations $R(g)$, $g \in G$, on X and a Borel field of conull sets $B(g) \subset X$ such that $R(g)$ is strictly invariant with respect to the action β_g, and $R_\beta(e)|_{B(g)} = R(g)|_{B(g)}$.*

Proof. Since every transformation $\beta_g(h)$ is inner with respect to $R_\beta(e)$, for all $(g, h) \in G \times H$ the strictness domain

$$U_h^g = \{x \in X \colon (\beta_g(h)x, x) \in R_\beta(e)\}$$

of the automorphism $\beta_g(h)$ is a Borel set of μ-measure 1. This implies that for every $g \in G$ the Borel set

$$A_g = \{(h, x) \in H \times X \colon (\beta_g(h)x, x) \in R_\beta(e)\}$$

is $\mu_H \times \mu$-conull, with μ_H being the Haar measure of H. Hence by Fubini's theorem the Borel set

$$M_x^g = \{h \in H: (\beta_g(h)x, x) \in R_\beta(e)\}$$

is conull in H at every fixed $g \in G$ for a.a. $x \in X$. Furthermore, since the set

$$A = \{(g, h, x) \in G \times H \times X: (\beta_g(h)x, x) \in R_\beta(e)\}$$

is Borel, it follows from the Borel nature of the map

$$(g, x) \mapsto \mu_H(\{h \in H: (g, h, x) \notin A\})$$

[9, §35, Theorem 1] that $B(g) = \{x \in X: \mu_H(H \backslash M_x^g) = 0\}$ form a Borel field of conull sets.

Associate to every pair $(x, y) \in X \times X$ and $g \in G$ the set

$$L_g(x, y) = \{h \in H: (\beta_g(h)x, \beta_g(h)y) \in R_\beta(e)\}.$$

This is a Borel field of sets because the set

$$K = \{(g, h, x, y) \in G \times H \times X \times X: (\beta_g(h)x, \beta_g(h)y) \in R_\beta(e)\}$$

is clearly Borel. It is straightforward to check the following relations:

(i) $L_g(x, x) = H$;
(ii) $L_g(x, y) = L_g(y, x)$;
(iii) $L_g(x, z) \supset L_g(x, y) \cap L_g(y, z)$;
(iv) $L_g(\beta_g(h)x, \beta_g(h)y) = L_g(x, y)h^{-1}$.

Now for every $g \in G$ form the set of pairs

$$R(g) = \{(x, y) \in X \times X: \mu_H(H \backslash L_g(x, y)) = 0\}.$$

It follows from the Borel nature of K and [9, §35, Theorem 1] that $R(g)$ form a Borel field of sets.

The relations (i)–(iii) imply that every $R(g)$ is an equivalence relation on X. (iv) ensures strict invariance of $R(g)$ with respect to the H-action β_g.

Now let $x, y \in B(g)$. If $(x, y) \in R_\beta(e)$, then $L_g(x, y) \supset M_x^g \cap M_y^g$, and hence $\mu_H(H \backslash L_g(x, y)) = 0$. On the contrary, if $(x, y) \notin R_\beta(e)$, then $H \backslash L_g(x, y) \supset M_x^g \cap M_y^g$. This means exactly that $R_\beta(e)|_{B(g)} = R(g)|_{B(g)}$.

Lemma 2.2. *There is a Borel field of sets $U(g) \subset X$, $g \in G$, such that*

(a) $\mu(U(g)) = 1$ *for all* $g \in G$;

(b) *$U(g)$ is invariant with respect to* $\alpha(g)^n$, $n \in \mathbf{Z}$;
(c) $\alpha(g)R_\beta(e)|_{U(g)} = R_\beta(e)|_{U(g)}$.

Proof. It follows from Lemma 2.1 that $\beta_g(h)$ is an inner automorphism of $R(g)$ for all $h \in H$; hence every

$$V_h^g = \{x \in X : (\beta_g(h)x, x) \in R(g)\}$$

is conull in X. Thus for every $g \in G$ the Borel set

$$V^g = \{(h, x) \in H \times X : (\beta_g(h)x, x) \in R(g)\}$$

is conull in $H \times X$. Now we apply as above Fubini's theorem in order to deduce that for every $g \in G$ the set

$$E_x^g = \{h \in H : (\beta_g(h)x, x) \in R(g)\}$$

is conull in H when x is in the conull subset

$$\mathscr{D}(g) = \{x \in X : \mu_H(H \backslash E_x^g) = 0\}$$

of X. Since the set

$$V = \{(g, h, x) \in G \times H \times X : (\beta_g(h)x, x) \in R(g)\}$$

is Borel, $\mathscr{D}(g)$ form a Borel field of sets [**9**, §35, Theorem 1].

