SOME AFFINE GEOMETRIC ASPECTS
OF OPERATOR ALGEBRAS

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Dedicated to the memory of Henry Abel Dye

The predual of a von Neumann algebra is shown to be a neutral strongly facially symmetric space, thereby suggesting an affine geometric approach to operator algebras and their non-associative analogues. Geometric proofs are obtained for the polar decompositions of normal functionals in ordered and non-ordered settings.

A fundamental problem in the operator algebraic approach to quantum mechanics is to determine those algebraic structures in Banach spaces which are characterized by a set of geometrical axioms defining the quantum mechanical measuring process. This problem was solved in the context of *ordered* Banach spaces by Alfsen, Hanche-Olsen, and Shultz ([2], [1]) and led to a characterization of the state spaces of JB^*-algebras and C^*-algebras. The main thrust of the present authors' recent research has been to find those algebraic structures induced on (unordered) Banach spaces in which such quantum mechanical axioms are satisfied. This project, which was initiated in [14] and [15] using the affine geometry of the dual unit ball, is used here to give a geometric proof of the Tomita-Sakai-Effros polar decomposition of a normal functional on a von Neumann algebra.

Thus, the purpose of this partially expository paper is to show the richness and power of the affine geometric structure of the dual space of an operator algebra, by working in a purely geometric model. Indeed, since this geometry can be described in terms of the underlying real structure, it can be used to obtain new results in the real structure of operator algebras and in the structure of real operator algebras. For example, by using this approach, Dang ([5]) has shown that a *real*-linear isometry of a C^*-algebra is the sum of a linear and a conjugate linear isometry, and hence is multiplicative, thereby obtaining a real analogue of Kadison's non-commutative extension of the Banach-Stone Theorem.

The category of strongly facially symmetric (SFS) spaces (simply called facially symmetric spaces in [14] and [15]) has been shown to

be an appropriate vehicle for the study of this problem. Strongly facially symmetric spaces include the preduals of von Neumann algebras (cf. §2 below), and more generally of JBW^*-triples (cf. §3 below), and can therefore serve as a geometric order-free model in which to study operator algebras and their non-associative analogues. This project may also be viewed as an attempt to place the characterizations of von Neumann algebras and JBW-algebras in terms of self-dual homogeneous cones ([4], [19]) in a context which is free of a global order structure.

A discrete spectral theorem for an arbitrary element in the dual of a reflexive SFS space was obtained in [14] by using the basic notions of *orthogonality, projective unit, norm exposed face, symmetric face, generalized tripotent, and generalized Peirce decomposition*, which were introduced and developed in this purely geometric setting. Moreover, a one-to-one correspondence was established between generalized tripotents and norm exposed faces, analogous to the correspondence between projections in a von Neumann algebra and norm closed faces in its normal state space (cf. [9]). This constitutes an important link between geometry and algebra in this context.

We next discuss the notion of compatibility. It is often useful to decompose a space simultaneously with respect to a family of individual decompositions. It is therefore important to know conditions under which this joint decomposition does not depend on the order, i.e. when the corresponding projections all commute. In this case we say that the members of the family are *compatible*.

In the globally ordered case, faces correspond to idempotents, and compatibility implies commutativity of the idempotents. Therefore, it is not possible to study non-commutative phenomena in this framework without abandoning compatibility, and thus increasing significantly the complexity of the problem. On the other hand, by dropping the requirement of a global order structure, it has been possible to describe non-commutative objects by using compatible families. This is illustrated by the proof of the classification of JBW^*-triple factors of type I in [6].

In §1 we review the basic affine geometric notions needed in the proof of the geometric polar decomposition ([15, Theorem 4.3]), and give necessary and sufficient conditions for compatibility (Theorem 1.2), considerably sharpening one of the main results of [15]. In §2 we develop enough elementary machinery of operator algebras in order to show that the predual of a von Neumann algebra is a neutral strongly

facially symmetric space (Theorem 2.11). It is then a simple matter to combine Theorems 1.3 and 2.11 to obtain a geometric proof of the Tomita-Sakai-Effros polar decomposition. A generalization of the results of §2 appears in §3, which introduces the JB^*-triples.

1. Compatibility in weakly facially symmetric spaces. In this section we shall give several necessary and sufficient conditions for compatibility of elements in a purely geometric setting. A consequence of this result constitutes one step in the proof of the geometric polar decomposition ([**15**, Theorem 4.3]), which we state here. All other definitions from [**14**] and [**15**] which are needed for our proof of compatibility will be recalled here for the convenience of the reader.

We shall say that elements f and g of a normed space Z are *orthogonal* if they satisfy

$$\|g + f\| = \|g - f\| = \|g\| + \|f\|,$$

or if one of them is zero (cf. [**14**, §1] or [**15**, §1]. We shall write $f \diamond g$ to indicate that f and g are orthogonal.

Let K be a convex set. A *face* of K is a non-empty convex subset F of K with the following property: if $f \in F$ and $g, h \in K$ satisfy $f = \lambda g + (1 - \lambda)h$ for some $\lambda \in (0, 1)$, then $g, h \in F$. An important example is: $K = Z_1 =$ the unit ball of Z and $F = \{f \in K : f(x) = 1\}$ for some element $x \in Z^*$ of norm 1. We shall denote this set F which is either empty or a face (called a *norm exposed face*), by F_x. Note that $F_x \cap F_y = F_z$ where $z = (x + y)/2$. The following is immediate from the definitions.

REMARK 1.1. *Let T be a linear isometry of a normed space Z onto itself. For each norm exposed face F_x in Z_1,*

$$T^{-1}(F_x) = F_{T^*x}.$$

We say that Z is *facially linearly complemented* if it is a real or complex normed space in which the orthogonal complement

$$F^\diamond = \{g \in Z : g \diamond f, \forall f \in F\}$$

of every norm exposed face F is a linear subspace. An element $u \in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $\langle u, F_u^\diamond \rangle = 0$. Let $\mathscr{F}$ and $\mathscr{U}$ denote the collections of norm exposed faces of Z_1 and projective units in Z^*, respectively. The map

$$\mathscr{U} \ni u \mapsto F_u \in \mathscr{F}$$

is not onto in general ([**14**, p. 269]). In order to obtain the bijection between distinguished subsets of $\mathscr{F}$ and $\mathscr{U}$, we need to recall the definitions of symmetric face and generalized tripotent.

Motivated by measuring processes in quantum mechanics, we define a *symmetric face* to be a norm exposed face F in Z_1 with the following property: there is a linear isometry S_F of Z onto Z, with $S_F^2 = I$, such that the fixed point set of S_F is $(\overline{\mathrm{sp}}F) \oplus F^\diamond$ (topological direct sum). In particular, $F^\diamond$ is a closed linear space.

