PACIFIC JOURNAL OF MATHEMATICS
Vol. 137, No. 1, 1989

ADDITION OF C*-ALGEBRA EXTENSIONS

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Dedicated to the memory of Henry A. Dye

Two extensions of a C*-algebra A by a C*-algebra B can be added, by the method of Brown, Douglas, and Fillmore, whenever the quotient multiplier algebra $M(A)/A$ contains two isometries with orthogonal ranges. If A is stable (i.e., if $A \cong A \otimes \mathscr{K}$) then such isometries can be found already in $M(A)$, but if A has a tracial state then this is not possible. (Hence in the case that A is a separable AF algebra, this is possible only if A is stable.) Here it is shown that, in the case that A is a separable AF algebra (assumed to have no nonzero unital quotient), there exist two isometries in $M(A)/A$ with orthogonal ranges if, and only if, the space $T(A)$ of tracial states of A is compact.

1. Introduction. In **[15]** and **[14]** a systematic study was made of extensions

$$0 \to A \to C \to B \to 0$$

where A and B are fixed separable approximately finite-dimensional C*-algebras. (Then C is also approximately finite-dimensional—see **[1]**.) An equivalence relation was considered on these extensions which can be described as the simplest possible notion of equivalence for the corresponding extensions of dimension ranges,

$$0 \to D(A) \to D(C) \to D(B) \to 0,$$

namely, isomorphism at the level of $D(C)$, inducing the identity on $D(A)$ and $D(B)$.

This equivalence relation was shown in **[3]** to be just the strong equivalence of Brown, Douglas, and Fillmore **[2]**. In the case $A = \mathscr{K}$, this was already shown in **[15]**. As pointed out in **[15]**, since $D(\mathscr{K}) = \mathbf{Z}^+$, it follows easily that the Brown-Douglas-Fillmore group $\mathrm{Ext}_s(B)$ depends only on the abelian group $K_0(B)$, and is the group $\mathrm{Ext}(K_0(B), \mathbf{Z})$. (Here we do not require extensions to be unital, or even that B have a unit.) The group $\mathrm{Ext}_s(B)$ was also computed by Pimsner and Popa, by a different method, in **[17]** and **[16]**.

It was observed in **[15]** and **[14]** that the equivalence classes of essential extensions of A by B, when computed as a set, often turn out to look like a semigroup, even when A is not stable. (In the case of stable A, this is of course to be expected.)

It is the purpose of this paper to give an explanation of this phenomenon.

Recall that the essential extensions of A by B may be thought of as the injective homomorphisms of the C*-algebra B into the quotient multiplier algebra $M(A)/A$. Strong equivalence is then unitary equivalence with respect to a unitary in $M(A)/A$ in the connected component of 1 (therefore, liftable to a unitary in the connected component of 1 in $M(A)$). Unitary equivalence with respect to an arbitrary unitary in $M(A)/A$ was called weak equivalence in **[2]**. Brown, Douglas, and Fillmore also considered a third, weaker, type of equivalence in **[2]** (Definition 3.1 of **[2]**). In the case $B = \mathbf{C}$, in which extensions are just projections in $M(A)/A$, this weakest equivalence is just equivalence of projections.

Accordingly, if A is such that equivalence classes of essential extensions of A by B can always be added, in some natural sense, then one would expect that the equivalence classes with respect to the third, weakest, relation could also be added, in a natural sense. In the case $B = \mathbf{C}$, when the equivalence classes are just the Murray-von Neumann equivalence classes of projections, one would expect that this natural addition coincided with the usual addition of equivalence classes with orthogonal representatives, and so universal existence of addition would mean that $D(M(A)/A)$, the abelian local semigroup of equivalence classes of projections in $M(A)/A$, was a semigroup.

While the argument of the preceding paragraph is not conclusive, we shall now show that if it does happen that $D(M(A)/A)$ is a semigroup, then there is a natural way of adding strong equivalence classes of essential extensions of A by B. Furthermore, this addition is compatible with weak equivalence, and also with the third, weaker, equivalence relation referred to above. Finally, while to prove that the addition of strong equivalence classes is independent of any choice made in the construction, we need to assume that A is a separable AF algebra, this is not necessary in the case of weak equivalence, or the third equivalence relation. In any case, no assumption is necessary on B.

THEOREM. *Let A and B be* C**-algebras. Suppose that the abelian local semigroup $D(M(A)/A)$, the range of the dimension on $M(A)/A$, is a semigroup. It follows that the weak equivalence classes of essential extensions of A by B form a semigroup, in a canonical way. This is* *also the case with respect to the third, weaker, equivalence relation of* [**2**]. *The strong equivalence classes can be made into a semigroup in a way which, while possibly not canonical in general, is canonical at least in the case that A is a separable AF algebra.*

Proof. The hypothesis that $D(M(A)/A)$ is a semigroup implies that $\dim(1)+\dim(1)$ is defined in $D(M(A)/A)$. (These two properties are in fact equivalent.) That $\dim(1)+\dim(1)$ is defined just says that there exist isometries v_1 and v_2 in $M(A)/A$ with orthogonal ranges, i.e. such that $v_2^*v_1=0$. Since also $\dim(1)+\dim(1)+\dim(1)$ is defined, we may choose v_1 and v_2 so that there exists an isometry v_3 with range orthogonal to the ranges of both v_1 and v_2. We shall always choose v_1 and v_2 in this way.

Let ρ_1 and ρ_2 be embeddings of B into $M(A)/A$. With $v_i\rho_iv_i^*$ denoting the embedding $b\mapsto v_i\rho_i(b)v_i^*$, the images of $v_1\rho_1v_1^*$ and $v_2\rho_2v_2^*$ in $M(A)/A$ are orthogonal, and so the sum $v_1\rho_1v_1^*+v_2\rho_2v_2^*$ is also an embedding of B into $M(A)/A$.

We shall show that this addition is compatible with each of the three equivalence relations, for fixed v_1 and v_2. We shall also show that the resulting addition of weak equivalence classes is independent of the choice of v_1 and v_2, and that the same holds for the third equivalence relation. We shall then show that in the case that A is a separable AF algebra, also addition of strong equivalence classes defined in this way is independent of the choice of v_1 and v_2. Finally, we shall show that addition is associative.

Let u_1 and u_2 be unitaries in $M(A)/A$. Then the extensions $v_1\rho_1v_1^*+v_2\rho_2v_2^*$ and $v_1(u_1\rho_1u_1^*)v_1^*+v_2(u_2\rho_2u_2^*)v_2^*$ are unitarily equivalent by the unitary

$$w=v_1u_1v_1^*+v_2u_2v_2^*+(1-v_1v_1^*-v_2v_2^*).$$

This shows that addition is compatible with weak equivalence. Furthermore, if u_1 and u_2 are connected to 1 in $M(A)/A$, then $v_iu_iv_i^*$ is connected to $v_iv_i^*$ inside $v_iv_i^*(M(A)/A)v_iv_i^*$, and hence w is connected to 1 in $M(A)/A$. This shows that addition is compatible with strong equivalence.

Before showing that addition is compatible with the third equivalence relation of [**2**], let us recall what this is. We shall say that essential extensions ρ and ρ' (considered as embeddings of B in $M(A)/A$) are equivalent in the third sense (of Brown, Douglas, and Fillmore) if there exists $x\in M(A)/A$ such that

$$(1-x^*x)\rho(B)=(1-xx^*)\rho'(B)=0,\quad \rho'(b)=x\rho(b)x^*,\qquad b\in B.$$

(In [2], it was specified that x should be a partial isometry, but then it is not immediate that the relation is transitive, and in the case that A is arbitrary, or even separable and AF, it is not at all clear how to prove this.) In this case, we shall write $\rho' = x\rho x^*$. (If ρ is an essential extension, and if $x \in M(A)/A$ is such that $(1-x^*x)\rho(B) = 0$, then $b \mapsto x\rho(b)x^*$ is also an extension, say ρ', and necessarily $(1-xx^*)\rho'(B) = 0$, so we may write $\rho' = x\rho x^*$.) Clearly, if $\rho_1' = x_1\rho_1x_1^*$ and $\rho_2' = x_2\rho_2x_2^*$, then

$$v_1\rho_1'v_1^* + v_2\rho_2'v_2^* = x(v_1\rho_1v_1^* + v_2\rho_2v_2^*)x^*$$

where $x = v_1x_1v_1^* + v_2x_2v_2^*$. This shows that addition is compatible with the third equivalence relation.

It is easy to show that for equivalence classes in the third sense, addition as defined above is independent of the choice of v_1 and v_2. If v_1' and v_2' are two other isometries with orthogonal ranges (and here we do not need the existence of a third, v_3'), then

$$v_1'\rho_1v_2'^* + v_2'\rho_2v_2'^* = x(v_1\rho_1v_1^* + v_2\rho_2v_2^*)x^*$$

where $x = v_1'v_1^* + v_2'v_2^*$.

To show that for weak equivalence, addition as defined above is independent of the choice of v_1 and v_2, we shall use that also v_3 exists with $v_3^*v_3 = 1$ and $v_3^*v_1 = v_3^*v_2 = 0$. If v_1', v_2', and v_3' are three other isometries in $M(A)/A$ with orthogonal ranges, we shall show that there is a unitary u in $M(A)/A$ with $uv_1 = v_1'$ and $uv_2 = v_2'$. It follows that

$$v_1'\rho_1v_1'^* + v_2'\rho_2v_2'^* = u(v_1\rho_1v_1^* + v_2\rho_2v_2^*)u^*,$$

as desired.

With v_i and v_i' as above, $i = 1, 2, 3$, showing that, for some unitary u in $M(A)/A$, $uv_1 = v_1'$ and $uv_2 = v_2'$ is the same as showing that the projections $1 - (v_1v_1^* + v_2v_2^*)$ and $1 - (v_1'v_1'^* + v_2'v_2'^*)$ are equivalent in $M(A)/A$. By Theorem 1.4 of [4], the set of all projections in $M(A)/A$ containing a projection equivalent to 1 maps into a group in $D(M(A)/A)$, isomorphic to $K_0(M(A)/A)$. (This is true in any unital C*-algebra for the set of projections containing two orthogonal projections each equivalent to 1, provided that this set is not empty.) Since the projections $1-(v_1v_1^*+v_2v_2^*)$ and $1-(v_1'v_1'^*+v_2'v_2'^*)$ majorize $v_3v_3^*$ and $v_3'v_3'^*$ respectively, and belong to the same class in $K_0(M(A)/A)$ (namely, the class $-[1]$), it follows that they are equivalent.

To show that for strong equivalence, addition as defined above is independent of the choice of v_1 and v_2 (provided that there exists v_3 with $v_3^*v_3 = 1$ and $v_3^*v_1 = v_3^*v_2 = 0$), it would be sufficient to show that if v'_1, v'_2, and v'_3 are three other isometries in $M(A)/A$ with orthogonal ranges, then a unitary $u \in M(A)/A$ such that $uv_1 = v'_1$ and $uv_2 = v'_2$, which exists by the preceding paragraph, can be chosen to be connected to 1. It is not difficult to see that u can be chosen to be trivial in $K_1(M(A)/A)$. (The projection $1-(v_1v_1^*+v_2v_2^*)$ contains an infinite sequence of orthogonal projections equivalent to 1, and so contains a unitary belonging to an arbitrary class in $K_1(M(A)/A)$, in particular, to the class $[u_0]$ where u_0 is some unitary with $u_0v_1 = v'_1$, $u_0v_2 = v'_2$; if u_1 is such a unitary inside $1-(v_1v_1^*+v_2v_2^*)$ then with $u = u_0u_1^{-1}$ we have $uv_1 = v'_1$, $uv_2 = v'_2$, and $[u] = [u_0]-[u_0] = 0$.) It is conceivable that by using the ideas of [**4**] it is possible to show (assuming the existence of v_1 and v_2) that any unitary u in $M(A)/A$ which is trivial in K_1 is connected to 1. What we can show is that this is true if A is a separable AF algebra (and then we do not use the existence of v_1 and v_2): If u is a unitary which is trivial in $K_1(M(A)/A)$, so that it is connected to 1 in some matrix algebra, then it is liftable to a unitary in some matrix algebra over $M(A)$. On the other hand, u can be lifted to a partial isometry in $M(A)$ (this part of Lemma 2.6 of [**8**] is valid for any AF algebra). The index of this partial isometry is invariant under perturbation by a matrix with entries in A, and is therefore the same as the index of the unitary lifting u in a matrix algebra, namely, 0. But if the kernel and cokernel of a partial isometry in $M(A)$ lie in A and have the same K_0-class, then as A is an AF algebra these projections are equivalent, and so the partial isometry extends to a unitary in $M(A)$ (with image u in $M(A)/A$). By Theorem 2.4 of [**8**] (which is valid for any separable AF algebra), the unitary group of $M(A)$ is connected. Therefore u is connected to 1 in $M(A)/A$.