One can readily deduce from the strict invariance of $R(g)$ with respect to the H-action β_g that every E_x^g is a subgroup in H, and so $E_x^g = H$ when $x \in \mathscr{D}(g)$. This means, in particular, that $R_\beta(g)|_{\mathscr{D}(g)} \subset R(g)|_{\mathscr{D}(g)}$.

Let

$$U'(g) = B(g) \cap B(g^{-1}) \cap \mathscr{D}(g) \cap \mathscr{D}(g^{-1}),$$

and then $U(g) = \bigcap_{n \in \mathbf{Z}} \alpha(g)^n U'(g)$. It is straightforward to check that $U(g)$ form a Borel field of sets and satisfy the conditions (a)–(b) of the lemma.

To prove that (c) also holds, let $x, y \in U(g)$ and $(x, y) \in R_\beta(e)$, that is, $y = \beta_e(h)x$ for some $h \in H$. Then

$$\alpha(g)y = \alpha(g)\beta_e(h)x = \beta_g(h)\alpha(g)x, \quad \text{i.e. } (\alpha(g)x, \alpha(g)y) \in R_\beta(g),$$

and therefore $(\alpha(g)x, \alpha(g)y) \in R(g)$ because $\alpha(g)x, \alpha(g)y \in \mathscr{D}(g)$. Note that one also has $\alpha(g)x$, $\alpha(g)y \in B(g)$, and hence $(\alpha(g)x, \alpha(g)y) \in R_\beta(e)$ (see Lemma 2.1).

Conversely, let $x, y \in U(g)$, but $(x, y) \notin R_\beta(e)$. We claim that $(\alpha(g)x, \alpha(g)y) \notin R_\beta(e)$. In fact, if $(\alpha(g)x, \alpha(g)y) \in R_\beta(e)$, then we argue as in the preceding paragraph with g being replaced by g^{-1}, and get $(x, y) \in R_\beta(e)$. This is a contradiction with our assumption.

THEOREM 2.3. *There exist a Borel equivalence relation R_α on X and a conull Borel set $B \subset X$ such that R_α is strictly invariant with respect to the G-action α, and $R_\alpha|_B = R_\beta(e)|_B$.*

Proof is essentially the same as that of Lemma 2.1. We shall sketch only the main stages. The Borel field of strictness domains $U(g)$ of the G-action α automorphisms constructed above is used to form the $\mu_G \times \mu$-conull Borel set $A = \{(g, x) \in G \times X : x \in U(g)\}$ with μ_G being the Haar measure of G. An application of Fubini's theorem permits one to choose a conull Borel subset B of X consisting of those points x for which $M_x = \{g \in G : x \in U(g)\}$ is a conull subset of G. Now form the family of sets

$$L(x, y) = \{g \in G : (\alpha(g)x, \alpha(g)y) \in R\}, \qquad (x, y) \in X \times X,$$

which possesses the following properties:

(i) $L(x, x) = G$ for all $x \in X$;
(ii) $L(x, y) = L(y, x)$, $x, y \in X$;
(iii) $L(x, z) \supset L(x, y) \cap L(y, z)$, $x, y, z \in X$;
(iv) $L(\alpha(h)x, \alpha(h)y) = L(x, y)h^{-1}$, $x, y \in X$, $h \in G$.

Finally, form an equivalence relation

$$R_\alpha = \{(x, y) \in X \times X : \mu_G(G \backslash L(x, y)) = 0\},$$

which satisfies all the necessary conditions.

3. We shall state here the main definitions concerning the actions of groupoids on Lebesgue spaces (see also **[6]**, **[12]**, **[13]**, **[14]**).

DEFINITION 3.1. An action of a groupoid $\mathcal{G}$ on the set X is a pair (p, a) where $p : X \to \mathcal{G}^{(0)}$ is a surjective map and a a map of the set $\mathcal{G} * X = \{(g, x) \in \mathcal{G} \times X : d(g) = p(x)\}$ into X, with the following conditions being satisfied: if $(g, x) \in \mathcal{G} * X$, and $(h, g) \in \mathcal{G}^{(2)}$, then

(i) $p(a(g, x)) = r(g)$,
(ii) $a(hg, x) = a(h, a(g, x))$.

In the case when $\mathcal{G}$ is a Borel groupoid, and X a Borel space, the action (p, a) is said to be Borel if p and a are Borel maps.

In the sequel we shall delete the action symbol a and write merely gx instead of $a(g, x)$.