For each symmetric face F we define contractive projections $P_k(F)$, $k = 0, 1, 2$ on Z as follows. First $P_1(F) = \frac{1}{2}(I - S_F)$ is the projection on the -1 eigenspace of S_F. Next we define $P_2(F)$ and $P_0(F)$ as the projections of Z onto $\overline{\mathrm{sp}}F$ and $F^\diamond$ respectively, so that $P_2(F) + P_0(F) = \frac{1}{2}(I + S_F)$ is the projection on the $+1$ eigenspace of S_F. These projections are called *generalized Peirce projections*. Note that

$$P_2(F) + P_1(F) + P_0(F) = I \quad \text{and} \quad S_F = P_2(F) - P_1(F) + P_0(F).$$

A real or complex normed space Z is said to be *weakly facially symmetric* (WFS) if every norm exposed face in Z_1 is symmetric. Hence a WFS space is facially linearly complemented.

A *generalized tripotent* is a projective unit $u \in \mathscr{U}$ with the property that F_u is a symmetric face and $S_{F_u}^* u = u$. The importance of this concept is explained by the following. Denote by $\mathscr{GT}$ and $\mathscr{SF}$ the collections of generalized tripotents and symmetric faces respectively. According to [**15**, Proposition 1.6], for any WFS space Z, the map

$$(1) \qquad\qquad \mathscr{GT} \ni u \mapsto F_u \in \mathscr{SF}$$

is a bijection of the set of generalized tripotents and the set of symmetric faces.

We now define the orthogonality of generalized tripotents and other elements of the dual space Z^* of a WFS space Z. Elements $a, b \in Z^*$, Z any normed space, are *orthogonal*[1] if there is a symmetric face $F \subset Z_1$ such that either

    1. $a \in P_2(F)^*(Z^*)$ and $b \in P_0(F)^*(Z^*)$; or

    2. $a \in P_0(F)^*(Z^*)$ and $b \in P_2(F)^*(Z^*)$.

We shall write $a \diamond b$ or $b \diamond a$ to indicate this relation.

We next recall the notion of neutrality. A contractive projection Q on a normed space X is said to be *neutral* if for each $\xi \in X$, $\|Q\xi\| = \|\xi\|$ implies $Q\xi = \xi$. A normed space Z is *neutral* if for every

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[1] Later we shall use the term orthogonal in connection with elements of a C^*-algebra. By Theorem 2.11, the two definitions are consistent.

symmetric face F, the projection $P_2(F)$ corresponding to some choice of symmetry S_F, is neutral. An interesting property of neutrality is that in a neutral WFS space, a symmetry S_F as well as the generalized Peirce projections, are uniquely determined by the norm exposed face F ([**15**, Theorem 2.4]).

We next define the notion of compatibility and prove the important characterization theorem. For an analogous result in a purely algebraic setting, see [**23**].

Two generalized tripotents u and v are said to be *compatible* if their generalized Peirce projections commute, i.e.

$$[P_k(F_u), P_j(F_v)] = 0 \quad \text{for } k, j \in \{0, 1, 2\}.$$

The following theorem gives a characterization of compatibility which sharpens a previous result of the authors. For convenience, we adopt the following notations, where u is a generalized tripotent in the dual of a WFS space Z: $P_k(u) = P_k(F_u)$, $k = 0, 1, 2$; $U = Z^*$, $Z_k(u) = P_k(u)(Z)$; $U_k(u) = P_k(u)^*(U)$; $S_u = S_{F_u}$. Also, the inverse of the map (1) will be denoted by $F \mapsto v_F$.

THEOREM 1.2. *Let Z be a neutral WFS space, and let $u, v \in \mathcal{GJ}$. The following are equivalent.*
(1) *u and v are compatible;*
(2) *$S_v^* u \in U_2(u)$;*
(2′) *$S_u^* v \in U_2(v)$;*
(3) *$u = u_2 + u_1 + u_0$ with $u_j \in U_j(v) \cap U_2(u)$, $j \in \{0, 1, 2\}$;*
(3′) *$v = v_2 + v_1 + v_0$ with $v_j \in U_j(u) \cap U_2(v)$, $j \in \{0, 1, 2\}$.*

Proof.
(1) $\Rightarrow$ (3): $P_i(v)^* u = P_i(v)^* P_2(u)^* u = P_2(u)^* P_i(v)^* u \in U_i(v) \cap U_2(u)$.

(3) $\Rightarrow$ (2): $S_v^* u = u_2 - u_1 + u_0 \in U_2(u)$.

(2) $\Rightarrow$ (1): By Remark 1.1, $S_v(F_u) = F_{S_v^* u}$, and since $S_v^* u \in U_2(u)$ we have by [**15**, Theorem 2.3] that $S_v(F_u) \subset Z_2(u)$. Thus

(2) $$P_2(u) S_v P_2(u) = S_v P_2(u).$$

Since $S_v^2 = I$, $S_v(Z_2(u)) = Z_2(u)$. Since by [**15**, Proposition 1.5], $Z_2(u) \diamond Z_0(u)$, we have $Z_0(u) \subset Z_2(u)^\diamond$. But $F_u \subset Z_2(u)$ implies $Z_0(u) = F_u^\diamond \supset Z_2(u)^\diamond$ so $Z_0(u) = Z_2(u)^\diamond$. Therefore

$$S_v(Z_0(u)) = S_v(Z_2(u)^\diamond) = S_v(Z_2(u))^\diamond = Z_2(u)^\diamond = Z_0(u),$$

from which we get

$$(3) \qquad\qquad P_0(u)S_v P_0(u) = S_v P_0(u).$$

For S_v^*, by [**15**, Theorem 2.3] and Remark 1.1, we obtain

$$\begin{aligned} S_v^*(U_k(u)) &= \overline{\mathrm{sp}}^{w^*}\{S_v^*(v_H)\colon H \in \mathscr{SF}, H \subset Z_k(u)\}, \\ &= \overline{\mathrm{sp}}^{w^*}\{v_{S_v(H)}\colon H \in \mathscr{SF}, \ H \subset Z_k(u)\}, \end{aligned}$$

for $k = 0$ or 2. Since (2) and (3) imply that $S_v(H) \subset Z_k(u)$, we have

$$(4) \qquad\qquad P_2(u)^* S_v^* P_2(u)^* = S_v^* P_2(u)^*,$$

and

$$(5) \qquad\qquad P_0(u)^* S_v^* P_0(u)^* = S_v^* P_0(u)^*.$$

Using (4) and (2) we have

$$\begin{aligned} P_2(u)^{**} S_v^{**} &= P_2(u)^{**} S_v^{**} P_2(u)^{**} \\ &= (P_2(u)S_v P_2(u))^{**} \\ &= (S_v P_2(u))^{**} = S_v^{**} P_2(u)^{**}. \end{aligned}$$

Thus $[P_2(u), S_v] = 0$.

