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ON THE VANISHING OF $H^n(\mathcal{A}, \mathcal{A}^*)$ FOR CERTAIN C^*-ALGEBRAS

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Dedicated to the memory of Henry A. Dye

The norm continuous Hochschild cohomology for a C^*-algebra $\mathcal{A}$ with coefficients in the dual space vanishes if either $\mathcal{A}$ is nuclear or $\mathcal{A}$ has no bounded traces. The norm continuous cyclic cohomology for a C^*-algebra with no bounded traces vanishes.

1. Introduction. There has been some success in computing Hochschild cohomology groups for von Neumann algebras, especially the result that any derivation on a von Neumann algebra is inner [**11**, **14**]. We also have the results by Johnson, Kadison and Ringrose [**10**] which through the work of Connes [**4**] can be phrased: The Hochschild cohomology for an injective von Neumann algebra with coefficients in a dual normal module vanishes. Conversely Connes has proved [**5**], that this property actually characterises the injective von Neumann algebras. Recently E. G. Effros and the present authors have computed some cohomology groups and shown that the completely bounded cohomology vanishes if the module is an injective von Neumann algebra which contains the algebra in question. If the algebra is a C^*-algebra and the coefficients come from another C^*-algebra, very little is known in general, and it is clear that in this case the bounded cohomology will not vanish unless the algebra is very "nice" [**8**]. In the present paper we will prove that the norm continuous Hochschild cohomology for a C^*-algebra $\mathcal{A}$ with coefficients in the dual space $\mathcal{A}^*$ does vanish, if $\mathcal{A}$ is nuclear or if $\mathcal{A}$ has no bounded traces. The result for nuclear C^*-algebras is not new, in the sense that it has been known to a number of people. It follows relatively easily from the fact, that the double dual of a nuclear C^*-algebra is an injective von Neumann algebra [**2**]. The result for infinite C^*-algebras is proved by methods which are closely related to the techniques developed by Johnson, Kadison and Ringrose [**10**] in order to reduce the norm continuous cohomology to the ultraweakly continuous cohomology. Their results do not fit exactly because $\mathcal{A}^*$ is not a dual normal module for the von Neumann algebra $\mathcal{A}^{**}$. Despite this a modification of well-known techniques gives a trick to base the proof which in essence goes like this. Let $\phi : \mathscr{A}^k \to \mathscr{A}^*$ be a cocycle which has been reduced by coboundaries to such an extent that there exist orthogonal elements x and y in the unit ball of $\mathscr{A}$ for which $\phi(x)$ and $\phi(y)$ are orthogonal and $\|\phi(x)\|$ and $\|\phi(y)\|$ both nearly attain the value $\|\phi\|$. Since we have an l^∞ sum for $(x+y)$ and an l^1 sum for $(\phi(x)+\phi(y))$ we get

$$\|\phi\| \geq \|\phi(x+y)\| = \|\phi(x)\| + \|\phi(y)\| \geq 2(\|\phi\| - \varepsilon).$$

Hence $\phi = 0$ and the original cocycle is actually a coboundary.

We end the article by proving that the norm continuous cyclic cohomology for a C^*-algebra without bounded traces vanishes. We also prove that the odd cyclic cohomology groups for a nuclear C^*-algebra all vanish, and that the even groups are all isomorphic to the space of bounded traces on $\mathscr{A}$.

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2. Notation and definitions. Let $\mathscr{A}$ be a C^*-algebra, $\mathscr{A}^*$ and $\mathscr{A}^{**}$ denote the dual and the bidual of $\mathscr{A}$. It is well known that these spaces are Banach-$\mathscr{A}$ bimodules. In fact $\mathscr{A}^{**}$ is a von Neumann algebra and the canonical embedding—π say—of $\mathscr{A}$ into $\mathscr{A}^{**}$ is a $*$-isomorphism of $\mathscr{A}$ onto $\pi(\mathscr{A})$, which is an ultraweakly dense subalgebra of $\mathscr{A}^{**}$. For $\mathscr{A}^*$ the multiplications are given by

$$(af)(b) = f(ba) \quad \text{and} \quad (fa)(b) = f(ab).$$

In order to be able to switch back and forth between forms on $\mathscr{A}$ and multilinear mappings of $\mathscr{A}$ into $\mathscr{A}^*$ we will fix the following notation.