Let $(\mathcal{G}, [\lambda])$ be a groupoid, and let the probability measure λ on $\mathcal{G}$ have the decomposition $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ with respect to d. Consider also a Lebesgue probability space (X, μ) where μ has the decomposition $\mu = \int \mu_u \, d\tilde{\mu}(u)$ with respect to the Borel surjection $p : X \to \mathcal{G}^{(0)}$.

DEFINITION 3.2. A pair (p, a) is called an action of a measure groupoid $(\mathscr{G}, [\lambda])$ on a Lebesgue space (X, μ) if

(i) the maps p and a are Borel;
(ii) the measures $\tilde{\mu}$ and $\tilde{\lambda}$ are equivalent;
(iii) there is an i.r. $\mathscr{G}|_{U_0}$ of the groupoid $\mathscr{G}$ such that the pair $(p|_{p^{-1}(U_0)}, a|_{\mathscr{G}|_{U_0} * p^{-1}(U_0)})$ is an action of the groupoid $\mathscr{G}|_{U_0}$ in the sense of Definition 3.1 on the set $p^{-1}(U_0)$;
(iv) the measures $g\mu_{d(g)}$ and $\mu_{r(g)}$ are equivalent for λ-a.e. $g \in \mathscr{G}$.

Let $(R, [\nu])$ be a measured equivalence relation on (X, μ). Suppose that ν admits the decomposition $\nu = \int \nu_x \, d\tilde{\nu}(x)$ with ν_x being supported on the equivalence class of x and $\tilde{\nu} \sim \mu$. Consider also the equivalence relation $\mathscr{E} = \{(x, y) \colon p(x) = p(y)\}$ on X.

DEFINITION 3.3. An equivalence relation $(R, [\nu])$ is said to be invariant with respect to the action (p, a) of the groupoid $(\mathscr{G}, [\lambda])$ if

(i) $R \subset \mathscr{E}$;
(ii) given any $g \in \mathscr{G}$ and $x, y \in X$ such that $p(x) = p(y) = d(g)$, then the conditions $(x, y) \in R$ and $(gx, gy) \in R$ are equivalent.

DEFINITION 3.4. An equivalence relation $(R, [\nu])$ is said to be invariant mod 0 with respect to the action (p, a) of the groupoid $(\mathscr{G}, [\lambda])$ if

(i) there is a conull Borel set $A \subset X$ such that $R|_A \subset \mathscr{E}|_A$;
(ii) for a.e. $g \in \mathscr{G}$ there is a $\mu_{d(g)}$-conull Borel subset $U(g) \subset p^{-1}(d(g))$ such that for $x, y \in U(g)$ the conditions $(x, y) \in R$ and $(gx, gy) \in R$ are equivalent.

There is certainly some ambiguity in the choice of the family of strictness domains $U(g)$.

LEMMA 3.5. *Let an action (p, a) of the groupoid $(\mathscr{G}, [\lambda])$ on (X, μ) which leaves invariant* mod 0 *the equivalence relation $(R, [\nu])$ be given, then there exists a Borel field $U(g)$ of strictness domains.*

Proof. Replace if necessary $\mathscr{G}$ by its i.r. $\mathscr{G}|_{p(A)}$ and X respectively by its conull subspace A in order to get the inclusion $R \subset \mathscr{E}$ to be satisfied strictly, not only mod 0.

Consider the measure $\tilde{\omega} = \int \mu_{d(g)} \, d\lambda(g)$ on $\mathscr{G} * X$. The Borel automorphism $\phi \colon \mathscr{G} * X \to \mathscr{G} * X$, $\phi(g, x) = (g^{-1}, gx)$ preserves the measure class of $\tilde{\omega}$, as one can easily see from Definition 3.2 (iv) and the symmetry of the measure class $[\lambda]$. Moreover, a routine verification shows that $\phi \circ \phi = \mathrm{id}$.

Impose on $\mathcal{G} * X$ the Borel equivalence relation

$$\Omega_0 = \{((g,x),(h,y)) \colon g = h,\ (x,y) \in R\}.$$

Then the measure $\omega = \iint \nu_x \, d\mu_{d(g)}(x) \, d\lambda(g)$ on $(\mathcal{G} * X)^2$ is concentrated on Ω_0. (Here ν_x are the conditional measures in the decomposition of ν as described above.)