Finally, let us show that addition is associative. This can be done by showing that if ρ_1, ρ_2, and ρ_3 are three extensions, and if $(v'_i)_{1\le i\le 4}$ and $(v''_i)_{1\le i\le 4}$ are two sets of four isometries in $M(A)/A$ with orthogonal ranges, then the various kinds of equivalence classes of the two extensions

$$v'_1\rho_1v'^*_1 + v'_2\rho_2v'^*_2 + v'_3\rho_3v'^*_3,$$
$$v''_1\rho_1v''^*_1 + v''_2\rho_2v''^*_2 + v''_3\rho_3v''^*_3$$

are the same. In the case of weak equivalence, or equivalence in the third sense, the proof can be given just as above for one less isometry. In the case of strong equivalence, the proof can be given as before if A is a separable AF algebra. If A is arbitrary, then we can prove that the strong equivalence classes are the same if the isometries (v'_i) and

(v_i'') all belong to a single sub-C*-algebra D of $M(A)/A$ isomorphic to the algebra $\mathscr{O}_\infty$ of Cuntz. Indeed, by **[4]**, the unitary group of $\mathscr{O}_\infty$ is connected, and so the unitary constructed for weak equivalence also works for strong equivalence if it can be constructed in the algebra D. This is the case since Theorem 1.4 of **[4]**, applied before to $M(A)/A$, applies equally well to D (and to the set of projections in D containing a copy of 1—we do not need to know this is the set of all nonzero projections in D).

The isometries v_i' and v_i'' that arise in proving associativity are just monomials in given isometries v_1, v_2, and v_3 with orthogonal ranges, namely,

$$v_1' = v_1^2, \quad v_2' = v_1v_2, \quad v_3' = v_2, \quad v_4' = v_3,$$
$$v_1'' = v_1, \quad v_2'' = v_2v_1, \quad v_3'' = v_2^2, \quad v_4'' = v_3.$$

Furthermore, replacing v_3 by v_3v_1 we have that v_1, v_2, and v_3 are part of an infinite sequence v_1, v_2, $v_3, \ldots$ of isometries with mutually orthogonal ranges, namely, in the original notation, $v_1, v_2, v_3v_1, v_3v_2v_1, v_3v_2^2v_1, \ldots$. These generate a sub-C*-algebra D of $M(A)/A$ isomorphic to the (simple) C*-algebra $\mathscr{O}_\infty$, as desired.

This completes the proof of the theorem.

With addition of extensions as defined above, the sum of two unital extensions is no longer unital (except possibly with respect to the third equivalence relation). If, in $D(M(A))$, $\dim(1)+\dim(1)$ not only is defined but also is equal to $\dim(1)$ (this happens much less often)—equivalently, if there exist two isometries in $M(A)/A$ with orthogonal ranges the sum of which is equal to 1—then the sum of two unital extensions constructed as above but using two such isometries instead is again a unital extension. It is easy to see that for weak equivalence, or for equivalence in the third sense, this addition of unital extensions is compatible with equivalence, and the resulting addition of equivalence classes is canonical and associative. Compatibility of addition with strong equivalence is also clear; however, the resulting addition of strong equivalence classes of unitary extensions is not in general canonical—even in the case $A = \mathscr{K}$ it depends on the choice of the two isometries with range projections summing to 1. Nevertheless, it is still associative. The reason for this is that in the C*-algebra generated by two isometries with orthogonal ranges with sum 1 (i.e., in the C*-algebra $\mathscr{O}_2$ of Cuntz), every unitary is connected to 1 **[4]**. Therefore, as before, the proof of associativity of addition of weak equivalence classes also works for strong equivalence classes.

Since, as we have seen, it is of interest whether $\mathcal{O}_\infty$, or $\mathcal{O}_2$, is contained in the quotient multiplier algebra $M(A)/A$ ($\mathcal{O}_\infty$ for adding arbitrary essential extensions of A by B, and $\mathcal{O}_2$ for adding unital ones), we shall give criteria for this in the case that A is a separable AF algebra. Theorem 3.1 gives criteria for embedding $\mathcal{O}_\infty$, and Theorem 3.3 for embedding $\mathcal{O}_2$. As a natural continuation, we shall go on to give criteria for embedding a more general Cuntz-Krieger algebra $\mathcal{O}_P$ in $M(A)/A$. (Such an embedding is, after all, an extension in its own right.)

These results are based on a computation of $K_0(M(A))$ (and $D(M(A))$) given in §2.

Some results concerning the semigroup of extensions in various cases are given in §4.

2. Calculation of $K_0(M(A))$.

2.1. THEOREM. *Let A be a separable AF algebra, and let e and f be projections in $M(A)$. The following two conditions are equivalent*:

(i) *e is equivalent to f in $M(A)$.*

(ii) *$D(eAe) = D(fAf)$.*

Proof. Ad (i) $\Rightarrow$ (ii). Let u be a partial isometry in $M(A)$ such that $ueu^* = f$. If e_1 is a projection in eAe then ue_1u^* is an equivalent projection in fAf, and any projection in fAf is obtained in this way. This shows that $D(eAe)$ and $D(fAf)$ are equal as subsets of $D(A)$.

Ad (ii) $\Rightarrow$ (i). Suppose that $D(eAe) = D(fAf)$, by which we mean equality as subsets of $D(A)$. By Theorem 3.1 of **[10]** there exist approximate units (e_i) and (f_i) of eAe and fAf, respectively, consisting of projections. As A is separable we may choose (e_i) and (f_i) to be sequential and increasing. Since $D(eAe) \subseteq D(fAf)$, for each i there is a j such that e_i is equivalent to part of f_j in A, i.e., $[e_i] \le [f_j]$ in $D(A)$. Similarly, since $D(fAf) \subseteq D(eAe)$, for each i there is a j such that $[f_i] \le [e_j]$. Hence, passing to subsequences of (e_i) and (f_i), we may suppose that, for each i,

$$[e_i] \le [f_{i+1}], \qquad [f_i] \le [e_{i+1}].$$

Thus,

$$[e_1] \le [f_2] \le [e_3] \le [f_4] \le \cdots .$$

Hence, for each i, there is a projection $e'_{2i} \in A$ such that $e_{2i-1} \le e'_{2i} \le e_{2i+1}$ and, moreover, $[e'_{2i}] = [f_{2i}]$. Similarly, for each i there is a projection $f'_{2i-1} \in A$ such that $f_{2i-2} \leq f'_{2i-1} \leq f_{2i}$ (here as above, $i \geq 1$; by f_0 we just mean 0), and, moreover, $[f'_{2i-1}] = [e_{2i-1}]$. Replacing e_{2i} by e'_{2i} and f_{2i-1} by f'_{2i-1} for each i, we have that the sequences (e_i) and (f_i), in addition to being increasing approximate units for eAe and fAf, respectively, such that $[e_1] \leq [f_2] \leq [e_3] \leq [f_4] \leq \cdots$, satisfy the stronger condition

$$[e_i] = [f_i], \quad \text{all } i.$$

It follows that with u_i, for each i, a partial isometry in A such that

$$u_i^* u_i = e_i - e_{i-1} \quad (\text{where } e_0 = 0),$$
$$u_i u_i^* = f_i - f_{i-1} \quad (\text{where, as above, } f_0 = 0),$$

the sum $\sum u_i$ converges in the strict topology of $M(A)$ to a partial isometry $u \in M(A)$, and $u^*u = e$, $uu^* = f$.

2.2. Lemma. *Let A be an AF algebra, let τ be a semifinite lower semicontinuous trace on A^+, and let e be a projection in $M(A)$. It follows that*

$$\tau(e) = \sup \tau(D(eAe)).$$

Proof. By 6.6.6 of **[6]** there is a representation of A canonically associated to τ, and as this is nondegenerate it extends uniquely by 2.10.4 of **[6]** to a representation of $M(A)$, with the same weak closure. By 6.6.5 of **[6]** τ extends uniquely to a faithful semifinite normal trace on the weak closure of the image of A in this representation, and by $\tau(e)$ we mean the value at e of this extended trace on $M(A)^+$.

By Theorem 3.1 of **[10]**, eAe has an approximate unit (e_i) consisting of projections. Then e_i converges to e in the strict topology of $M(A)$, and hence in the weak topology of the bidual of A. Since a normal trace is weakly lower semicontinuous, and $\tau(e) \geq \tau(e_i)$ for all i, it follows that $\tau(e_i)$ converges to $\tau(e)$.

2.3. Theorem. *Let A be an AF algebra, and let e and f be projections in $M(A)$. Suppose that eAe has no nonzero unital quotient. The following two conditions are equivalent*:

(ii) $D(eAe) \subseteq D(fAf)$.

(iii) $\tau(e) \leq \tau(f)$ *whenever τ is a semifinite lower semicontinuous trace on A^+.*

Proof. Ad (ii) $\Rightarrow$ (iii). This follows immediately from Lemma 2.2.

Ad (iii) $\Rightarrow$ (ii). Let e_1 be a projection in eAe, and, assuming (iii), let us prove that e_1 is equivalent to some projection f_1 in fAf.

First let us prove that for any semifinite lower semicontinuous trace τ on A^+ such that $\tau(e_1)$ is finite and not zero, $\tau(e_1) < \sup \tau(D(eAe))$. To see this, note that if $\sup \tau(D(eAe))$ is finite and equal to $\tau(e_1)$, then the quotient of eAe by the kernel of $\tau|eA^+e$ is unital. (By Theorem 3.1 of **[10]**, eAe is an AF algebra and therefore so is the quotient. Passing to the quotient we have that τ is a faithful trace and its supremum on projections is attained at e_1. In particular, for every projection e_2 bigger than e_1, $\tau(e_2 - e_1) = 0$ and so $e_2 = e_1$. Thus e_1 is a maximal projection, but in an AF algebra a maximal projection must be a unit.) It follows by hypothesis that $\tau|eA^+e$ is zero, and in particular $\tau(e_1) = 0$. This proves the assertion.

Next denote by T the compact space of tracial states of e_1Ae_1. By the result of the preceding paragraph, for every $\tau \in T$ there exists a projection $e_1' \in eAe$ such that $\tau(e_1) < \tau(e_1')$, where by $\tau(e_1')$ we mean the value at e_1' of the smallest extension of τ to a semifinite lower semicontinuous trace on A^+—which is constructed as in the proof of Lemma 2.2.

By Lemma 2.2, for each semifinite lower semicontinuous trace τ on A^+,

$$\sup \tau(D(eAe)) = \tau(e) \leq \tau(f) = \sup \tau(D(fAf)).$$

It follows from this and the preceding paragraph that for each $\tau \in T$ there exists a projection $g \in fAf$ such that $\tau(e_1) < \tau(g)$.

Let us verify that, for each projection g in A, the map $\tau \mapsto \tau(g)$ from T to $\mathbf{R}^+ \cup \{+\infty\}$ is lower semicontinuous on T. For each $\tau \in T$, from normality of τ in the trace representation, in which the closed two-sided ideal I of A generated by e_1 is nondegenerate, it follows that $\tau(g) = \sup \tau(D(gIg))$. It is therefore sufficient to show that $\tau \mapsto \tau(g)$ is lower semicontinuous in the case that $g \in I$. Since e_1 generates the closed two-sided ideal I, if g is a projection in I then g is equivalent to a projection in $M_n(e_1Ae_1)$ inside $M_n(I)$ for some $n = 1, 2, \ldots$. (This presumably is true even if A is not an AF algebra, but to establish it we appeal to the bijective correspondence between closed two-sided ideals of an AF algebra and order ideals of its dimension group; see Section 5.1 of **[11]**. As a consequence of this, $g \in I$ if, and only if, the equivalence class $[g]$ of g belongs to the order ideal generated by the equivalence class $[e_1]$ of e_1, i.e., if and only if $[g] \leq n[e_1]$ for some $n = 1, 2, \ldots$. This correspondence of ideal structures is perhaps best known for separable AF algebras, but follows in general by using the fact that the collection of separable AF subalgebras of an AF algebra is upward directed.) It follows that the case $g \in I$ is equivalent to the case $g \in M_n(e_1Ae_1)$ (the function $\tau \mapsto \tau(g)$ is the same). In this case, then, the function $\tau \mapsto \tau(g)$ is in fact continuous on T.