2.1. DEFINITIONS. For a Banach space $\mathscr{X}$ let $C^k(\mathscr{A}, \mathscr{X})$ denote the Banach space of bounded k linear operators from $\mathscr{A}$ into $\mathscr{X}$. For each $\phi \in C^k(\mathscr{A}, \mathscr{A}^*)$, then $\omega_\phi \in C^{k+1}(\mathscr{A}, \mathbf{C})$ is defined by

$$\omega_\phi(a_0, \ldots, a_k) = \langle a_0, \phi(a_1, \ldots, a_k)\rangle.$$

Analogously for each $\omega \in C^{k+1}(\mathscr{A}, \mathbf{C})$, $\phi_\omega \in C^k(\mathscr{A}, \mathscr{A}^*)$ is given by

$$\langle a_0, \phi_\omega(a_1, \ldots, a_k)\rangle = \omega(a_0, \ldots, a_k).$$

Following [**13**, Theorem 5.3, p. 402] we know that any form $\omega \in C^{k+1}(\mathscr{A}, \mathbf{C})$ has a unique extension to a separately ultraweakly continuous form on $\mathscr{A}^{**}$. For a von Neumann algebra $\mathscr{R}$ we will let $C_\sigma^{k+1}(\mathscr{R}, \mathbf{C})$ denote the space of all separately ultraweakly continuous

$k+1$ linear forms on $\mathscr{R}$. For any form $\omega \in C^{k+1}(\mathscr{A},\mathbf{C})$ we will let $\overline{\omega}$ denote the unique form in $C_\sigma^{k+1}(\mathscr{A}^{**},\mathbf{C})$ which satisfies

$$\overline{\omega}(\pi(a_0),\dots,\pi(a_k)) = \omega(a_0,\dots,a_k).$$

For any $\phi \in C^k(\mathscr{A},\mathscr{A}^*)$ we will let $\bar{\phi} \in C^k(\mathscr{A}^{**},\mathscr{A}^*)$ be defined by $\phi_{\overline{\omega}_\phi}$. We remark that ϕ is uniquely determined by the fact that $\omega_{\bar{\phi}}$ is separately ultraweakly continuous on $\mathscr{A}^{**}$ and extends ω_ϕ.

We shall now turn to the special situation where $\mathscr{A}$ is a von Neumann algebra. In order to keep this continuously present we will let the algebra be denoted by $\mathscr{R}$. The predual of $\mathscr{R}$ is denoted by $\mathscr{R}_*$ and it is well known **[15]** that $\mathscr{R}^* = \mathscr{S} \oplus_{l^1} \mathscr{R}_*$, meaning that the dual is an l^1-sum of the spaces of singular and ultraweakly continuous functionals respectively. Similarly $\mathscr{R}^{**}$ is decomposed via a central projection p in $\mathscr{R}^{**}$ such that $\mathscr{R}^{**}p$ is isomorphic to $\mathscr{R}$ and all the elements of $\mathscr{R}_*$ vanish on $\mathscr{R}^{**}(I-p)$. We will let Φ denote the isomorphism of $\mathscr{R}$ onto $\mathscr{R}^{**}p$ given by $\Phi(r) = p\pi(r)$. The decomposition of $\mathscr{R}^* = \mathscr{S} \oplus_{l^1} \mathscr{R}_*$ carries over to multilinear forms. For $\omega \in C^{k+1}(\mathscr{R},\mathbf{C})$ we define $\omega^n \in C^{k+1}(\mathscr{R},\mathbf{C})$ by

$$\omega^n(r_0,\dots,r_k) = \overline{\omega}(\Phi(r_0),\dots,\Phi(r_k)).$$

For $\omega \in C_\sigma^{k+1}(\mathscr{R}^{**},\mathbf{C})$ we define $\omega^n \in C^{k+1}(\mathscr{R},\mathbf{C})$ by the same formula.

2.2. LEMMA. *If $\omega \in C^{k+1}(\mathscr{R},\mathbf{C})$, then $\omega^n \in C_\sigma^{k+1}(\mathscr{R},\mathbf{C})$. If $\omega \in C_\sigma^{k+1}(\mathscr{R},\mathbf{C})$, then $\omega^n = \omega$ and $\overline{\omega}$ vanishes whenever any of the arguments is of the form $(I-p)x$.*