The map ϕ may be lifted to a Borel automorphism

$$\overline{\phi} \colon (\mathcal{G} * X)^2 \to (\mathcal{G} * X)^2,$$
$$\overline{\phi}((g,x),(h,y)) = ((g^{-1}, gx), (h^{-1}, hy)).$$

Set $\Omega = \Omega_0 \cap \overline{\phi}(\Omega_0)$; then Ω is a Borel equivalence relation contained in Ω_0. Moreover, since for a.a. $g \in \mathcal{G}$ $\overline{\phi}(\Omega_0|_{U(g)}) = \Omega_0|_{U(g)}$ (see Definition 3.4), one has for these g $\int \nu_x(\Omega) \, d\mu_{d(g)}(x) = 1$, and hence $\omega(\Omega) = 1$.

Thus, Ω is a conull subgroupoid of Ω_0, and hence by **[12**, Lemma 5.2] contains an i.r. $\Omega_0|_{V_0}$ of Ω_0 onto some $\tilde{\omega}$-conull Borel subset V_0 of $\mathcal{G} * X$. Let $V = V_0 \cap \phi(V_0)$; then V not only possesses the same properties as V_0, but is also invariant with respect to ϕ, and hence the reduction $\Omega_0|_V$ is invariant with respect to $\overline{\phi}$. Set $V(g) = \{x \in p^{-1}(d(g)) \colon (g,x) \in V\}$.

$V(g)$ form a Borel field of sets and work as strictness domains for elements of $\mathcal{G}$.

We shall denote below a Borel field of strictness domains by $U(g)$, $g \in \mathcal{G}$, just as in Definition 3.4.

THEOREM 3.6. *Let $(\mathcal{G}, [\lambda])$ be a measure groupoid which acts on a probability Lebesgue space (X, μ) and leaves invariant* mod 0 *a measured equivalence relation $(R, [\nu])$ on X. Then there exists a Borel equivalence relation $\tilde{R}$ on X which*

(a) *coincides with R on some conull Borel set $B \subset X$;*
(b) *is invariant (strictly, and not only* mod 0*) with respect to the action (of some i.r.) of $(\mathcal{G}, [\lambda])$.*

Proof. Consider the subset $U = \{(g,x) \in \mathcal{G} * X \colon x \in U(g)\}$ of $\mathcal{G} * X$. By Lemma 3.5 one may assume that U is Borel, and $\tilde{\omega}(U) = 1$ since for a.a. $g \in \mathcal{G}$, $\mu_{d(g)}(U(g)) = 1$. Note that the measure $\tilde{\omega} = \int \mu_{d(g)} \, d\lambda(g)$ may be written in the form

$$d\tilde{\omega}(u, g, x) = d\mu_u(x) \, d\lambda_u(g) d\tilde{\lambda}(u).$$

It is equivalent to the measure

$$d\mu_u(x)\,d\lambda_u(g)\,d\tilde{\mu}(u) = d\lambda_{p(x)}(g)\,d\mu(x)$$

since $\tilde{\lambda} \sim \tilde{\mu}$. Thus, if we set

$$M_x = \{g \in d^{-1}(p(x)) \subset \mathcal{G} \colon x \in U(g)\},$$

then $\lambda_{p(x)}(M_x) = 1$ for all x in some conull Borel set $B \subset X$.

Associate with each pair $(x, y) \in \mathcal{E}$ the set

$$L(x, y) = \{g \in d^{-1}(p(x)) \subset \mathcal{G} \colon (gx, gy) \in R\}.$$

$L(x, y)$ form a Borel field of sets since the set

$$C = \{(g, x, y) \in \mathcal{G} \times X \times X \colon d(g) = p(x) = p(y),\ (gx, gy) \in R\}$$

is obviously Borel. Furthermore, $L(x, y)$ possess the following evident properties:

(i) $L(x, x) = d^{-1}(p(x))$ for all $x \in X$;
(ii) $L(x, y) = L(y, x)$ for all $(x, y) \in \mathcal{E}$;
(iii) $L(x, z) \supset L(x, y) \cap L(y, z)$ for all $(x, y), (y, z) \in \mathcal{E}$;
(iv) $L(gx, gy) = L(x, y)g^{-1}$ for all $(x, y) \in \mathcal{E}$, $g \in d^{-1}(p(x)) \subset \mathcal{G}$.

It follows from the decomposition of measures that the function $f \colon \mathcal{E} \to \mathbf{R}$, $f(x, y) = \lambda_{p(x)}(L(x, y))$ is Borel, and hence so is the set $\tilde{R} = \{(x, y) \in \mathcal{E} \colon \lambda_{p(x)}(L(x, y)) = 1\}$.