Similarly, (3) and (5) imply $[P_0(u), S_v] = 0$. Since $P_1(u) = I - P_2(u) - P_0(u)$ and $P_1(v) = (I - S_v)/2$, we have

$$(6) \qquad [P_j(u), S_v] = 0 = [P_j(u), P_1(v)] \quad \text{for } j \in \{0, 1, 2\}.$$

Since $P_2(v) + P_0(v) = (I + S_v)/2$, (6) implies

$$(7) \qquad (P_2(v) + P_0(v))(Z_j(u)) \subset Z_j(u), \quad \text{for } j \in \{0, 1, 2\}.$$

Using the fact that $Z_2(v) \diamond Z_0(v)$, from [**15**, Remark 1.3] (for $j = 0$) and [**15**, Remark 3.2] (for $j = 2$) we have

$$P_i(v)(Z_j(u)) \subset Z_j(u) \quad \text{for } i, j \in \{0, 2\},$$

i.e.

$$(8) \qquad P_i(v)P_j(u) = P_j(u)P_i(v)P_j(u), \quad \text{for } i, j \in \{0, 2\}.$$

Moreover $Q := P_i(v)P_j(u)$ is a neutral contractive projection with range $Z_i(v) \cap Z_j(u)$, for $i, j \in \{0, 2\}$.

Obviously, the subspace $Q^*(Z)$ contains $U_i(v) \cap U_j(u)$. If these spaces are not equal, we may choose a $\varphi \in Q(Z)$ such that $\|\varphi\| = 1$ and

$$\varphi(U_i(v) \cap U_j(u)) = 0.$$

Then there exists $x \in Q^*(U)$, $\|x\| = 1$ with $\varphi \in F_x$. By neutrality of Q, $F_x \subset Q(Z)$ and by [15, Theorem 2.3], $v_{F_x} \in U_i(v) \cap U_j(u)$. Hence $\varphi(v_{F_x}) = 0$, contradicting $\varphi \in F_x$. Thus $Q^*(U) = U_i(v) \cap U_j(u)$.

Now let $R := P_i(v)P_j(u)P_i(v)$. From (8) it follows that

$$R^2 = P_i(v)P_j(u)P_i(v)P_j(u)P_i(v) = R$$

so that R is a neutral contractive projection. For any $\varphi \in Z$ we have

$$\|R\varphi\| = \|R^2\varphi\| = \|P_i(v)P_j(u)R\varphi\| \leq \|P_j(u)R\varphi\| \leq \|R\varphi\|.$$

Since $P_j(u)$ is neutral for $j = 0$ or 2, $R\varphi = P_j(u)R\varphi$. Thus $R(Z) = Z_j(u) \cap Z_i(v)$. This implies, as above, that $R^*(U) = U_i(v) \cap U_j(u)$ and thus by [15, Lemma 2.2], $Q = R$, i.e.

$$(9) \qquad P_i(v)P_j(u) = P_i(v)P_j(u)P_i(v), \quad \text{for } i,j \in \{0,2\}.$$

From this it follows that (for $i \in \{0,2\}$)

$$
\begin{aligned}
P_i(v)P_1(u) &= P_i(v)[P_2(u) + P_1(u) + P_0(u)]P_i(v)P_1(u) \\
&= (P_i(v)P_1(u))^2,
\end{aligned}
$$

implying, by neutrality of $P_i(v)$,

$$(10) \qquad P_i(v)P_1(u) = P_1(u)P_i(v)P_1(u), \quad \text{for } i \in \{0,2\}.$$

Finally, using (8) and (10), we have, for $i,j \in \{0,2\}$,

$$
\begin{aligned}
P_j(u)P_i(v) &= P_j(u)P_i(v)[P_2(u) + P_1(u) + P_0(u)] \\
&= P_j(u)P_i(v)P_j(u) \\
&= P_i(v)P_j(u),
\end{aligned}
$$

i.e., $[P_i(v), P_j(u)] = 0$ for $i,j \in \{0,2\}$. This fact, together with (6) shows that u and v are compatible.

By symmetry of (1), (2′) and (3′) are equivalent to (1). □

As a consequence of Theorem 1.2 we obtain [15, Theorem 3.3], one of the main results of [15].

A WFS space Z is *strongly facially symmetric* (SFS) if for every norm exposed face F, and every $y \in Z^*$ with $\|y\| = 1$ and $F \subset F_y$, we have $S_F^* y = y$, where S_F denotes a symmetry corresponding to F. In a SFS space, the generalized tripotents coincide with the projective units.

Another main result of [15] is the following, which is a geometric polar decomposition.

THEOREM 1.3 (*Theorem 4.3 of* [15]). *Let* Z *be a neutral SFS space.*
For any $f \in Z$ *with* $\|f\| = 1$, *there is a unique generalized tripotent* v
such that
  (a) $f \in F_v$;
  (b) $\langle v, \{f\}^\diamond \rangle = 0$.
Moreover,
  (c) F_v *is the smallest norm exposed face containing* f;
  (d) f *is faithful in the sense that*

$$f(u) = \|P_2(u)f\| > 0 \quad \text{for any } u \in \mathscr{GT} \text{ with } F_u \subset F_v.$$

2. Application to operator algebras. In this section we will show that
the predual of a von Neumann algebra is a neutral strongly facially
symmetric space, and that the set of generalized tripotents coincides
with the set of partial isometries in the von Neumann algebra. These
facts will be used to give a geometric proof of the Tomita-Sakai-Effros
polar decomposition of a normal functional.

If v is a partial isometry in a C^*-algebra A, then with $l = vv^*$ and
$r = v^*v$, projections $E(v)$, $F(v)$, and $G(v)$ on A are defined by

$$\left. \begin{array}{l} E(v)x = lxr; \\ F(v)x = (1 - l)x(1 - r); \\ G(v)x = lx(1 - r) + (1 - l)xr \end{array} \right\}, \quad \text{for } x \in A,$$

and yield the familiar "matrix" representation of x:

$$x \sim \begin{bmatrix} E(v)x & lx(1 - r) \\ (1 - l)xr & F(v)x \end{bmatrix}.$$

We call $E(v)$, $F(v)$, $G(v)$ *Peirce projections* corresponding to v.

If g is a linear functional on A then, by abuse of notation, $E(v)g$
will denote $g \circ E(v)$, etc.