It follows from the preceding two paragraphs and compactness of T that there exists a finite set P of projections in fAf such that

$$\tau(e_1) < \sup \tau(P), \qquad \tau \in T.$$

Since $D(fAf)$ is upward directed, there exists a single projection $g \in fAf$ such that $\tau(e_1) < \tau(g)$, $\tau \in T$.

Let g be such a projection, and let us prove that e_1 is equivalent to a subprojection of g. Denote by J the closed two-sided ideal of A generated by e_1 and g. Since $D(J)$ is upward directed, there is a projection p in J such that both e_1 and g are equivalent to subprojections of p. Replacing e_1 and p by equivalent projections (recall that equivalence of projections in A is the same as unitary equivalence in $M(A)$), we may suppose that e_1 and g are both majorized by p. To show that e_1 is equivalent to part of g in pAp, an AF algebra with unit, by Theorem 1.4 of **[7]** (see also Lemma 4.1 of **[12]**) it is sufficient to show that $\tau'(e_1) < \tau'(g)$ for every tracial state τ' of pAp.

Let τ' be a tracial state of $pAp = pJp$, and let us prove that $\tau'(e_1) < \tau'(g)$. First, if $\tau'(e_1) = 0$ then $\tau'(g) \neq 0$, since $\tau'(p) \neq 0$ and p belongs to J, the closed two-sided ideal generated by e_1 and g. Therefore in this case, $\tau'(e_1) < \tau'(g)$. Second, if $\tau'(e_1) \neq 0$, denote the restriction of $\tau'(e_1)^{-1}\tau'$ to e_1Ae_1 by τ. Then $\tau \in T$, and if as above we denote by τ also the least extension of τ to a semifinite lower semicontinuous trace on pJ^+p, then $\tau \leq \tau'(e_1)^{-1}\tau'$, since $\tau'(e_1)^{-1}\tau'$ is a semifinite lower semicontinuous (in fact, finite continuous) trace on pJ^+p agreeing with τ on e_1Je_1. In particular, $\tau(g) \leq \tau'(e_1)^{-1}\tau'(g)$. But since $\tau \in T$, by the choice of g we have $\tau(e_1) < \tau(g)$. Therefore in this second case,

$$\tau'(e_1) = \tau'(e_1)\tau(e_1) < \tau'(e_1)\tau(g) \leq \tau'(g).$$

This shows that e_1 is equivalent to a projection f_1 in gAg. Since $g \in fAf$, $f_1 \in fAf$ as desired.

2.4. COROLLARY. *Let A be a separable AF algebra, and let e and f be projections in $M(A)$. Suppose that neither eAe nor fAf has a nonzero unital quotient. The following two conditions are equivalent:*

(i) *e is equivalent to f in $M(A)$.*

(iii) *$\tau(e) = \tau(f)$ whenever τ is a semifinite lower semicontinuous trace on A^+.*

Proof. This follows immediately from Theorems 2.1 and 2.3.

2.5. COROLLARY. *Let A be a separable AF algebra with no nonzero unital quotient, and let e and f be projections in $M(A)$. The following three conditions are equivalent*:

(i) $1 \oplus e$ *is equivalent to* $1 \oplus f$ *in* $M_2(M(A))$.
(iii) $\tau(e) = \tau(f)$ *for all* $\tau \in T(A) \subseteq T(M(A))$.
(iv) $\tau(e) = \tau(f)$ *for all* $\tau \in T(M(A))$.

Proof. The implications (i) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (iii) are immediate.

Ad (iii) $\Rightarrow$ (i). We shall deduce this from (iii) $\Rightarrow$ (i) of Corollary 2.4, by showing that the present condition (iii) implies Condition 2.4(iii) with $M_2(A)$ in place of A, and $1 \oplus e$, $1 \oplus f$ in place of e, f. Let τ be a semifinite lower semicontinuous trace on $M_2(A)^+$. If $\tau(1 \oplus 0)$ is infinite, then $\tau(1 \oplus e)$ and $\tau(1 \oplus f)$ are both infinite and therefore equal. If $\tau(1 \oplus 0)$ is zero, then $\tau(1 \oplus e)$ and $\tau(1 \oplus f)$ are both zero and therefore equal. If $\tau(1 \oplus 0)$ is finite and not zero, then we may suppose that $\tau(1 \oplus 0) = 1$ and so $\tau|A \in T(A)$. Hence in this case $\tau(1 \oplus e) = \tau(1 \oplus f)$ follows from (iii).

2.6. THEOREM. *Let A be an AF algebra. If e is a projection in $M(A)$ then $D(eAe)$ and $D((1-e)A(1-e))$ are intervals in $D(A)$, and*

$$D(eAe) + D((1-e)A(1-e)) = D(A).$$

Conversely, if D_1 and D_2 are intervals in $D(A)$ such that

$$D_1 + D_2 = D(A),$$

and if A is separable, then there exists a projection e in $M(A)$ such that

$$D_1 = D(eAe), \quad D_2 = D((1-e)A(1-e)).$$

Proof. Let e be a projection in $M(A)$. By Theorem 3.1 of **[10]** both $A_1 = eAe$ and $A_2 = (1-e)A(1-e)$ are AF algebras. In particular, $D(A_1)$ and $D(A_2)$ are intervals in $D(A)$. (The only property of an interval which is not obviously possessed by $D(A_i)$ is upward directedness, but this holds in the dimension range of an AF algebra.) Since $A_1A_2 = A_2A_1 = 0$, clearly $D(A_1) + D(A_2)$ is contained in $D(A)$.

On the other hand, $A_1 + A_2$ contains an approximate unit for A, so every element g of $D(A)$ is majorized by $f_1 + f_2$ for some $f_1 \in D(A_1)$, $f_2 \in D(A_2)$. By the Riesz decomposition property, $g \le f_1 + f_2$ implies $g = g_1 + g_2$ with $g_1 \le f_1$, $g_2 \le f_2$, in particular, with $g_1 \in D(A_1)$, $g_2 \in D(A_2)$. This shows that $D(A)$ is contained in $D(A_1) + D(A_2)$.

Now let D_1 and D_2 be intervals in $D(A)$ such that $D_1 + D_2 = D(A)$. Suppose that A is separable. Let us show that there exist mutually orthogonal projections

$$e_1, e_2, \ldots, \quad f_1, f_2, \ldots$$

in A such that the finite sums $(e_1 + f_1) + \cdots + (e_k + f_k)$ are an approximate unit for A, the finite sums $[e_1] + \cdots + [e_k]$ belong to D_1 and generate D_1 as an interval, and the finite sums $[f_1] + \cdots + [f_k]$ belong to D_2 and generate D_2 as an interval. Since D_1 and D_2 are countable, there exist sequences

$$g_1, g_2, \ldots, \quad h_1, h_2, \ldots$$

in D_1 and D_2 respectively such that the finite sums $g_1 + \cdots + g_k$ belong to D_1 and generate D_1 as an interval, and the finite sums $h_1 + \cdots + h_k$ belong to D_2 and generate D_2 as an interval. Since $D_1 + D_2 = D$ it follows that the finite sums $(g_1 + h_1) + \cdots + (g_k + h_k)$ belong to D and generate D as an interval (i.e., eventually majorize any element of D). Choose inductively mutually orthogonal projections

$$e'_1, f'_1, e'_2, f'_2, \ldots$$

in A such that $[e'_1] = g_1$, $[f'_1] = h_1, \ldots$. Denote by B the hereditary sub-C*-algebra of A generated by $e'_1, f'_1, e'_2, f'_2, \ldots$. Since B is the closure of the union of the sequence $p_1Ap_1 \subseteq p_2Ap_2 \subseteq \cdots$ where $p_k = (e'_1 + f'_1) + \cdots + (e'_k + f'_k)$, $D(B) = \bigcup D(p_kAp_k)$. Since $[p_1], [p_2], \ldots$ generate $D(A)$ as an interval, $D(B) = D(A)$. Hence by Theorem 4.3 of **[9]**, B is isomorphic to A, and by an isomorphism acting as the identity map from $D(B)$ to $D(A)$. It follows that with $e_1, e_2, \ldots, f_1, f_2, \ldots$ the images in A, by such an isomorphism, of the projections $e'_1, e'_2, \ldots, f'_1, f'_2, \ldots$ in B, the required conditions are fulfilled.

2.7. COROLLARY. *Let A be a separable AF algebra. There is a bijective correspondence between unitary equivalence classes of projections in $M(A)$ and ordered pairs (D_1, D_2) where D_1 and D_2 are intervals in $D(A)$ such that $D_1 + D_2 = D(A)$.*

Proof. This follows immediately from Theorems 2.1 and 2.6.

2.8. LEMMA. *Let A be a C*-algebra, and let I be a closed two-sided ideal of A. The map which to each state of I (i.e., to each positive linear* *functional on I of norm equal to one) associates the unique extension to a state of A is continuous with respect to the weak* topologies.*

Proof. Let (ω_i) be a net of states of I converging to the state ω of I, on each element of I. We must show that ω_i converges to ω on each element of A. (Here we use the same notation for the unique state extensions to A.) It is sufficient to pass to an arbitrary subnet of (ω_i) and then show that there exists a subnet converging to ω on A. By compactness of the unit ball of the dual of A, there is a subnet of (ω_i) converging on A to a positive linear functional ϕ on A of norm at most one. Since ϕ agrees with ω on I and $\|\phi\| \leq 1$, and $\|\omega|I\| = 1$, ϕ must agree with ω on A.

2.9. Lemma. *Let A be a separable AF algebra with no nonzero unital quotient. Let $f \in \operatorname{Aff} T(M(A))$ be such that for some $\varepsilon > 0$,*

$$\varepsilon \leq f(\tau) \leq 1 - \varepsilon, \quad \tau \in T(A) \subseteq T(M(A)).$$

It follows that there exists a projection e in $M(A)$ such that

$$f(\tau) = \tau(e), \quad \tau \in T(A) \subseteq T(M(A)).$$

Proof. Set $D(A) = D$. We must show that there are intervals D_1 and D_2 in D such that

$$D_1 + D_2 = D, \quad f(\tau) = \sup \tau(D_1), \qquad \tau \in T(A).$$

To see that it is sufficient to do this (and also necessary), refer to Theorem 2.6 and Lemma 2.2.

Define subsets D_1 and D_2 of D as follows:

$$D_1 = \{g \in D; \tau(g) < f(\tau) \text{ for all } \tau \in T(A)\},$$
$$D_2 = \{g \in D; \tau(g) < 1 - f(\tau) \text{ for all } \tau \in T(A)\}.$$

Let us prove that D_1 and D_2 are intervals in D fulfilling the requirements.

Note that the property $f(\tau) = \sup \tau(D_1)$, $\tau \in T(A)$, is a consequence of the definition of D_1 and D_2 together with the property $D \subseteq D_1 + D_2$ (to be proved): for each $\tau \in T(A)$ we have, successively,

$$f(\tau) \geq \sup \tau(D_1), \quad 1 - f(\tau) \geq \sup \tau(D_2),$$

$$1 = f(\tau) + (1 - f(\tau)) \geq \sup \tau(D_1) + \sup \tau(D_2)$$
$$\geq \sup \tau(D_1 + D_2) \geq \sup \tau(D) = 1,$$

$$f(\tau) = \sup \tau(D_1), \quad 1 - f(\tau) = \sup \tau(D_2).$$

Thus, what needs to be proved is that D_1 and D_2 are upward directed, and $D_1 + D_2 = D$.

Note first that no closed two-sided ideal of A has a nonzero unital quotient: If I and J are closed two-sided ideals of A such that $J \subseteq I$ and I/J is unital, then I/J is a direct summand of A/J, and therefore at the same time a quotient of A/J, and hence of A. Since by hypothesis any unital quotient of A is zero, I/J is zero.