Properties (i)–(iii) of $L(x, y)$ mean that $\tilde{R}$ is an equivalence relation. Note that for the measure groupoid $(\mathcal{G}, [\lambda])$ one can choose an inessential reduction onto a Borel subset $U_0 \subset \mathcal{G}^{(0)}$ such that $g\lambda_{d(g)} \sim \lambda_{r(g)}$ for all $g \in \mathcal{G}|_{U_0}$ **[12**, Lemma 2.4]. Replace now $\mathcal{G}$ by its i.r. $\mathcal{G}|_{U_0}$ and respectively X by its conull subspace $p^{-1}(U_0)$. Then (iv) implies strict invariance of $\tilde{R}$ with respect to the action of $(\mathcal{G}, [\lambda])$.

Let $x, y \in B$, and $(x, y) \in \mathcal{E}$. If $(x, y) \in R$, then it is straightforward to check that $L(x, y) \supset M_x \cap M_y$, and hence $\lambda_{p(x)}(L(x, y)) = 1$. On the contrary, if $(x, y) \notin R$, then one has $d^{-1}(p(x)) \backslash L(x, y) \supset M_x \cap M_y$, and so $\lambda_{p(x)}(L(x, y)) = 0$. This means exactly that $\tilde{R}|_B = R|_B$, which was to be proved.

The theorem we have just proved implies a result on point realization of groupoid homomorphisms into the automorphism group of measured equivalence relations.

Let $\alpha \colon \mathcal{G} \to \mathrm{Aut}(R)$ be a homomorphism of a measure groupoid $(\mathcal{G}, [\lambda])$ into the automorphism group of a measured equivalence relation $(R, [\nu])$ on the Lebesgue space (X, μ). Denote by $\mathcal{B}(X)$ the Boolean algebra of sets on (X, μ) **[10]**, **[12]**. The natural inclusion map

$\mathrm{Aut}(R) \to \mathrm{Aut}\,\mathscr{B}(X)$ permits one to associate with α a homomorphism $\tilde{\alpha}\colon \mathscr{G} \to \mathrm{Aut}\,\mathscr{B}(X)$. The homomorphism $\alpha\colon \mathscr{G} \to \mathrm{Aut}(R)$ is said to be Borel if for every $b \in \mathscr{B}(X)$ the map $\mathscr{G} \to \mathscr{B}(X)$, $g \mapsto \alpha(g)b$, is Borel.

DEFINITION 3.7. A homomorphism $\alpha\colon \mathscr{G} \to \mathrm{Aut}(R)$ is said to have a point realization if there exists an action (p, a) of the groupoid $(\mathscr{G}, [\lambda])$ on the space $\mathscr{G}^{(0)} \times X$ such that

(i) $p(u, x) = u$;

(ii) the action (p, a) leaves invariant some Borel equivalence relation $\tilde{R}$ on $\mathscr{G}^{(0)} \times X$ which coincides mod 0 with the equivalence relation

$$\{((u,x),(v,y)) \in (\mathscr{G}^{(0)} \times X)^2\colon u = v,\ (x,y) \in R\};$$

(iii) for a.a. $g \in \mathscr{G}$ the map $a(g,\cdot)\colon d(g) \times X \to r(g) \times X$ represents the class $\alpha(g)$ in $\mathrm{Aut}(R)$.

THEOREM 3.8. *Every Borel homomorphism $\alpha\colon \mathscr{G} \to \mathrm{Aut}(R)$ of a measure groupoid $(\mathscr{G}, [\lambda])$ into the automorphism group of a measured equivalence relation $(R, [\nu])$ admits a point realization (see also* **[6]**).

Proof. Apply the same argument as in the proofs of Theorem 3.3 and Lemma 3.2 of **[12]** and the construction of a universal $\mathscr{G}$-space in order to get the Borel map $g\colon \mathscr{G} \times X \to X$ which has the property $g(h_1h_2, x) = g(h_1, g(h_2, x))$ for $(h_1, h_2) \in \mathscr{G}^{(2)}$ and such that the automorphism $g(h,\cdot)$ of (X, μ) represents the class $\alpha(h)$ in $\mathrm{Aut}(R)$, $h \in \mathscr{G}$.

Consider the action (p, a) of $\mathscr{G}$ on $\mathscr{G}^{(0)} \times X$ with $p\colon \mathscr{G}^{(0)} \times X \to X$ being the projection, $p(u, x) = u$, and the map $a\colon \mathscr{G} * (\mathscr{G}^{(0)} \times X) \to \mathscr{G}^{(0)} \times X$ being given by $a(h, d(h), x) = (r(h), g(h, x))$. Clearly, the action (p, a) leaves invariant mod 0 the equivalence relation

$$\{((u,x),(v,y)) \in (\mathscr{G}^{(0)} \times X)^2\colon u = v,\ (x,y) \in R\}.$$

Apply now Theorem 3.6 and obtain the desired statement.

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