The following lemma is an easy consequence of the matrix repre-
sentation.

LEMMA 2.1. *For a partial isometry* v *in a* C^*-algebra A,
  (a) $\|(E(v) + F(v))x\| = \max\{\|E(v)x\|, \|F(v)x\|\} \leq \|x\|$, $x \in A$;
  (b) $\|E(v)g\| + \|F(v)g\| = \|(E(v) + F(v))g\| \leq \|g\|$, $g \in A^*$.

The next lemma puts an abstract C^*-algebra structure on the Peirce
space $E(v)A$ of a partial isometry v. Its proof consists of straightfor-
ward calculations. For the third assertion, the following well-known
result of Effros is needed: for a normal functional f on a von Neu-
mann algebra and a projection e in the algebra, $f = f \cdot e \Leftrightarrow \|f\| =$
$\|f \cdot e\|$, where $f \cdot e$ is the functional $x \mapsto f(xe)$ (see [26, p. 140]).

LEMMA 2.2. *Let v be a partial isometry in a C^*-algebra A.*

1. $A_v := v^*Ar$, *with $r = v^*v$, is a C^*-subalgebra of A, with unit r. If A is a von Neumann algebra, so is A_v.*

2. *The map $x \mapsto vx$ is a linear isometric bijection of A_v onto $E(v)A$ with inverse $a \mapsto v^*a$. Thus $E(v)A$ becomes a C^*-algebra with unit v and operations*

$$a \cdot b := av^*b, \quad a^\# := va^*v.$$

If A is a von Neumann algebra, the adjoint of $x \mapsto vx$ is an isometry of $E(v)A_*$ onto $(A_v)_*$.

3. *The map $f \mapsto f|_{E(v)A}$ is an affine isometry of $\{f \in A^*: f(v) = \|f\|\}$ onto $(E(v)A)^*_+$. If A is a von Neumann algebra, this map restricts to an affine isometry of $\{f \in A_*: f(v) = \|f\|\}$ onto $(E(v)A)_{*,+}$.*

The Peirce space $E(v)A$ will occur frequently in the sequel. It will be denoted by $A_2(v)$. If A is a von Neumann algebra, then by Lemma 2.2, the normal state space of $A_2(v)$ is affinely isometric to the norm exposed face F_v defined by $F_v = \{f \in A_*: f(v) = \|f\| = 1\}$.

The next lemma examines the relation between partial isometries in a von Neumann algebra A and norm exposed faces in the unit ball $A_{*,1}$ of its predual. Note that if u is a non-zero partial isometry in A, then by Lemma 2.2, $F_u \neq \varnothing$.

Partial isometries u and v are *orthogonal* if their left and right supports are orthogonal, i.e. $uu^*vv^* = 0$ and $u^*uv^*v = 0$. More generally, elements x, y in a C^*-algebra are orthogonal if $xy^* = 0 = y^*x$.

LEMMA 2.3. *For each x in a von Neumann algebra A with $\|x\| = 1$ and $F_x \neq \varnothing$, there is a partial isometry $w \in A$ with $F_x = F_w$. Moreover, $x = y + w$ with y orthogonal to w.*

Proof. Let $|x| = \int_0^1 \lambda \, d\lambda$ be the spectral decomposition of $|x| = (x^*x)^{1/2}$. One can verify that $w := u \int_{\{1\}} d\lambda$ satisfies the requirements of the lemma, where $x = u|x|$ is the polar decomposition of x. For details, see [7, Lemma 3]. $\square$

Lemma 2.3 shows that the map $u \mapsto F_u$ from the set of partial isometries in a von Neumann algebra A to the set of norm exposed faces in the unit ball $A_{*,1}$ of the predual A_* is onto. In fact, this map is also one-to-one.[2] Indeed, by the Jordan decomposition of hermitian functionals on a von Neumann algebra and Lemma 2.2, for any partial isometry u, $E(u)^*A_* = \text{sp} \, F_u$. Also, $u \in E(u)A$. Therefore,

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[2]This will follow from [15, Proposition 1.6] once Theorem 2.11 is proved.

u is determined by its values on $\operatorname{sp} F_u$. It follows that if $F_u = F_w$, then $u = w$.

We next prove that the Peirce projections $E(v)$ and $F(v)$ corresponding to a partial isometry $v \in A$ are neutral. Although this follows from the result of Effros mentioned above, we prefer to give a direct proof of this fact. This proof introduces some tools which will be used later. In the first place, by defining a "triple product" $\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$, and letting $A_j(v)$ denote the Peirce spaces corresponding to the partial isometry v, i.e.

$$A_2(v) = E(v)A, \quad A_1(v) = G(v)A, \quad A_0(v) = F(v)A,$$

we have by a simple matrix calculation,

$$(11) \qquad \{A_i(v), A_j(v), A_k(v)\} \subset A_{i-j+k}(v),$$

where $A_l(v) = \{0\}$ if $l \notin \{0, 1, 2\}$. This notation will be justified by Theorem 2.11. Having introduced the triple product $\{xyz\}$ we also define an operator $D(x, y)\colon A \to A$ by $D(x, y)z = \{xyz\}$ and the "cube" of x as $x^{(3)} = D(x, x)x = \{xxx\} = xx^*x$. More generally, odd "powers" of elements of A are defined inductively by $x^{(1)} = x$, $x^{(m)} = \{xx^{(m-2)}x\}$ for m odd, $m \geq 3$. $D(x)$ will denote $D(x, x)$.

Note that x and y are orthogonal if and only if $D(x, y) = 0$.

LEMMA 2.4. *Let v be a partial isometry in a C^*-algebra A and let λ be a complex number, $|\lambda| = 1$. Define a linear map $S_\lambda = S_\lambda(v)\colon A \to A$ by*

$$S_\lambda = S_\lambda(v) = \lambda^2 E(v) + \lambda G(v) + F(v).$$

Then

(a) S_λ preserves the triple product $\{xyz\}$, i.e.

$$S_\lambda(\{xyz\}) = \{S_\lambda x, S_\lambda y, S_\lambda z\} \quad \text{for } x, y, z \in A;$$

(b) S_λ is an isometry of A onto A.