Note next that if I is a closed two-sided ideal of A, then the restriction map $T(M(I)) \to T(M(A))$ is affine and continuous, so we may view f as in $\operatorname{Aff} T(M(I))$. Furthermore, there is a natural injection of $T(I)$ into $T(A)$. Let us show that

$$D_1 \cap D(I) = \{g \in D(I); \tau(g) < f(\tau) \text{ for all } \tau \in T(I)\},$$
$$D_2 \cap D(I) = \{g \in D(I); \tau(g) < 1 - f(\tau) \text{ for all } \tau \in T(I)\}.$$

Since $T(I) \subseteq T(A)$, the left sides are contained in the right sides. Suppose now that $g \in D(I)$ and $\tau(g) < f(\tau)$ for all $\tau \in T(I)$, and let $\tau \in T(A)$. If $\|\tau|I\| = 1$ then $\tau \in T(I)$, so we may assume that $\|\tau|I\| < 1$. If $\tau|I = 0$ then $\tau(g) = 0$, and by hypothesis $0 < f(\tau)$, so $\tau(g) < f(\tau)$. Assuming $\tau|I \neq 0$, set $\|\tau|I\|^{-1}\tau|I = \tau_0$; then $\tau_0 \in T(I)$. Considering the image of τ_0 in $T(A)$, recall that τ_0 is the least extension of $\|\tau|I\|^{-1}\tau|I$ to a continuous trace on A^+. In particular, $\tau_0 \leq \|\tau|I\|^{-1}\tau$ on A^+, so $\tau - \|\tau|I\|\tau_0$ is a trace on A^+. Since $\|\tau|I\| < 1$, $\tau - \|\tau|I\|\tau_0 \neq 0$. Set $\|\tau - \|\tau|I\|\tau_0\|^{-1}(\tau - \|\tau|I\|\tau_0) = \tau_1$; then $\tau_1 \in T(A)$, and $\tau = \lambda_0\tau_0 + \lambda_1\tau_1$ where $\lambda_0 = \|\tau|I\|$, $\lambda_1 = \|\tau - \|\tau|I\|\tau_0\|$. Evaluating at 1 gives $\lambda_0 + \lambda_1 = 1$, and hence since f is affine and positive,

$$f(\tau) = \lambda_0 f(\tau_0) + \lambda_1 f(\tau_1) \geq \lambda_0 f(\tau_0).$$

On the other hand, since $\tau_0 \in T(I)$, we have $\tau_0(g) < f(\tau_0)$. Since $\tau_1|I = 0$, combining these inequalities gives

$$\tau(g) = \lambda_0\tau_0(g) < \lambda_0 f(\tau_0) \leq f(\tau),$$

as desired. Together with a similar argument (or the conclusion) with f replaced by $1 - f$, this shows that the left sides contain the right sides and so are equal to them.

Let us prove that D_1 and D_2 are upward directed. Let g_1 and g_2 be in D_1; we must find $g \in D_1$ majorizing both g_1 and g_2. Clearly it is sufficient to do this inside the ideal of $K_0(A)$ generated by g_1 and g_2, which we denote by $D(I)$—I being the corresponding closed two-sided ideal of A. By the results of the preceding two paragraphs we may just pass to I and suppose that $I = A$. In particular, $K_0(A)$ now has an order unit (e.g. $g_1 + g_2$). Choose an order unit u of $K_0(A)$; since D is upward directed and $u \in nD$ for some $n = 1, 2, \ldots$, it is possible to choose u in D. Denote the state space $S(K_0(A), u)$ by S. For each $\tau \in S$ set $\sup \tau(D) = \|\tau\|$; since $u \in D$ and $\tau(u) = 1$, $\|\tau\| \geq 1$. If $\tau \in S$ and $\|\tau\| < +\infty$ then $\|\tau\|^{-1}\tau \in T(A)$, so we may define an extended positive real-valued function $\overline{f}$ on S by

$$\overline{f}(\tau) = \begin{cases} +\infty, & \|\tau\| = +\infty, \\ f(\|\tau\|^{-1}\tau)\|\tau\|, & \|\tau\| < +\infty. \end{cases}$$

We shall prove that $\overline{f}$ is affine and lower semicontinuous on S. By Lemma 2.8, the map $T(A) \to T(M(A))$ is continuous, so we may view f as a continuous affine function on $T(A)$. We shall also denote by f the unique affine extension of f to the convex cone $\mathbf{R}^+T(A)$ which is 0 at 0. Let us first show that f is lower semicontinuous on $\mathbf{R}^+T(A)$. Let $\lambda_i\tau_i$ converge to $\lambda\tau$, where τ_i and τ belong to $T(A)$ and λ_i and λ to $\mathbf{R}^+$. To show that $\liminf f(\lambda_i\tau_i) \geq f(\lambda\tau)$ it is sufficient to pass to an arbitrary subnet and then show that this holds at least for some subnet, so we may suppose that λ_i converges to $\lambda' \in \mathbf{R}^+ \cup \{+\infty\}$. If $\lambda' = +\infty$ then as $f(\lambda_i\tau_i) = \lambda_i f(\tau_i) \geq \lambda_i\varepsilon$, $f(\lambda_i\tau_i) \to +\infty$. If $\lambda' = 0$ then as $\|\tau_i\| = 1$, $\lambda_i\tau_i \to 0$ and $f(\lambda\tau) = f(0) = 0$. In both these cases the inequality $\liminf f(\lambda_i\tau_i) \geq f(\lambda\tau)$ is trivial. If $0 < \lambda' < +\infty$, then $\lambda'\tau_i$ converges to $\lambda\tau$. Since $T(M(A))$ is compact we may pass to a subnet and suppose that τ_i converges in $T(M(A))$, say to τ'. Since $\lambda'\tau'$ agrees with $\lambda\tau$ on A, $\lambda'\tau' \geq \lambda\tau$ on $M(A)$ (i.e. on $M(A)^+$). By continuity of f on $T(M(A))$, $f(\tau_i) \to f(\tau')$. Hence

$$f(\lambda_i\tau_i) = \lambda_i f(\tau_i) \to \lambda' f(\tau') = f(\lambda'\tau') \geq f(\lambda\tau),$$

as desired. (Here the inequality $f(\lambda'\tau') \geq f(\lambda\tau)$ is in terms of the affine extension of f to the cone $\mathbf{R}^+T(M(A))$, equal to 0 at 0. The inequality holds since f is still positive, and since $\lambda'\tau' = \lambda\tau + (\lambda'\tau' - \lambda\tau)$, and $\lambda'\tau' - \lambda\tau \in \mathbf{R}^+T(M(A))$.)

Now let us prove that $\overline{f}$ is affine and lower semicontinuous on S. Note that if $\tau \in S$ and $\overline{f}(\tau) < +\infty$, then $\overline{f}(\tau) = f(\tau)$. It follows that $\overline{f}$ is affine and lower semicontinuous where it is finite. Since $\overline{f}(\tau) = +\infty$ exactly when $\|\tau\| = +\infty$, it follows that $\overline{f}$ is affine on S. (We must use the convention $0 \cdot (+\infty) = 0$.) Since $f \geq \varepsilon$ on $T(A)$, we have $f(\tau) \geq \varepsilon\|\tau\|$ for all $\tau \in \mathbf{R}^+T(A)$, and so $\overline{f}(\tau) \geq \varepsilon\|\tau\|$ for all $\tau \in S$. Since $\tau \mapsto \|\tau\| = \sup \tau(D)$ is lower semicontinuous on S, $\overline{f}$ is lower semicontinuous at each point of S where it is infinite.

Since $D_1 = \{g \in D(A); \tau(g) < \overline{f}(\tau)$ for all $\tau \in S\}$, it follows by Lemma 2.10 of **[14]** that D_1 is upward directed. Replacing f by $1-f$ we deduce that D_2 is upward directed.

Next let us prove that $D_1 + D_2 \subseteq D$. If $g_1 \in D_1$ and $g_2 \in D_2$ then $g_1 + g_2 \in K_0(A)^+$ and, for all $\tau \in T(A)$,

$$\tau(g_1 + g_2) = \tau(g_1) + \tau(g_2) < f(\tau) + (1 - f(\tau)) = 1.$$

To prove that $g_1 + g_2$ belongs to D we may pass to the closed two-sided ideal of A generated by a projection with dimension $g_1 + g_2$. (Cf. above.) Then $g_1 + g_2$ is an order unit for $K_0(A)$, and it follows by Proposition 7.7 of **[14]** that $g_1 + g_2 \in D$. (The hypothesis of Proposition 7.7 of **[14]** that D be soft holds by Proposition 7.5 of **[14]**, as A has no nonzero unital quotient.)

Finally, let us prove that $D \subseteq D_1 + D_2$. Let $g \in D$. To prove that $g \in D_1 + D_2$ we may pass to the closed two-sided ideal of A generated by a projection with dimension g. (Cf. above.) Then g is an order unit for $K_0(A)$, and we may apply Theorem 4.3(a) of **[14]** with $G_1 = K_0(A)$, $G_2 = \mathbf{Z}\oplus\mathbf{Z}$, $u_1 = g$, $G = G_1 \oplus G_2$, and $l\colon \pi^{-1}(G_2^+) \to \Lambda(S_1)$ the additive map which on G_1 is the canonical map, on $(1,0) \in G_2$ is the function $\overline{f}$ defined above, and on $(0,1) \in G_2$ is the function $(1-f)^-$ obtained by replacing f by $1-f$ in the definition of f^-. The map l is judiciously infinite in the sense of **[14]** because A has no nonzero unital quotient. By Theorem 4.3(a) of **[14]**, G becomes a dimension group in an order with respect to which

$$G_1 \cap [0, (1,0)] = D_1,$$
$$G_1 \cap [0, (0,1)] = D_2,$$

where $[0, (1,0)]$ denotes the interval in G between 0 and $(1,0) \in G_2$. Furthermore, in this order,

$$0 \leq g \leq (1,1)$$

(this holds for any element of D). In this step we have again used Proposition 7.7 of **[14]**, which is applicable as D is soft. Since $(1,1) = (1,0) + (0,1)$, by Riesz interpolation in G we have $g = g_1 + g_2$ with $g_1 \in [0, (1,0)]$, $g_2 \in [0, (0,1)]$. Since $g \in G_1^+$, also g_1 and g_2 are in G_1, which is an ideal of G. Therefore $g_1 \in D_1$, $g_2 \in D_2$.

2.10. THEOREM. *Let A be a separable AF algebra with no nonzero unital quotient. $T(A)$ is dense in $T(M(A))$ in the weak* topology.*

Proof. (The result may hold for any separable AF algebra A, but our proof depends on Corollary 2.5 and Lemma 2.9, which are not valid in the general case.)

Note first that $K_0(M(A))$ separates the points of $T(M(A))$. This follows immediately from the fact, established in the proof of Theorem 3.1 of **[8]**, that $M(A)$ is generated as a Banach space by its projections. (More precisely, it was shown in **[8]** that, for any separable AF algebra A, each element of A is a sum of four elements—one in A, and each of the three others commuting with some sequential increasing approximate unit for A consisting of projections, and therefore a limit of linear combinations of projections.)

It follows by the Hahn-Banach theorem that $\mathbf{Q}$ times the image of $K_0(M(A))$ in $\operatorname{Aff} T(M(A))$ is norm dense.

By Lemma 2.9, if an element of $K_0(M(A))$ is greater than ε on $T(A)$ for some $\varepsilon > 0$, it is equal on $T(A)$ to the class of some projection in a matrix algebra over $M(A)$. Let $[e] - [f]$ be such an element of $K_0(M(A))$. Passing to a matrix algebra over $M(A)$, we have orthogonal projections e, f, and g in $M(A)$ with $[e] - [f]$ and $[g]$ agreeing on $T(A)$, i.e. with e and $f + g$ agreeing on $T(A)$. By (iii) $\Rightarrow$ (iv) of Corollary 2.5, e and $f + g$ agree on $T(M(A))$. This shows that $[e] - [f]$ is positive (in fact, greater than ε) on all of $T(M(A))$.