Proof. (a) By the standard polarization formula

$$\{abc\} = \frac{1}{8} \sum_{\alpha^4 = \beta^2 = 1} \alpha\beta(a + \alpha b + \beta c)^{(3)},$$

it suffices to prove (a) in the special case $x = y = z$. Write $x = \sum_{j=0}^2 x_j$ with $x_j \in A_j(v)$. Then $S_\lambda x = \sum \lambda^j x_j$, $\{S_\lambda x, S_\lambda x, S_\lambda x\} =$

$\sum_{i,j,k} \lambda^{i-j+k}\{x_i x_j x_k\}$, and $x^{(3)} = \sum_{i,j,k}\{x_i x_j x_k\}$. By (11), $\{x_i x_j x_k\} \in A_{i-j+k}$. Therefore

$$S_\lambda(x^{(3)}) = \sum_{i,j,k} \lambda^{i-j+k}\{x_i x_j x_k\} = (S_\lambda x)^{(3)}.$$

(b)

$$\|S_\lambda(x)\|^3 = \|\{S_\lambda(x), S_\lambda(x), S_\lambda(x)\}\| = \|S_\lambda(x^{(3)})\|$$
$$\leq \|S_\lambda\|\|x^{(3)}\| = \|S_\lambda\|\|x\|^3.$$

Thus $\|S_\lambda\| \leq 1$ and since $S_\lambda S_{\bar\lambda} = I$, S_λ is isometric. $\qquad\square$

LEMMA 2.5. *Let v be a partial isometry in a C^*-algebra A, and let $x \in A_2(v) \cup A_0(v)$, $y \in A_1(v)$, $t \in \mathbf{C}$. Then for each positive integer n*

$$(x + ty)^{(3^n)} = x^{(3^n)} + t2^n D(x^{(3^{n-1})}) \cdots D(x^{(3)})D(x)y + O(|t|^2).$$

Proof. By (11), $\{xyx\} = 0$, and therefore

$$(x + ty)^{(3)} = x^{(3)} + t\{xyx\} + 2t\{xxy\} + O(|t|^2)$$
$$= x^{(3)} + 2tD(x)y + O(|t|^2).$$

The result now follows by induction: with

$$x_n = x^{(3^n)} \quad \text{and} \quad y_n = 2^n D(x_{n-1}) \cdots D(x)y,$$

we have

$$(x + ty)^{(3^{n+1})} = (x_n + ty_n + O(|t|^2))^{(3)} = (x_n + ty_n)^{(3)} + O(|t|^2)$$
$$= x_n^{(3)} + 2tD(x_n)y_n + O(|t|^2)$$
$$= x_{n+1} + ty_{n+1} + O(|t|^2). \qquad\square$$

The following proposition will imply the neutrality of the predual of a von Neumann algebra.

PROPOSITION 2.6. *Let v be a partial isometry in a C^*-algebra A and let $f \in A^*$.*
(a) *If $\|E(v)f\| = \|f\|$, then $E(v)f = f$;*
(b) *If $\|F(v)f\| = \|f\|$, then $F(v)f = f$.*

Proof. (a) Since $\|E(v)f\| + \|F(v)f\| = \|E(v)f + F(v)f\| \leq \|f\|$, $F(v)f = 0$. It remains to prove that $G(v)f = 0$. Toward this end let $y \in G(v)A$. We are to prove that $f(y) = 0$. We may assume $\|f\| = 1$,

$f(y) \geq 0$, and $\|y\| \leq 1$. For $\varepsilon > 0$ choose $x \in E(v)A$ with $\|x\| = 1$ and $f(x) \geq 1 - \varepsilon$. Then for $t \in \mathbf{R}$,

$$\|x + ty\| \geq f(x + ty) = f(x) + tf(y) \geq 1 - \varepsilon + tf(y).$$

Therefore by Lemma 2.5

$$(1 - \varepsilon + tf(y))^{3^n} \leq \|x + ty\|^{3^n} = \|(x + ty)^{(3^n)}\|$$
$$\leq \|x^{(3^n)}\| + t2^n\|y\| + O(|t|^2)$$

and so

$$(1 - \varepsilon)^{3^n} + 3^n tf(y)(1 - \varepsilon)^{3^n - 1} + O(|t|^2) \leq 1 + t2^n\|y\| + O(|t|^2).$$

Letting $\varepsilon \to 0$, and dividing by $|t|$ results in

$$f(y) + O(|t|) \leq \left(\frac{2}{3}\right)^n \|y\| + O(|t|).$$

Letting $t \to 0$, then $n \to \infty$ yields $f(y) = 0$. The proof of (b) is similar. $\square$

The following gives a useful characterization of the relation $f \diamond g$.

LEMMA 2.7. *Let f and g be normal functionals on a von Neumann algebra A. The following are equivalent.*
  (a) $f \diamond g$;
  (b) *There exist orthogonal partial isometries p, q in A such that $f(p) = \|f\|$ and $g(q) = \|g\|$;*
  (c) *There exist orthogonal partial isometries p, q in A such that $f = E(p)f$ and $g = E(q)g$.*

Proof. (a) $\Rightarrow$ (b): Suppose that $f \diamond g$. By the weak*-continuity of $f + g$ there is an $x \in A$, $\|x\| = 1$ with $\langle f + g, x \rangle = \|f + g\|$. Choose a partial isometry w by Lemma 2.3 with $F_x = F_w$. Then

$$\|f\| + \|g\| = \|f + g\| = \langle f + g, w \rangle \leq |\langle f, w \rangle| + |\langle g, w \rangle| \leq \|f\| + \|g\|.$$

Therefore $f(w) = \|f\|$, $g(w) = \|g\|$ so that $\varphi := f|_{A_2(w)}$, $\psi := g|_{A_2(w)}$ are positive normal functionals on the von Neumann algebra $A_2(w)$ with $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$. If p and q denote the support projections in $A_2(w)$ of φ and ψ respectively, then, as is well known, p and q are orthogonal projections in $A_2(w)$, i.e.

$$p = p \cdot p = pw^*p, \quad q = q \cdot q = qw^*q;$$
$$p = p^{\#} = wp^*w, \quad q = q^{\#} = wq^*w;$$
$$p \cdot q = pw^*q = 0.$$

From these equations we have

$$p = p \cdot p^{\#} \cdot p = pw^*(wp^*w)w^*p = p(ww^*pw^*w)^*p = pp^*p$$

and

$$pq^* = p(q^{\#})^* = p(wq^*w)^* = pw^*qw^* = 0.$$

Therefore p and q are orthogonal partial isometries in A.

We have $f(p) = \varphi(p) = \|\varphi\| = \|f\|$ and similarly, $g(q) = \|g\|$.

(b) $\Rightarrow$ (c): This follows from Proposition 2.6.

(c) $\Rightarrow$ (a): If there exist orthogonal partial isometries p, q in A such that $f = E(p)f$ and $g = E(q)g$, then since $E(q) = F(p)E(q)$, it follows from Lemma 2.1 that $f \diamond g$. □

We are now ready to show that the predual of a von Neumann algebra is WFS. Let F_x be a norm exposed face in $A_{*,1}$. With a partial isometry w given by Lemma 2.3 such that $F_x = F_w$, let S be the "Peirce reflection" with respect to w, i.e.

$$S = E(w) - G(w) + F(w).$$

Since the fixed point set of S is $E(w)A_* \oplus F(w)A_*$, the fact that A_* is WFS will follow from the following two lemmas.

LEMMA 2.8. *With S as above,*
(1) $S^2 = I$;
(2) S *is an isometry of A_* onto itself*;
(3) $\text{sp } F_w = E(w)A_* \ (= \{g \in A_* : E(w)g = g\})$.

Proof. (1) is obvious and (2) follows from Lemma 2.4 with $\lambda = -1$. To prove (3), take $g \in F_w$. Then

$$1 = g(w) = \langle E(w)g, w \rangle \leq \|E(w)g\| \leq \|g\| = 1,$$

so that by Proposition 2.6, $g = E(w)g$. Therefore $F_w \subset E(w)A_*$ and

$$\text{sp } F_w \subset E(w)A_*.$$

Since F_w is the normal state space of the von Neumann algebra $A_2(w)$ and $E(w)A_*$ is its predual, (3) now follows from the Jordan and Cartesian decompositions of normal functionals. □

We now have the following consequence of Lemmas 2.8 and 2.4.

COROLLARY 2.9. *Let u be a partial isometry in a von Neumann algebra A and suppose that $F_u \subset F_x$ for some $x \in A$ with $\|x\| = 1$. Then $E(u)x = u$ and $G(u)x = 0$.*

Proof. Let

$$x = x_2 + x_1 + x_0$$

be the Peirce decomposition of x with respect to u, i.e. $x_2 = E(u)x$, $x_1 = G(u)x$, $x_0 = F(u)x$.

Since $F_u \subset F_x$,

$$\langle F_u, x_2 \rangle = \langle F_u, x \rangle = 1 = \langle F_u, u \rangle.$$

On the other hand, u and x_2 both vanish on $G(u)(A_*) \oplus F(u)(A_*)$. Thus $x_2 = u$.

Let $y = u - ix_1 - x_0$. Then $y = -S_i(u)x$ so that $\|y\| = 1$ by Lemma 2.4. Therefore

$$z := \tfrac{1}{2}(x + y) = u + \mu x_1 \quad \text{with } \mu = \tfrac{1}{2}(1 - i)$$

and $\|z\| \leq 1$. We now have $\|\{zzz\}\| = \|z\|^3 \leq 1$, and therefore

$$\|u + 2|\mu|^2 \{x_1 x_1 u\}\| = \|P_2(u)\{zzz\}\| \leq \|\{zzz\}\| \leq 1.$$

Since $\{x_1 x_1 u\}$ is a positive element of $A_2(u)$, it must be zero, and then it follows by a simple matrix calculation that $x_1 = 0$. $\qquad\square$

LEMMA 2.10. *For any partial isometry w in a von Neumann algebra A*

$$F_w^\diamond = F(w)A_* \quad (= \{g \in A_* : F(w)g = g\}).$$

Proof. Let $g \in F(w)A_*$, $f \in F_w$. Then since $f \in E(w)A_*$, we have, by Lemma 2.1, $\|f \pm g\| = \|f\| + \|g\|$, proving that $F_w^\diamond \supset F(w)A_*$.

Before proving the converse inclusion, note that since $A_2(w)$ is a von Neumann algebra with unit w,

$$w = \sup\{e_\varphi : \varphi \in A_2(w)_{*,+}\},$$

where e_φ is the support projection in $A_2(w)$ of φ. By Zorn's lemma, there is an orthogonal family $\{e_\alpha\}$ such that

$$(12) \qquad\qquad w = \sum e_\alpha,$$

where e_α is the support of a normal functional φ_α on $A_2(w)$. As in the proof of Lemma 2.7, $\{e_\alpha\}$ is an orthogonal family of partial isometries in A.

Now let $g \in F_w^\diamond$. We must show that $g \in F(w)A_*$. Consider $f_\alpha := \varphi_\alpha \circ E(w) \in E(w)A_*$. Then $f_\alpha(w) = \varphi_\alpha(w) = \|\varphi_\alpha\| = \|f_\alpha\|$ so that $\|f_\alpha\|^{-1} f_\alpha \in F_w$ and therefore $f_\alpha \diamond g$ for all α. By Lemma 2.7, there exist orthogonal partial isometries p_α, q_α such that $f_\alpha(p_\alpha) = \|f_\alpha\|$, $g(q_\alpha) = \|g\|$ and $g = E(q_\alpha)g$. We now have

$$\|f_\alpha\|^{-1} f_\alpha \in F_w \cap F_{p_\alpha} = F_{w_\alpha},$$

for some partial isometry w_α (by Lemma 2.3). By Corollary 2.9, since $F_{w_\alpha} \subset F_{p_\alpha}$,

$$p_\alpha = w_\alpha + b_\alpha$$

with b_α orthogonal to w_α. Therefore w_α is orthogonal to q_α. Again by Corollary 2.9

$$w = w_\alpha + c_\alpha$$

with c_α orthogonal to w_α. By direct calculation then, $w_\alpha \in A_2(w)$ and

$$w_\alpha^\# = w_\alpha \cdot w_\alpha = w_\alpha.$$

Therefore w_α is a projection in $A_2(w)$ with $\varphi_\alpha(w_\alpha) = \|\varphi_\alpha\|$. Since e_α is the support projection of φ_α, we have $e_\alpha \leq w_\alpha$ as projections in $A_2(w)$. Thus $e_\alpha \leq w_\alpha$ as partial isometries in A and so e_α and q_α are orthogonal for all α.

For the rest of this proof, it will be convenient to use the notation $l(u) = uu^*$ and $r(u) = u^*u$ for a partial isometry u.

By (12)

$$l(w) = \sum l(e_\alpha) \quad \text{and} \quad r(w) = \sum r(e_\alpha).$$

Therefore, for $x \in A$ and all α, since $g = E(q_\alpha)g$

$$g(l(w)x(1 - r(w)) = \lim \sum g(l(q_\alpha)l(e_\alpha)x(1 - r(w))) = 0,$$

and similarly

$$g((1 - l(w)xr(w)) = 0 \quad \text{and} \quad g(l(w)xr(w)) = 0,$$

i.e. $G(w)g = 0$. Similarly, $E(w)g = 0$. $\qquad\qquad\square$

THEOREM 2.11. *The predual of a von Neumann algebra A is a neutral strongly facially symmetric space in which the set of generalized tripotents coincides with the set of non-zero partial isometries. If u is a partial isometry, then the generalized Peirce projections defined by u agree with the Peirce projections defined by u, i.e.*

(13) $\qquad P_2(u) = E(u), \quad P_1(u) = G(u) \quad and \quad P_0(u) = F(u).$

Proof. It follows from Lemmas 2.8 and 2.10 that A_* is a WFS space.