It follows that if the product of an element of $\mathbf{Q}$ and an element of the image of $K_0(M(A))$ in $\operatorname{Aff} T(M(A))$ is greater than some $\varepsilon > 0$ on $T(A)$ then it is positive on $T(M(A))$. By the density of such elements in $\operatorname{Aff} T(M(A))$, we deduce that if an arbitrary element of $\operatorname{Aff} T(M(A))$ is positive on $T(A)$ then it is positive on $T(M(A))$. (If $f \in \operatorname{Aff} T(M(A))$ is positive on $T(A)$ and $\varepsilon > 0$ then $f + \varepsilon \geq 0$ on $T(A)$ and so $g \geq \varepsilon/2$ on $T(A)$ whenever $\|f - g\| \leq \varepsilon/2$. If in addition g belongs to $\mathbf{Q}$ times the image of $K_0(M(A))$, then it follows (as shown above) that $g \geq 0$ on $T(M(A))$. Hence $f + \varepsilon/2 \geq 0$ on $T(M(A))$, for arbitrary $\varepsilon > 0$, i.e. $f \geq 0$ on $T(M(A))$.)

By the Hahn-Banach separation theorem (applied to the dual of $\operatorname{Aff} T(M(A))$ with the weak* topology) it follows that $T(A)$ is dense in $T(M(A))$.

2.11. THEOREM. *Let A be a separable AF algebra with no nonzero unital quotient. Let $f \in \operatorname{Aff} T(M(A))$ be such that*

$$0 < f(\tau) < 1, \qquad \tau \in T(M(A)).$$

It follows that there exists a projection e in $M(A)$ such that

$$\begin{aligned} \tau(e) &= f(\tau), && \tau \in T(M(A)), \\ \tau(e) &= +\infty, && \tau \text{ an infinite semifinite lower semicontinuous trace on } A^+. \end{aligned}$$

Proof. By compactness of $T(M(A))$ there exists $\varepsilon > 0$ such that

$$\varepsilon \leq f(\tau) \leq 1 - \varepsilon, \qquad \tau \in T(M(A)).$$

Hence by Lemma 2.9 there exists a projection $e \in M(A)$ such that

$$f(\tau) = \tau(e), \qquad \tau \in T(A) \subseteq T(M(A)).$$

By Theorem 2.10, $T(A)$ is weak* dense in $T(M(A))$, so the equality holds for all $\tau \in T(M(A))$.

The construction of Lemma 2.9 in fact yields a projection e satisfying the second condition of the theorem as well. To see this, choose $n \in \{2, 3, \ldots\}$ such that $1/n < \min(\varepsilon, 1 - \varepsilon)$, and denote $\{g \in D; \tau(g) < 1/n \text{ for all } \tau \in T(A)\}$ by $D_{1/n}$. From the construction of e it is immediate that $D_{1/n} \subseteq D(eAe)$. On the other hand, the proof of Theorem 2.9 shows that $D_{1/n}$ is an interval, and the sum of $D_{1/n}$ taken n times is equal to D. It follows that if $\tau(1) = +\infty$, i.e. $\sup \tau(D) = +\infty$ (recall $D = D(A)$), then $\sup \tau(D_{1/n}) = +\infty$ and hence $\tau(e) = +\infty$.

2.12. Corollary. *Let A be a separable AF algebra with no nonzero unital quotient. The canonical map*

$$K_0(M(A)) \to \operatorname{Aff} T(M(A))$$

is bijective. It takes $K_0(M(A))^+$ into a semigroup of positive elements of Aff $T(M(A))$, *containing the semigroup of strictly positive elements, together with* 0. *In the case that A is simple, the image of $K_0(M(A))^+$ is equal to the latter semigroup if, and only if, $T(A) = T(M(A))$.*

Proof. Let us first show that the map is injective. We must show that if e and f are projections in $M(A)$ (or in a matrix algebra over $M(A)$) such that $\tau(e) = \tau(f)$ for every $\tau \in T(M(A))$, then the classes of e and f in $K_0(M(A))$ are equal. This follows immediately from (iv) $\Rightarrow$ (i) of Corollary 2.5.

Let us next show that any strictly positive element of Aff $T(M(A))$ is the function on $T(M(A))$ determined by a projection in some matrix algebra over $M(A)$. Since $T(M(A))$ is compact there exist

$n \in \{1, 2, \dots\}$ and $\varepsilon > 0$ such that the given function lies between ε and $n - \varepsilon$ on $T(M(A))$. The existence of a suitable projection in $M_n(M(A))$ follows from Theorem 2.11 with $M_n(A)$ in place of A.

Let us show next that the map is surjective. This follows from the preceding paragraph, since any element of $\operatorname{Aff} T(M(A))$ is a difference of strictly positive elements (one of which, for example, is a constant function).

Finally, if A is simple then every nonzero element of A^+ is strictly positive on $T(A)$. Hence, if A is simple, every nonzero projection in $M(A)$ is strictly positive on $T(A)$, and, if $T(A) = T(M(A))$, therefore also on $T(M(A))$.

2.13. Remark. If A has a nonzero unital quotient then of course Corollary 2.12 fails. The map $K_0(M(A)) \to \operatorname{Aff} T(M(A))$ is not surjective, and need not be injective.

In any case, however, as shown in the proof of Theorem 2.10, $K_0(M(A))$ separates the points of $T(M(A))$, so $\mathbf{Q}$ times its image is norm-dense in $\operatorname{Aff} T(M(A))$.

Furthermore, the canonical pre-order in $K_0(M(A))$ is an order (even if the map into $\operatorname{Aff} T(M(A))$ is not injective). To see this, it is enough to prove, after passing to a matrix algebra, that if e, f, and g are orthogonal projections in $M(A)$ such that $e + f + g$ is equivalent to e then also $e + f$ is equivalent to e. If $e + f + g$ is equivalent to e in $M(A)$ then

$$\begin{aligned} D(eAe) &\subseteq D((e+f)A(e+f)) \\ &\subseteq D((e+f+g)A(e+f+g)) = D(eAe), \end{aligned}$$

whence by Theorem 2.1, $e + f$ is equivalent to e.

It seems to be an interesting question whether the image of $K_0(M(A))^+$ always contains the strictly positive elements in the image of $K_0(M(A))$. (This is true if A is unital by Theorem 1.4 of [7].)

A closely related question is whether every state of $K_0(M(A))$ (normalized on 1) comes from a tracial state of $M(A)$. Indeed, if the tracial states of $M(A)$ separate points of $K_0(M(A))$, and the property of the preceding paragraph holds (e.g., by Theorem 2.12, if A has no nonzero unital quotient), then the present property follows. (A state on $K_0(M(A))$ in this case determines a functional on $\mathbf{Q}$ times the image of $K_0(M(A))$ which is positive on $\mathbf{Q}$ times the image of $K_0(M(A))^+$. The latter subset contains the strictly positive elements in $\mathbf{Q}$ times the image of $K_0(A)$, and these are dense in the set of all strictly positive elements of $\operatorname{Aff} T(M(A))$. Hence this functional is continuous on $\mathbf{Q}$ times the image of $K_0(M(A))$ and its extension by continuity to $\operatorname{Aff} T(M(A))$ is positive. This functional is therefore evaluation at some point of $T(M(A))$, which is then a tracial state on $M(A)$ extending the given state of $K_0(M(A))$.)

Conversely, if every state of $K_0(M(A))$ comes from a tracial state of $M(A)$, then it follows by Theorem 1.4 of [7] that for any element g of $K_0(M(A))$ which is strictly positive in $\operatorname{Aff} T(M(A))$, there exists $n \in \{1, 2, \ldots\}$ such that $ng \in K_0(M(A))^+$. (Does this imply that $g \in K_0(M(A))^+$?)

2.14. EXAMPLE. It is interesting, in view of Corollary 2.12, to decide when $T(M(A))$ is equal to $T(A)$. This condition is necessary for the map from $K_0(M(A))$ to the ordered group $\operatorname{Aff} T(M(A))$ with the strict pointwise order to be an order isomorphism, whether A is simple or not. (If $T(A) \neq T(M(A))$, then $T(A) \neq \varnothing$ (since if $T(A) = \varnothing$, A is stable and $T(M(A)) = \varnothing$), and by (i) $\Rightarrow$ (ii) of Theorem 3.1 below, $T(M(A)/A) \neq \varnothing$, so for some projection $e \in A$, $[e]$ is not zero on $T(A) \subseteq T(M(A))$ but of course $[e]$ is zero on $T(M(A)/A) \subseteq T(M(A))$.)

As we shall show in Theorem 3.1 below, $T(A) = T(M(A))$ if and only if $T(A)$ is compact (in the weak* topology from the duality with A). (To show this, we shall assume that A has no nonzero unital quotient.) If $T(A)$ is finite-dimensional, then of course $T(A)$ is compact and so $T(A) = T(M(A))$. If $T(A)$ is of infinite dimension, however, then, as we shall show in the following theorem, there exists a separable AF algebra B such that A is isomorphic to a full hereditary sub-C*-algebra of B and $T(B)$ is not compact.

Using [7] and [9], it is possible to construct such an example with A simple, and even with $T(A)$ isomorphic to the simplest infinite-dimensional simplex $S(c)$, the state space of the C*-algebra c of convergent complex sequences. (In this last case, the convex hull of $T(B) \cup \{0\}$ must be isomorphic to the same simplex, and it follows that $T(B)$ is the set of all infinite convex combinations of its extreme points, which form a discrete set.)

2.15. THEOREM. *Let G be a dimension group, and let D be an interval in G. Denote by $S(G, D)$ the space of all positive functionals τ on G with* $\sup \tau(D) = 1$. *The following two conditions are equivalent:*

(i) *For every interval $D' \supseteq D$, the state space $S(G, D')$ is compact.*

(ii) *$S(G, D)$ is finite-dimensional.*

Proof. Ad (ii) $\Rightarrow$ (i). For every interval $D' \supseteq D$, the restriction map $S(G, D') \to \mathbf{R}^+ S(G, D)$ is injective (since $G = \mathbf{Z}D$) and affine, so if $S(G, D)$ is finite-dimensional so also is $S(G, D')$.

Ad (i) $\Rightarrow$ (ii). Our proof is indirect. Suppose that $S(G, D)$ is of infinite dimension. Since the convex hull of $S(G, D)$ and 0 is compact, it follows that there exists an infinite sequence (τ_n) of distinct extreme points of $S(G, D)$. For each $n = 1, 2, \ldots$ choose $e_n \in G^+$ such that

$$\tau_n(e_n) > n, \quad \tau_k(e_n) < 2^{-n}, \qquad 1 \leq k < n.$$

To see that such $e_n \in G^+$ exists, for each fixed n, note that by Theorem 4.8 of **[13]**, for each $g \in G^+$ the ideal $H = H(g)$ of G generated by g has dense image in the subgroup of $\operatorname{Aff} S(H, g)$ comprising those functions which at each extreme point $\tau \in S(H, g)$ take values in $\tau(H) \subseteq \mathbf{R}$. Choose $g_n \in G^+$ such that $\tau_n(g_n) > n$. Since $0 \in \tau(H)$ for any $\tau \in S(H, g_n)$, and for each $k = 1, 2, \ldots, n$, either $\tau_k(H) = 0$ or $\tau_k(g_n)^{-1}\tau_k \in S(H, g_n)$, there exists $e_n \in H^+$ such that $\tau_k(e_n)$ is close to 0 for $1 \leq k < n$ and $\tau_n(e_n)$ is close to $\tau_n(g_n)$. If these approximations are close enough, then $\tau_n(e_n) > n$ and $\tau_k(e_n) < 2^{-n}$, $1 \leq k < n$, as desired.