Let w be a non-zero partial isometry in A. Then $\|w\| = 1$ and $F_w \neq \varnothing$. Since $F_w^\diamond = F(w)A_*$, and $w \in A_2(w)$, $\langle w, F_w^\diamond \rangle = \langle w, F(w)A_* \rangle = 0$. Thus w is a projective unit. But clearly, $S_w^* w = w$ so that w is a generalized tripotent. Conversely let u be a generalized tripotent. By Lemma 2.3 there is a partial isometry w with $F_u = F_w$ and $y := u - w$ orthogonal to w. We show that $y = 0$ by showing that $g(y) = 0$ for all $g \in A_*$. Since $y \in A_0(w)$ we may assume that $g \in F(w)A_*$ $(= F_w^\diamond = F_u^\diamond)$, and hence $g(w) = 0$. Since $\langle u, F_u^\diamond \rangle = 0$, $g(u) = 0$. Therefore $g(y) = 0$. Thus, the set of non-zero partial isometries coincides with the set of generalized tripotents.

It follows from Proposition 2.6 that A_* is neutral.

By [**15**, Theorem 2.4], (13) holds.

To prove SFS let $F_u \subset F_x$ for some generalized tripotent u and some element x in A. Let

$$x = x_2 + x_1 + x_0$$

be the generalized Peirce decomposition of x with respect to u, i.e. $x_j = P_j(u)x$ for $j = 0, 1, 2$. We shall show that $x_2 = u$ and then that $x_1 = 0$, which implies $S_u^* x = x$, as required for SFS.

Since $x_2 = P_2(u)^* x$ and $F_u \subset F_x$,

$$\langle F_u, x_2 \rangle = \langle F_u, x \rangle = 1 = \langle F_u, u \rangle.$$

On the other hand, since also $u = P_2(u)^* u$, u and x_2 both vanish on $P_1(u)(A_*) \oplus P_0(u)(A_*)$. Thus $x_2 = u$.

The proof that $x_1 = 0$ is identical to the last paragraph in the proof of Corollary 2.9.   □

We can now combine Theorems 1.3 and 2.11 to obtain a geometric proof of the Tomita-Sakai-Effros polar decomposition of a normal functional. We state the version of this theorem from the book of Takesaki ([**26**, p. 140]).

THEOREM 2.12. *Let A be a von Neumann algebra, f an element of the predual.*

(1) *There exists a pair (φ, v) with the following properties:*

    (a) *φ is a positive normal functional on A, and $\|\varphi\| = \|f\|$;*

    (b) *v is a partial isometry of A with $v^* v = \text{supp } \varphi$ (:= the support of φ);*

    (c) *$f(x) = \varphi(v^* x)$, $\varphi(x) = f(vx)$, for $x \in A$.*

(2) *Let ψ be a positive normal functional on A, u a partial isometry of A with $u^*u = \operatorname{supp}\psi$ and $f(x) = \psi(u^*x)$, for $x \in A$. Then $\psi = \varphi$, $u = v$.*

Proof. For notation's sake, assume $\|f\| = 1$.

(1) Given f, choose a generalized tripotent v satisfying the conditions in Theorem 1.3. Define $\varphi \in A_*$ by the rule $\varphi(x) = f(vx)$, for $x \in A$. Then $\|\varphi\| \leq \|f\|$ and $\varphi(v^*v) = f(v) = \|f\|$ so that $\|\varphi\| = \|f\|$. Also, v is a partial isometry and, with $r = v^*v$, we have $\varphi(1) = \varphi(r + (1 - r)) = \varphi(r) = 1$ so that $\varphi \in A_{*,+}$ and $\operatorname{supp}\varphi \leq r$. If p is a projection in A with $p \leq r$, then $vp \leq v$ by Lemma 2.2. Hence by (d) of Theorem 1.3, $\varphi(p) = f(vp) > 0$, proving that $r = \operatorname{supp}\varphi$. Finally, $\varphi(v^*x) = f(vv^*x) = f(x)$ for $x \in A$, by neutrality.

(2) Obviously, $\|f\| \leq \|\psi\|$. On the other hand, $\|\psi\| = \psi(u^*u) = f(u)$ so $\|f\| = \|\psi\|$. Since u is a partial isometry, it is a generalized tripotent with $f(u) = \psi(u^*u) = 1$, i.e. $f \in F_u$. Using the isomorphism of Lemma 2.2, we see that ψ on A_u corresponds to f on $E(u)A$ and since ψ is faithful on A_u, f is faithful on $E(u)A$.

By [14, Corollary 1.3b], if $g \in \{f\}^\diamond$, $\|g\| = 1$, there exist orthogonal norm exposed faces F_x, F_y with $f \in F_x$, $g \in F_y$. Thus $f \in F_x \cap F_u = F_w$ for some projection w in $E(u)A$. By the faithfulness of f on $E(u)A$, $w = u$ and therefore $F_u \subset F_x$, and $F_u^\diamond \supset F_x^\diamond \supset F_y$. Since $\langle u, F_u^\diamond \rangle = 0$ and $g \in F_y$, $g(u) = 0$. Therefore $\langle u, \{f\}^\diamond \rangle = 0$. By the uniqueness part of Theorem 1.3, $u = v$ and $\psi(x) = f(ux) = f(vx) = \varphi(x)$. $\qquad\square$

3. Application to JB^*-triples.

A JB^*-triple (or Banach Jordan Triple System) is a complex Banach space with a certain ternary algebraic structure arising in the study of bounded symmetric domains in finite and infinite dimensions ([21]). The class includes C^*-algebras and JB^*-algebras as well as the range of an arbitrary contractive projection on a C^*-algebra ([11]). A JBW^*-triple is a JB^*-triple which is the dual of a Banach space.

Recent results in the analytic and algebraic theory of Jordan triple systems have led to a good understanding of their structure. The algebraic structure of a JB^*-triple is uniquely determined by the isometric structure of its unit ball. It is also uniquely determined by the holomorphic structure of its unit ball. Hence, results on the geometry of JB^*-triples underlie much of this structure theory. Some of the principal advances in this direction include the following.

Geometric Structure

(1) Affine geometric structure of a JBW^*-triple and its predual: facial structure ([10]), and state space properties ([12]).