If $S(G, D)$ is not compact, then (i) is violated with D in place of D'. It remains to consider the case that $S(G, D)$ is compact. In this case, since D is upward directed there exist $e_0 \in D$ and $\varepsilon > 0$ such that

$$\tau(e_0) > \varepsilon \quad \text{for all } \tau \in S(G, D).$$

Now denote by D' the interval of G generated by the set

$$\{e + e_1 + \cdots + e_m; e \in D, m = 1, 2, \ldots\}.$$

For each $n = 1, 2, \ldots$ and each $g \in D'$, choosing $e \in D$ and $m \geq 1$ such that $g \leq e + e_1 + \cdots + e_m$, we have

$$\begin{aligned} \tau_n(g) &\leq \tau_n(e) + \tau_n(e_1) + \cdots + \tau_n(e_m) \\ &\leq 1 + \tau_n(e_1) + \cdots + \tau_n(e_n) + 2^{-(n+1)} + \cdots + 2^{-m} \\ &\leq 2 + \tau_n(e_1 + \cdots + e_n). \end{aligned}$$

Since $e_n \in D'$ and $\tau_n(e_n) > n$ we have

$$n < \sup \tau_n(D') \leq 2 + \tau_n(e_1 + \cdots + e_n) < +\infty.$$

It follows that with $\tau'_n = (\sup \tau_n(D'))^{-1}\tau_n$, we have $\tau'_n \in S(G, D')$ and

$$\tau'_n(e_0) \leq n^{-1}\tau_n(e_0) \leq n^{-1}.$$

On the other hand, for any $\tau' \in S(G, D')$, τ' is not zero on D (since $\mathbf{Z}^+ D = G^+$), and so $\tau'(e_0) > 0$ (in fact $\tau'(e_0) \geq \sup \tau'(D)\varepsilon > 0$). If $S(G, D')$ were compact then there would exist $\varepsilon' > 0$ with

$$\tau'(e_0) > \varepsilon' \quad \text{for all } \tau' \in S(G, D').$$

This would contravene the estimates $\tau'_n(e_0) \leq n^{-1}$.

3. Embedding $\mathscr{O}_n$ in $M(A)/A$.

3.1. THEOREM. *Let A be a separable AF algebra with no nonzero unital quotient. The following nine statements are equivalent*:

(i) $T(M(A)/A) = \varnothing$.
(ii) $T(A) = T(M(A))$.
(iii) *The image of $T(A)$ in $T(M(A))$ is compact.*
(iv) *$T(A)$ is compact.*
(v) *For some $g \in K_0(A)$, $\tau(g) \geq 1$ for all $\tau \in T(A)$.*
(vi) *The image of $K_0(A)$ in $\operatorname{Aff} T(M(A))$ is dense.*
(vii) *There exists a unital morphism $\mathscr{O}_\infty \to M(A)/A$.*
(viii) *$D(M(A)/A)$ is a semigroup.*
(ix) $S(K_0(M(A)/A), [1]) = \varnothing$.

Proof. Ad (i) $\Rightarrow$ (ii). Let $\tau \in T(M(A))$. Denote by τ' the smallest extension of $\tau|A$ to a trace on $M(A)$. Then $\tau' - \tau$ is a positive bounded trace on $M(A)$ which is zero on A and therefore, if (i) holds, zero. This shows that τ belongs to the image of $T(A)$ in $T(M(A))$.

Ad (ii) $\Rightarrow$ (iii). This implication is immediate.

Ad (iii) $\Rightarrow$ (iv). This implication follows from the continuity of the restriction map from $T(M(A))$ into the dual of A, which takes the image of $T(A)$ in $T(M(A))$ back into $T(A)$.

Ad (iv) $\Rightarrow$ (v). If $T(A)$ is compact, then as $D(A)$ is upward directed, there exist $n \in \{1, 2, \ldots\}$ and a projection $e \in A$ such that

$$\tau(e) \geq 1/n, \qquad \tau \in T(A).$$

Hence if g denotes $n[e]$ in $K_0(A)$, (v) holds.

Ad (v) $\Rightarrow$ (i). Let $g \in K_0(A)$ be such that $\tau(g) \geq 1$ for all $\tau \in T(A)$. By Theorem 2.10, $\tau(g) \geq 1$ for all $\tau \in T(M(A))$. (i) follows immediately.

Ad (ii) $\Rightarrow$ (vi). (This implication clearly needs some restriction on A, but perhaps only that A have no finite-dimensional quotient. All the other implications considered may hold for any separable AF algebra A.) Suppose that $T(A) = T(M(A))$. Let $f \in \operatorname{Aff} T(M(A))$. To show that f can be approximated on $T(M(A))$ by an element of the group $K_0(A)$, it is sufficient to consider the case that for some $\varepsilon > 0$, $\varepsilon \leq f \leq 1 - \varepsilon$. (Instead of approximating f, approximate the functions $(f + 2\|f\|)/k$ and $2\|f\|/k$ where k is an integer greater than $3\|f\|$; this yields an approximation of f/k and hence of $f = k(f/k)$.) In this case, by Lemma 2.9 there exists a projection $e \in M(A)$ such that $f(\tau) = \tau(e)$, $\tau \in T(M(A))$. By Lemma 2.2, for each $\tau \in T(A)$, $\tau(e) = \sup \tau(D(eAe))$. By Theorem 3.1 of **[10]**, $D(eAe)$ is upward directed. Since $T(A)$ is compact (as $T(A) = T(M(A))$), by Dini's theorem e can be approximated uniformly on $T(A)$ by $[e_0] \in D(eAe)$. Since $T(A) = T(M(A))$, this says that $[e_0]$ approximates f on $T(M(A))$.

Ad (vi) $\Rightarrow$ (v). This implication is immediate.

Ad (iv) $\Rightarrow$ (vii). Suppose that $T(A)$ is compact. We must construct an infinite sequence of isometries in $M(A)/A$ with pairwise orthogonal range projections. It is of course enough to construct two isometries with orthogonal ranges. (If u_1 and u_2 are such, then $u_2, u_1u_2, u_1^2u_2, \ldots$ have pairwise orthogonal ranges.)

As $T(A)$ is compact, and $D(A)$ is upward directed, there exists a projection $p \in A$ such that

$$\tau(p) > 0, \qquad \tau \in T(A).$$

Again by compactness of $T(A)$, there exists $\varepsilon > 0$ such that

$$\tau(p) \geq 2\varepsilon, \qquad \tau \in T(A).$$

By Lemma 2.2, for any $\tau \in T(A)$, $\sup \tau(D(A)) = \tau(1) = 1$. As $T(A)$ is compact and $D(A)$ is upward directed, by Dini's theorem there exists a projection $q \in A$ such that

$$\tau(q) \geq 1 - \varepsilon, \qquad \tau \in T(A).$$

Combining these inequalities, we have

$$\tau((1-p) \oplus (1-q)) \leq 1 - 2\varepsilon + \varepsilon = 1 - \varepsilon, \qquad \tau \in T(A).$$

Since A has no nonzero unital quotient,

$$\tau(p) < 1, \qquad \tau \in T(A).$$

(If $\tau(p) = 1$ then since $\tau(1) = 1$ it follows that the quotient of A by the largest closed two-sided ideal I of A on which τ is zero is unital (see proof of Theorem 2.3), whence $A/I = 0$; this contravenes $\tau \neq 0$.) Since $T(A) = T(M(A))$ (by (iv) $\Rightarrow$ (ii)), we have

$$0 < \tau((1-p) \oplus (1-q)) < 1, \qquad \tau \in T(M(A)).$$

Hence by Theorem 2.11 there exists a projection $e \in M(A)$ such that

$$\tau(e) = \tau((1-p) \oplus (1-q)), \qquad \tau \in T(M(A)),$$
$$\tau(e) = +\infty, \tau \text{ an infinite semifinite lower semicontinuous trace on } A^+.$$

It follows by (iii) $\Rightarrow$ (i) of Corollary 2.4 (with $A = M_2(A)$, $e = e \oplus 0$, and $f = (1-p) \oplus (1-q))$ that $e \oplus 0$ is equivalent to $(1-p) \oplus (1-q)$ in $M(M_2(A))$.

This shows that $(1-p) \oplus (1-q)$ is equivalent to part of $1 \oplus 0$ in $M(M_2(A))$. Since $D(A)$ is upward directed, p and q are both equivalent to part of a single projection r in A. It follows that $(1-r) \oplus (1-r)$ is equivalent to part of $1 \oplus 0$ in $M(M_2(A))$. In other words, there are two orthogonal projections e_1 and e_2 in $M(A)$, each equivalent to $1-r$ in $M(A)$. Since $r \in A$, the images of e_1 and e_2 in $M(A)/A$, which are of course orthogonal, are each equivalent to 1. This shows that there are two isometries in $M(A)/A$ with orthogonal ranges.

Ad (vii) $\Rightarrow$ (viii). Let u and v be isometries in $M(A)/A$ with orthogonal ranges. In other words, $u^*u = v^*v = 1$, and uu^*, vv^*, and $1 - uu^* - vv^*$ are orthogonal projections. Equivalently, in the local semigroup $D(M(A)/A)$, $\dim(1) + \dim(1)$ is defined. It follows that $D(M(A)/A)$ is a semigroup.

Ad (viii) $\Rightarrow$ (ix). If $D(M(A)/A)$ is a semigroup then, in the preordered group $K_0(M(A)/A)$, $[1] + [1] \leq [1]$. Applying any $\tau \in S(K_0(M(A)/A), [1])$ to this inequality would yield $1 + 1 \leq 1$ in $\mathbf{R}$.

Ad (ix) $\Rightarrow$ (i). This implication is immediate.

3.2. LEMMA. *Let G be a dimension group and let D, D_1, and D_2 be intervals in G^+ (i.e. upward directed, hereditary subsets of G^+). The following implications hold:*

(1) $D_1 + D_1 \subseteq D_2 + D_2 \Rightarrow D_1 \subseteq D_2$.

(2) $D + D + D_1 \subseteq D + D + D_2 \Rightarrow D + D_1 \subseteq D + D_2$.

Proof. Ad 1. Let g_1 be in D_1, and suppose that $g_1 + g_1 = g_2' + g_2''$ with $g_2', g_2'' \in D_2$. Since D_2 is upward directed there exists $g_2 \in D_2$ with $g_2 \geq g_2', g_2''$. Then $2g_1 \leq 2g_2$, whence $g_1 \leq g_2$. Since D_2 is hereditary, $g_1 \in D_2$.

Ad 2. If $D + D + D_1 \subseteq D + D + D_2$, then

$$D + D + D_1 + D_1 \subseteq D + D + D_2 + D_1 \subseteq D + D + D_2 + D_2.$$

Hence by Part 1, with D_1 replaced by $D+D_1$ and D_2 replaced by $D+D_2$ (note that $D+D_i$ is upward directed and hereditary), $D+D_1 \subseteq D+D_2$.

3.3. THEOREM. *Let A be a separable AF algebra. For each $n = 1, 2, \ldots$ the following three statements are equivalent*:

(i) $(n-1)[1] = 0$ *in* $K_0(M(A)/A)$.

(ii) $(n-1)[1]$ *belongs to the image of* $K_0(A)$ *in* $K_0(M(A))$.

(iii) *There exists a unital morphism* $\mathcal{O}_n \to M(A)/A$.

Proof. Ad (i) $\Rightarrow$ (ii). By elementary algebraic K-theory, the canonical sequence of K_0-groups corresponding to the short exact sequence

$$0 \to A \to M(A) \to M(A)/A \to 0$$

is exact at the midpoint. (i) $\Rightarrow$ (ii) follows immediately.

Ad (ii) $\Rightarrow$ (iii). Suppose first that $n > 1$ and that $(n-1)[1]$ belongs to the image of $K_0(A)$ in $K_0(M(A))$. In other words, for some $d = 1, 2, \ldots$ there exist projections p and q in $M_d(A)$ such that, in $K_0(M(A))$,

$$(n-1)[1] = [p] - [q].$$

This means that, for some $k = 1, 2, \ldots,$ the orthogonal sum of q and $n - 1 + k$ copies of 1 is equivalent in $M_{d+n-1+k}(M(A))$ to the orthogonal sum of p and k copies of 1.

In the case $k = 1$, this says that the orthogonal sum of q and n copies of 1 is equivalent in $M_{n+d}(M(A))$ to the orthogonal sum of p and 1. Since p and q belong to A, this implies that the orthogonal sum of n copies of $1 \in M(A)/A$ is equivalent in $M_n(M(A)/A)$ to 1. This of course is the same as saying that $1 \in M(A)/A$ is the sum of n orthogonal projections in $M(A)/A$, each equivalent to 1. In other words, there is a unital morphism $\mathcal{O}_n \to M(A)/A$.