(2) Geometric tools: stability under contractive projections ([22], [25]), and weak* continuity of the triple product ([3],[8],[16]).

(3) Foundations in holomorphy ([20], [21], [27], [30]).

Algebraic Structure

(4) Gelfand-Naimark Theorem ([13]).

(5) Classification Theory for JBW^*-triples: atomic ([6], [24]), type I ([17]), and continuous ([18]).

This theory has had and continues to have significant connections to other areas of analysis. (An introductory survey of several applications of Jordan theory to analysis, operator theory, and the foundations of quantum mechanics has been given by Upmeier [29].) A Riemann Mapping Theorem for the bounded symmetric domains in complex Banach spaces was proven by Kaup ([21]), who showed that there is a one-to-one correspondence between the biholomorphic equivalence classes of these domains and the algebraic isomorphism classes of JB^*-triples. Upmeier has developed a definitive theory of multivariable Toeplitz operators over finite dimensional bounded symmetric domains using the correspondence between these domains and JB^*-triples (cf. [29, Lecture 8]).

Since the analog of the polar decomposition is known in the context of JBW^*-triples, and since the latter are intrinsically geometric objects, it should not be surprising that this polar decomposition is a consequence of the theory of facially symmetric spaces. Indeed, we have the following two theorems, which are analogues of the main results of §2. In order to state these results we now recall the definitions and basic properties of JB^*-triples (cf. [28]).

A JB^*-triple is a complex Banach space U endowed with a continuous sesqui-linear map $D: U \times U \to \mathscr{B}(U)$, such that for $x \in U$, $D(x, x)$ is Hermitian positive, $\|D(x, x)\| = \|x\|^2$, and, setting $\{xyz\} := D(x, y)z$, one has

$$\{xyz\} = \{zyx\},$$

and

$$\{xy\{uvz\}\} + \{u\{yxv\}z\} = \{\{xyu\}vz\} + \{uv\{xyz\}\}.$$

For example, a C^*-algebra is a JB^*-triple with

$$\{xyz\} := \tfrac{1}{2}(xy^*z + zy^*x),$$

and a JB^*-algebra is a JB^*-triple with

$$\{xyz\} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (z \circ x) \circ y^*,$$

where $\circ$ denotes the Jordan product in the JB^*-algebra.

Denote by Q the *quadratic operator* on the JB^*-triple U, i.e. $Q(x)y = \{xyx\}$, for $x, y \in U$. Then set

$$Q(x, z) = \tfrac{1}{2}(Q(x + z) - Q(x) - Q(z))$$

so that $Q(x, z)y = \{xyz\}$, for $x, y, z \in U$.

A *tripotent* in a JB^*-triple is an element e with $e = \{eee\}$. Define the *Peirce projections* $P_k(e)$, $k = 0, 1, 2$ relative to a tripotent e by

$$P_2(e) = Q(e)^2, \quad P_1(e) = 2(D(e, e) - Q(e)^2),$$
$$P_0(e) = I - 2D(e, e) + Q(e)^2.$$

Note that $\sum P_k(e) = I$ and $D(e, e) = P_2(e) + \tfrac{1}{2}P_1(e)$. It follows from purely algebraic considerations that each $P_j(e)$ is idempotent and that $P_k(e)P_j(e) = 0$ if $k \neq j$. Let $U_k(e)$ be the range of $P_k(e)$. The *Peirce decomposition* is

$$U = U_2(e) \oplus U_i(e) \oplus U_0(e)$$

and $U_k(e)$ is the $k/2$-eigenspace of $D(e, e)$. We have the following fundamental properties:

$$\{U_i(e)U_j(e)U_k(e)\} \subset U_{i-j+k}(e);$$
$$\{U_2(e)U_0(e)U\} = \{U_0(e)U_2(e)U\} = 0.$$

The Peirce 2-space $U_2(e)$ is a complex Jordan *-algebra, with product $x \circ y = \{xey\}$, unit e, and involution $z^\# = \{eze\}$. If U is a JB^*-triple, then $U_2(e)$ is a JB^*-algebra with these operations.

THEOREM 3.1. *The predual U_* of a JBW^*-triple U is a neutral strongly facially symmetric space in which the set of generalized tripotents coincides with the set of tripotents. If u is a generalized tripotent, then the generalized Peirce projections associated with u agree with the adjoints of the Peirce projections associated with u.*

Proof. If F_x is a norm exposed face in $U_{*,1}$ then by [12, Proposition 8], there is a tripotent u in U with $F_x = F_u$. By [12, Lemma 1.1], $S_{-1}(u) := P_2(u) - P_1(u) + P_0(u)$ is an isometric symmetry. By the

Jordan decomposition of hermitian normal functionals on a JBW^*-algebra, $P_2(u)U_* = \operatorname{sp} F_u$. By an algebraic analog of Lemma 2.7 one can show that $P_0(u)U_* = F_u^\diamond$. Therefore U_* is *WFS*.

By an argument similar to that in Theorem 2.11 one can show that the set of generalized tripotents coincides with the set of non-zero tripotents.

By [12, Proposition 1], U_* is neutral.

By [15, Theorem 2.4], the generalized Peirce projections agree with the adjoints of the Peirce projections. Now by [12, Lemma 1.6] and the argument of Theorem 2.11 it follows that U_* is SFS.   □

THEOREM 3.2 (*Proposition 2 of* [13]). *For each functional* f *in the predual of a* JBW^*-*triple, there is a unique tripotent* e *such that* $f = P_2(e)^*f$ *and* $f|_{U_2(e)}$ *is a faithful normal positive functional on the* JBW^*-*algebra* $U_2(e)$.

Proof. Since U_* is a neutral strongly facially symmetric space, given f (with $\|f\| = 1$), there is a generalized tripotent e satisfying the conditions in Theorem 1.3. Then e is a tripotent and since $f \in F_e$, $\|f\| = \|P_2(e)^*f\|$ so by [12, Proposition 1], $f = P_2(e)^*f$. Since $f|_{U_2(e)}(e) = 1$, f is positive, and since f is normal, so is $f|_{U_2(e)}$. Let p be a projection in $U_2(e)$. Then p is a tripotent in U with $p \leq e$ and therefore $f(p) > 0$ by (d) of Theorem 1.3. The existence is proved. Here we have used [15, Lemma 4.2].

To prove the uniqueness, it suffices, by Theorem 1.3 to show that any tripotent u satisfying the above conditions also satisfies $f \in F_u$ and $\langle u, \{f\}^\diamond \rangle = 0$.

Since f is a positive functional on the JB^*-algebra $U_2(u)$, $f \in F_u$. Now continue as in the last paragraph of the proof of Theorem 2.12.

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