We shall now reduce the case $k > 1$ to the case $k = 1$ by using Lemma 3.2. We have, in $K_0(A)$, that the sum of the interval $D(A)$ taken $n - 1 + k$ times and the interval $[0, [q]]$ is equal to the sum of the interval $D(A)$ taken k times and the interval $[0, [p]]$. (This is an immediate consequence of our data, and in fact by Theorem 2.1 is a reformulation of it.) If $k > 1$, we may apply Lemma 3.2, Part 2, with $D = D(A)$, D_1 the sum of $D(A)$ taken $n - 1 + k - 2$ times and $[0, [q]]$, and D_2 the sum of $D(A)$ taken $k - 2$ times and $[0, [p]]$. From $D + D + D_1 = D + D + D_2$, which we have, thus follows $D + D_1 = D + D_2$, i.e., the sum of $D(A)$ taken $n - 1 + k - 1$ times and $[0, [q]]$ is equal to the sum of $D(A)$ taken $k - 1$ times and $[0, [p]]$. If $k - 1 > 1$, we may repeat this argument with $k - 1$ in place of k, and continue in this way until finally we deduce that the sum of $D(A)$ taken $n - 1 + 1 = n$ times and $[0, [q]]$ is equal to the sum of $D(A)$ taken once and $[0, [p]]$.

By Theorem 2.1, it follows that the orthogonal sum of q and n copies of 1 in $M_{d+n}(M(A))$ is equivalent in $M_{d+n}(M(A))$ to the orthogonal sum of p and 1. This is the case $k = 1$, dealt with in the preceding paragraph.

Finally, we must show that the statements (i), (ii), and (iii) are equivalent in the case $n = 1$. Actually, in this case they are all true. (i) and (ii) are true trivially, and (iii) is true since $\mathcal{O}_1$ $(= C(\mathbf{T}))$ has a quotient isomorphic to $\mathbf{C}$.

Ad (iii) $\Rightarrow$ (i). A unital morphism $\mathcal{O}_n \to M(A)/A$ induces a morphism $(K_0(\mathcal{O}_n), [1]) \to (K_0(M(A)/A), [1])$. Clearly $n[1] = [1]$ in $K_0(\mathcal{O}_n)$.

3.4. REMARK. If it is assumed that A has no nonzero unital quotient, then the equivalent statements of Theorem 3.3 are also equivalent to the following one:

(iv) $n - 1$ belongs to the image of $K_0(A)$ in $\operatorname{Aff} T(A)$.

The implication (ii) $\Rightarrow$ (iv) is obvious. The implication (iv) $\Rightarrow$ (ii) follows from Theorem 2.10 and the statement of injectivity in Corollary 2.12.

Statement (iv) should be compared with Statement (v) of Theorem 3.1.

3.5. THEOREM. *Let A be a separable AF algebra with no nonzero unital quotient. Let P be a $d \times d$ matrix of zeros and ones with no row or column consisting entirely of zeros. Assume that $T(A)$ is compact. It follows that every morphism of groups*

$$K_0(\mathcal{O}_P) \to K_0(M(A)/A)$$

is induced by a morphism of C*-*algebras*

$$\mathcal{O}_P \to M(A)/A,$$

and if the group morphism takes [1] *into* [1] *then the* C*-*algebra morphism may be chosen to take* 1 *into* 1.

Proof. Using the hypotheses on A (including that $T(A)$ is compact), we shall construct an additive map from $K_0(M(A)/A)$ to the abelian local semigroup $D(M(A)/A)$ of Murray-von Neumann equivalence classes of projections in $M(A)/A$ (i.e. the dimension range of $M(A)/A$), which is a right inverse to the canonical map $D(M(A)/A) \to K_0(M(A)/A)$, and which takes the class of 1 into the class of 1. As we shall show, the theorem follows immediately from the existence of such a map $K_0(M(A)/A) \to D(M(A)/A)$.

By the proof of Lemma 2.9 in the case $f = \frac{1}{2}$, there exists a projection $e_{1/2} \in M(A)$ such that $e_{1/2}$ is equivalent to $1 - e_{1/2}$. Let us show that the canonical map

$$d(e_{1/2} + A) + D(M(A)/A) \to K_0(M(A)/A)$$

is bijective. The inverse of this map,

$$K_0(M(A)/A) \to d(e_{1/2} + A) + D(M(A)/A) \subseteq D(M(A)/A),$$

then clearly has the properties stipulated above.

Let us first show that the map is injective. By **[1]**, any projection in $M(A)/A$ is the image $e + A$ of a projection e in $M(A)$. We shall denote $e + A$ by $\dot{e}$. Since $(1 - e_{1/2})A(1 - e_{1/2})$ is also AF (**[10]**, Theorem 3.1), the same holds with this algebra in place of A. In other words, any projection majorizing $\dot{e}_{1/2}$ in $M(A)/A$ is the image of a projection majorizing $e_{1/2}$ in A. Therefore, what we must show is that if $e_{1/2} + e$ and $e_{1/2} + f$ are projections in $M(A)$ with the same image in $K_0(M(A)/A)$, then $\dot{e}_{1/2} + \dot{e}$ and $\dot{e}_{1/2} + \dot{f}$ are equivalent in $M(A)/A$. Recall that, as pointed out in the proof of (i) $\Rightarrow$ (ii) of Theorem 3.3, $K_0(M(A)/A) \supseteq K_0(M(A))/K_0(A)$. Hence, in $K_0(M(A))$,

$$[e_{1/2} + e] - [e_{1/2} + f] \in K_0(A).$$

In other words, there exist projections e' and f' in some matrix algebra over A such that

$$[e_{1/2} + e + e'] = [e_{1/2} + f + f'] \quad \text{in } K_0(M(A)).$$

In particular, for every $\tau \in T(M(A))$,

$$\tau(e_{1/2} + e + e') = \tau(e_{1/2} + f + f').$$

Hence by Corollary 2.5,with $e_{1/2}$ in place of 1,

$$e_{1/2} + e + e' \text{ is equivalent to } e_{1/2} + f + f',$$

in a matrix algebra over $M(A)$. Passing to the quotient, as e' and f' map into zero in $M(A)/A$, we have

$$\dot{e}_{1/2} + \dot{e} \text{ is equivalent to } \dot{e}_{1/2} + \dot{f} \text{ in } M(A)/A,$$

as desired.

Let us now show that the map is surjective. For this we shall have to use the hypothesis that $T(A)$ is compact, which, by Theorem 3.1, is equivalent to density of the image of $K_0(A)$ in $\operatorname{Aff} T(M(A))$. By the six-term exact sequence of Bott periodicity applied to the extension $A \to M(A) \to M(A)/A$, since $K_1(A) = 0$, we have $K_0(M(A)/A) = K_0(M(A))/K_0(A)$. Therefore, to prove surjectivity, we must prove that for any $g \in K_0(M(A))$ there exists $h \in K_0(A)$ such that $g + h$ is the class in $K_0(M(A))$ of a projection in $M(A)$ majorizing $e_{1/2}$. Let $g \in K_0(M(A))$. By density of $K_0(A)$ in $\mathrm{Aff}\,T(M(A))$, there exists $h \in K_0(A)$ such that

$$\tfrac{1}{4} < \tau(g+h) - \tfrac{1}{2} < \tfrac{1}{2}, \qquad \tau \in T(M(A)).$$

Hence by Lemma 2.9, with the AF algebra $(1 - e_{1/2})A(1 - e_{1/2})$ (see Theorem 3.1 of **[10]**) in place of A, there exists a projection $e \in M((1 - e_{1/2})A(1 - e_{1/2}))$ such that

$$\begin{aligned} &\tau(e) = \tau(g+h) - \tfrac{1}{2}, && \tau \in T(M(A)), \quad \text{i.e.,} \\ &\tau(e_{1/2} + e) = \tau(g+h), && \tau \in T(M(A)). \end{aligned}$$

By Corollary 2.12, this says that $[e_{1/2} + e] = g + h$ in $K_0(M(A))$, as desired.

Finally, let us note that the existence of an additive map

$$\psi \colon K_0(M(A)/A) \to D(M(A)/A)$$

taking [1] into [1] and acting as a right inverse to the canonical map $D(M(A)/A) \to K_0(M(A)/A)$, which we have just established, implies Theorem 3.5. By definition [5], $\mathcal{O}_P$ is the C*-algebra generated universally by finitely many partial isometries with certain additive relations among their range and support projections (finitely many relations, each one involving only finite sums). Such relations may be expressed in $D(\mathcal{O}_P)$, and by mapping canonically into $K_0(\mathcal{O}_P)$, from there by a given map into $K_0(M(A)/A)$, and from there by ψ into $D(M(A)/A)$, one has similar relations in $D(M(A)/A)$. This means that one has partial isometries with the appropriate relations in $M(A)/A$ (inducing these relations in $D(M(A)/A)$). By universality, one has a morphism $\mathcal{O}_P \to M(A)/A$, giving rise to the given map $K_0(\mathcal{O}_P) \to K_0(M(A)/A)$. Furthermore, if the given map takes [1] into [1], then, since the sum of the range projections of the generating partial isometries in $\mathcal{O}_P$ is equal to 1, the sum of the corresponding classes in $D(M(A)/A)$ is equal to [1]. Hence partial isometries with the appropriate relations in $M(A)/A$ can be chosen as above and in addition with the sum of the range projections equal to 1. Then the morphism $\mathcal{O}_P \to M(A)/A$ is unital.

3.6. Theorem 3.3 is not a special case of Theorem 3.5, since in Theorem 3.3 A is allowed to have a unital quotient. Note that, by the proof of (i) $\Rightarrow$ (iv) of Theorem 3.1, compactness of $T(A)$ is a consequence of any of the conditions of Theorem 3.3. Hence, although it is not clear whether the assumption that $T(A)$ be compact is necessary in general in Theorem 3.5, at least it is superfluous in the case that the class $[1] \in K_0(M(A)/A)$ is of finite order. This leads to the following result.

COROLLARY. *Let A be a separable AF algebra with no nonzero unital quotient, and let P be a $d \times d$ matrix of zeros and ones with no row or column consisting entirely of zeros. Assume that the range of $1 - P^{\mathrm{tr}}$ on $\mathbf{Z}^d$ contains a nonzero vector with all entries equal. (This holds if* $\det(1 - P) \neq 0$.) *It follows that every morphism of groups*

$$K_0(\mathcal{O}_P) \to K_0(M(A)/A)$$

taking $[1]$ *into* $[1]$ *is induced by a unital morphism of* C*-*algebras*

$$\mathcal{O}_P \to M(A)/A.$$

Proof. The assumption on P is exactly that the class $[1] \in K_0(\mathcal{O}_P)$ $(= \operatorname{coker}(1 - P^{\mathrm{tr}})|\mathbf{Z}^d)$ is of finite order. (See **[4]** and **[18]**.) Hence by hypothesis, $[1] \in K_0(M(A)/A)$ is of finite order. The conclusion follows by Theorems 3.3, 3.1, and 3.5 as observed above.

4. Calculation of nonstable Ext.

4.1. Our first observation is that if A is a separable AF algebra and B is a unital C*-algebra, and if $M(A)/A$ contains two isometries with orthogonal ranges, then for (essential) extensions of A by B for which the complement of the image of the unit of B in $M(A)/A$ contains a projection equivalent to the unit of $M(A)/A$ (for instance, extensions for which the image of the unit of B in $M(A)/A$ is contained in the range projection of one of the two isometries with orthogonal ranges), all three notions of equivalence coincide.

This is seen as follows. As pointed out in the proof of Theorem 1, it follows from Theorem 1.4 of **[4]** that cancellation holds for the equivalence classes of projections in $M(A)/A$ containing a copy of $1 \in M(A)/A$ (this uses the hypothesis that $\mathcal{O}_\infty$ is unitally embedded in $M(A)/A$). If ρ_1 and ρ_2 are two embeddings of B in $M(A)/A$ such that $\rho_1(1)$ and $\rho_2(1)$ (here 1 denotes the unit of B) are each orthogonal to copies of $1 \in M(A)/A$, then there exist isometries v_1 and

v_2 in $M(A)/A$ with ranges orthogonal to each other and to $\rho_1(1)$, and isometries v_1' and v_2' with ranges orthogonal to each other and to $\rho_2(1)$. Suppose that ρ_1 and ρ_2 are equivalent in the third sense (the weakest). Since B is unital, this means that there exists a partial isometry u_0 intertwining ρ_1 and ρ_2 such that $u_0^*u_0 = \rho_1(1)$ and $u_0u_0^* = \rho_2(1)$. Then, with $u_1 = u_0 + v_1'v_1^*$, u_1 is a partial isometry with support $\rho_1(1) + v_1^*v_1$ and range $\rho_2(1) + v_1'^*v_1'$. By cancellation, it follows that the complements of these projections (each of which contains a copy of $1 \in M(A)/A$—the first $v_2^*v_2$ and the second $v_2'^*v_2'$) are also equivalent; in other words, there exists a unitary $u \in M(A)/A$ extending u_1 (i.e. agreeing with u_1 on its support). As in the proof of Theorem 1, we may multiply u on the right by a unitary equal to 1 on the support of u_1 to obtain that u is in the connected component of 1 in the unitary group of $M(A)/A$. (This uses that A is a separable AF algebra.) Since ρ_1 and ρ_2 are intertwined by u, they are strongly equivalent, as desired.

4.2. The preceding observation has the following consequence. Let A and B be as in 4.1, and consider the subset of extensions of A by B described in 4.1 (i.e. those for which there is a copy of $1 \in M(A)/A$ orthogonal to the image of B in $M(A)/A$). Note that the sum of any two extensions of A by B, as defined in Theorem 1, is one of these. In particular, the equivalence classes of these extensions, in any sense (all are the same, by 4.1), form a semigroup—a subsemigroup of the semigroup of equivalence classes of all extensions of A by B, in the same sense. Denote this semigroup by $E(B, A)$. $E(B, A)$ is in fact equal to the semigroup of equivalence classes of all extensions of A by B in the third sense. (An arbitrary extension, ρ, is equivalent in the third sense of Brown, Douglas, and Fillmore to $v_1\rho v_1^*$ where v_1 and v_2 are isometries with orthogonal range.) On considering equivalence in the third sense, one sees immediately that $E(B, A)$ is a subsemigroup of $E(B, A \otimes \mathscr{K})$. Hence $E(B, A)$ is a subsemigroup of $\mathrm{Ext}(B, A)$, or $\mathrm{Ext}_s(B, A)$, the usual semigroup of (nonunital) extensions of $A \otimes \mathscr{K}$ by B with respect to weak or strong equivalence. (It is the subsemigroup consisting of all extensions of $A \otimes \mathscr{K}$ by B for which the image of B in $M(A \otimes \mathscr{K})/A \otimes \mathscr{K}$ is contained in the corner algebra determined by the projection $1 \otimes e_{11} \in M(A \otimes \mathscr{K})$, where e_{11} denotes a fixed minimal projection in $\mathscr{K}$.) It is also a subsemigroup of $\mathrm{Ext}(B \otimes \mathscr{K}, A)$, or $\mathrm{Ext}_s(B \otimes \mathscr{K}, A)$.

4.3. The preceding result permits the semigroup $E(B, A)$ (of extensions of A by B such that there is a copy of $1 \in M(A)/A$ orthogonal to the image of B in $M(A)/A$—with any kind of equivalence (see 4.1)) to be computed in the case that also B is a separable AF algebra. Since $E(B, A)$ is a certain subsemigroup of $\mathrm{Ext}_s(B \otimes \mathscr{K}, A)$ (the semigroup of strong equivalence classes of extensions of $A \otimes \mathscr{K}$ by $B \otimes \mathscr{K}$), by **[1]** and **[3]** we are looking at a certain subsemigroup of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$, the semigroup of (equivalence classes of) dimension group extensions of $K_0(A)$ by $K_0(B)$. If we stick to the case that A is simple (or consider only stenotic extensions for more general A—assuming that A contains a full projection), the semigroup $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ is computed in Theorem 4.3 and Proposition 6.4 of **[14]**.

As a result, one deduces that $E(B, A)$ has a zero, and that if every trace on A is finite, then $E(B, A)$ is a group. (Here we are assuming that A and B are separable AF, A is simple, $\mathscr{O}_\infty$ embeds unitally in $M(A)/A$, and B is unital; presumably the assumption that B is unital is unnecessary—but it is not clear how to remove it.)

These facts are deduced as follows. One has to describe $E(B, A)$ as a subset of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ in terms of the description of this larger set given in Theorem 4.3 of **[14]**. We shall obtain such a description of $E(B, A)$ by modifying Theorem 7.8 of **[14]**, which describes the subset of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ corresponding to the unital extensions of A by B, i.e. of $D(A)$ by $D(B)$. Actually, the desired modification of Theorem 7.8 of **[14]** is just an application of this theorem. What we want are the unital extensions of eAe by B for various projections e in $M(A)$ ($e \notin A$) such that the image of $1 - e$ in $M(A)/A$ contains a copy of 1, which is the same as to say that $1 - e$ and 1 can both be made smaller, the latter just by subtracting a projection in A, so that they become equivalent (compare 4.4 below). In other words (see Theorem 2.6), what we want are the unital extensions of D_1 by $D(B)$ for (nonunital) intervals D_1 with $D_1 + D_2 + D_3 = D(A)$ for some intervals D_2 and D_3 such that $D_3 + D_4 = D(A)$ for some unital D_4, i.e. $D_4 = [0, g]$ for some $g \in D(A)$.

With this description of $E(B, A)$, let us deduce the existence of a zero element. Note first that the semigroup $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ has a zero element, namely, the group-split dimension group extension constructed as in Theorem 7.11 of **[14]** with the map $\lambda\colon K_0(B)^+ \to \Lambda(S_1)$ of that theorem taken to be zero. Now note that the construction in Theorem 7.11 of **[14]** in fact yields a unital extension of $D(A)$ by $D(B)$, and so the zero of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ belongs to the subsemigroup $E(B, A)$.

Finally, suppose that every trace on A is finite, and let us show that $E(B, A)$ is a group. The hypothesis means (in the case that A is simple, as assumed) that every positive additive map $K_0(A) \to \mathbf{R}$ is bounded on $D(A)$, and therefore also bounded on any $D_1 \subseteq D(A)$. Hence (by Theorem 7.8 of **[14]**) for any element of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ arising from a unital extension of D_1 by $D(B)$ with $D_1 \subseteq D(A)$, any additive map $l \colon \pi^{-1}(K_0(B)^+) \to \Lambda(S_1)$ associated as in Theorem 4.3 of **[14]** with this element must be finite-valued, i.e., have values in $\mathrm{Aff}\, S_1$. In particular by the above description of $S(B, A)$ this holds for any element of $E(B, A)$.

Conversely, let us show that any element of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ for which the associated additive map $l \colon \pi^{-1}(K_0(B)^+) \to \Lambda(S_1)$ of Theorem 4.3 of **[14]** takes values in $\mathrm{Aff}\, S_1$ (this property is independent of the choice of l) belongs to $E(B, A)$. Fix a decomposition $D(A) = D_1 + D_2 + D_3$ where D_1 is nonunital and $D_3 + D_4 = D(A)$ for some unital D_4. (Equivalently (by 4.4 below and Theorem 2.6—see above), fix a projection e in $M(A)$, $e \notin A$, such that the image of $1 - e$ in $M(A)/A$ contains a copy of 1.) Then in Theorem 7.8 of **[14]**, if there is no u with $\pi(u) = [1] \in K_0(B)$ such that $l(u) = d$ (where d is the map $S_1 \ni \tau \mapsto \sup \tau(D(A))$), anyway choose u with $l(u)$ strictly positive, and subtract an element of $K_0(A)$ from u so that also $l(u)$ is strictly less than d_1 (the map $\tau \mapsto \sup \tau(D_1)$). (Recall that as A is simple and not elementary, by Theorem 4.8 of **[13]** the image of $K_0(A)$ is dense in $\mathrm{Aff}\, S_1$.) By Theorem 2.6, there exists a projection $f_1 \in M(A)$ such that $D(f_1 A f_1) = D_1$. Since $d_1 \in \mathrm{Aff}\, S_1$, and every trace on $f_1 A f_1$ is finite, we have that $T(f_1 A f_1)$ is compact, and so for some $\varepsilon > 0$, $\varepsilon \leq l(u) \leq 1 - \varepsilon$ on $T(f_1 A f_1)$, where now $l(u)$ denotes the homogeneous extension of $l(u) \in \mathrm{Aff}\, S_1$. Hence by Lemma 2.9, there exists a projection $f \in f_1 M(A) f_1$ such that $\tau(f) = l(u)$ for $\tau \in T(f_1 A f_1)$, i.e. $\tau(f) = l(u)$ for $\tau \in S_1$. Furthermore, with f as constructed in Lemma 2.9, $f \notin A$. In other words (see Theorem 2.6), we have $D_1 = D_5 + D_6$ with D_5 nonunital, and $d_5 = l(u)$. Thus, the given element of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ arises from a unital extension of D_5 by $D(B)$, and, furthermore, $D_5 + (D_6 + D_2) + D_3 = D(A)$ and $D_3 + D_4 = D(A)$ with D_4 unital. This shows that the given element of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ belongs to $E(B, A)$.

Thus, in the case that every trace on A is finite, the description of the semigroup $E(B, A)$ becomes especially simple; it is just the set of elements of $\mathrm{Ext}_{\mathrm{dim}}(K_0(B), K_0(A))$ for which the additive map l of Theorem 4.3 of **[14]** is finite-valued. Inspection of the description of addition in the semigroup $\mathrm{Ext}_{\dim}(K_0(B), K_0(A))$ given in Proposition 6.4 of **[14]** shows that the subsemigroup of such elements is in fact a subgroup. This shows that, in this case, $E(B, A)$ is a group.

4.4. Let us consider the special case that $B = \mathbf{C}$. Then, assuming that $D(M(A)/A)$ is a semigroup, we have $E(\mathbf{C}, A) \cong D(M(A)/A)\backslash\{0\}$. (As pointed out in 4.2, $E(B, A)$ is isomorphic to the semigroup of all equivalence classes of (essential) extensions with respect to equivalence in the third sense—i.e., since B is unital, equivalence with respect to a partial isometry.) From 4.3 we deduce that, in the case that A is simple, the semigroup $D(M(A)/A) \setminus \{0\}$ has a zero element, and if every trace on A is finite, then $D(M(A)/A) \setminus \{0\}$ is a group.

Let us show that, for any AF algebra A,

$$D(M(A)/A) \cong D(M(A))/D(A).$$

By the right hand side we mean $D(M(A))$ modulo the equivalence relation generated by identifying g with $g+h$ if $h \in D(A)$ (and $g, g+h \in D(M(A))$). By **[1]** we know that every element of $D(M(A)/A)$ is the canonical image of an element of $D(M(A))$. We shall show that if f and g are elements of $D(M(A))$ such that the images of f and g in $D(M(A)/A)$ are equal, then there exist $k \in D(M(A))$ and $x, y \in D(A)$ such that $f = k + x$ and $g = k + y$. (The assertion $D(M(A)/A) \cong D(M(A))/D(A)$ follows.) Let p and q be projections in $M(A)$ such that the images of p and q in $M(A)/A$ are equivalent, by a partial isometry $v \in M(A)/A$. By Theorem 3.1 of **[10]**, pAp is approximately finite-dimensional. Hence by the proof of Lemma 2.6 of **[8]** (compare the second last paragraph of the proof of Theorem 1 above), there exists a partial isometry v_0 in $M(A)$ lifting v such that $v_0^*v_0 \leq p$ and $v_0v_0^* \leq q$. Since v_0 lifts v, $p - v_0^*v_0$ and $q - v_0v_0^*$ belong to A, as desired.

Let us just point out that this relation, or also the relation

$$K_0(M(A)/A) \cong K_0(M(A))/K_0(A),$$

together with §2, makes it possible to distinguish between $M(A_1)/A_1$ and $M(A_2)/A_2$ for certain pairs of finite matroid C^*-algebras A_1, A_2—a problem left open in **[8]**. (For A finite matroid, the invariant $D(M(A)/A)$ is the group $\mathbf{R}/K_0(A)$, which tells which prime numbers divide $K_0(A)$.)

4.5. Finally, let B be an arbitrary separable commutative C*-algebra. By analogy with the case that B is AF, one might ask whether the semigroup $E(B, A)$ (with A simple separable AF such that $\mathcal{O}_\infty \subseteq M(A)/A$)

has a zero element, and whether, if every trace on A is finite, it is a group. If B is not AF, we must leave this question completely open.

Acknowledgments. This work was partially supported by the Natural Sciences and Engineering Research Council of Canada, the Carlsberg Foundation, the Danish Natural Science Research Council, the Centre National de la Recherche Scientifique of France (Laboratoire de Mathématiques, Université d'Aix-Marseille à Luminy), and the Mathematical Sciences Research Institute, Berkeley.

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Received December 30, 1987.

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