Proof. The mapping Φ is an isomorphism of $\mathscr{R}$ onto a central summand in $\mathscr{R}^{**}$ and hence ultraweakly continuous. Furthermore $\overline{\omega} \in C_\sigma^{k+1}(\mathscr{R}^{**},\mathbf{C})$ so $\omega^n = \overline{\omega} \circ \Phi$ is ultraweakly continuous. Suppose $\omega \in C_\sigma^{k+1}(\mathscr{R},\mathbf{C})$. There exists a net $(e_\lambda)_\lambda$ in the unit ball $\mathscr{R}_1$ such that $\pi(e_\lambda)$ converges ultrastrongly to p in $\mathscr{R}^{**}$, hence in particular e_λ converges ultrastrongly to I in $\mathscr{R}$ and

$$\begin{aligned}\omega^n(r_0,\dots,r_k) &= \lim_{\lambda_0}\cdots\lim_{\lambda_k}\overline{\omega}(\pi(e_{\lambda_0}r_0),\dots,\pi(e_{\lambda_k}r_k)) \\ &= \lim_{\lambda_0}\cdots\lim_{\lambda_k}\omega(\pi(e_{\lambda_0}r_0),\dots,\pi(e_{\lambda_k}r_k)) = \omega(r_0,\dots,r_k).\end{aligned}$$

We can now see, that the separately ultraweakly continuous functional on $\mathscr{R}^{**}$ given by

$$\overline{\omega}(x_0,\dots,x_k) - \overline{\omega}(px_0,\dots,px_k)$$

vanishes on $\pi(\mathscr{R})$, hence it vanishes on $\mathscr{R}^{**}$. In particular if $x_i = (I-p)x_i$ then

$$\overline{\omega}(x_0, \ldots, x_i, \ldots, x_k) = \overline{\omega}(px_0, \ldots, px_i, \ldots, px_k) = 0.$$

Continuing the tour through the notation we will let $C^k_\sigma(\mathscr{R}, \mathscr{R}_*)$ denote the set of $\phi \in C^k(\mathscr{R}, \mathscr{R}^*)$ for which $\omega_\phi \in C^{k+1}_\sigma(\mathscr{R}, \mathbf{C})$. For $\phi \in C^k(\mathscr{R}, \mathscr{R}^*)$ we define $\phi^n \in C^k_\sigma(\mathscr{R}, \mathscr{R}_*)$ by $\phi^n = \phi_{(\overline{\omega}_\phi)^n}$. As an easy consequence of Lemma 2.2 we have $\phi^n = \phi$ for $\phi \in C^k_\sigma(\mathscr{R}, \mathscr{R}_*)$. We will close this section by proving a couple of results that show that certain algebraic properties for a form ω are preserved by the normal part ω^n.

2.3. LEMMA. *If $\mathscr{V} \subseteq \mathscr{R}$ and $\omega \in C^{k+1}(\mathscr{R}, \mathbf{C})$ satisfies*

$$\omega(r_0, \ldots, r_i v, r_{i+1}, \ldots, r_k) = \omega(r_0, \ldots, r_i, v r_{i+1}, \ldots, r_k)$$

for any $r_0, \ldots, r_k \in \mathscr{R}$ and any $v \in \mathscr{V}$. Then

$$\omega^n(r_0, \ldots, r_i x, r_{i+1}, \ldots, r_k) = \omega^n(r_0, \ldots, r_i, x r_{i+1}, \ldots, r_k)$$

for any $x \in \overline{\mathscr{V}}$, the ultraweak closure of $\mathscr{V}$, and any $r_0, \ldots, r_k \in \mathscr{R}$.

Proof. As above we use the fact that there exists a net (e_λ) in $\mathscr{R}_1$ such that $\pi(e_\lambda)$ converges ultrastrongly to p in $\mathscr{R}^{**}$. Since ω^n is spearately ultraweakly continuous it is of course enough to prove the relation for elements $x \in \mathscr{V}$. Now

$$\begin{aligned}
&\omega^n(r_0, \ldots, r_i x, r_{i+1}, \ldots, r_k) \\
&\quad = \lim_{\lambda_0} \cdots \lim_{\lambda_k} \overline{\omega}(\pi(r_0 e_{\lambda_0}), \ldots, \pi(r_i e_{\lambda_i} x), \pi(r_{i+1} e_{\lambda_{i+1}}), \ldots, \pi(r_k e_{\lambda_k})) \\
&\quad = \lim_{\lambda_0} \cdots \lim_{\lambda_k} \omega(r_1 e_{\lambda_1}, \ldots, r_i e_{\lambda_i} x, r_{i+1} e_{\lambda_i}, \ldots, r_k e_{\lambda_k}) \\
&\quad = \lim_{\lambda_0} \cdots \lim_{\lambda_k} \omega(r_0 e_{\lambda_0}, \ldots, r_i e_{\lambda_i}, x r_{i+1} e_{\lambda_{i+1}}, \ldots, r_k e_{\lambda_k}) \\
&\quad = \omega^n(r_0, \ldots, r_i, x r_{i+1}, \ldots, r_k).
\end{aligned}$$

The Hochschild coboundary operator

$$\Delta : C^{k-1}(\mathscr{R}, \mathscr{R}^*) \to C^k(\mathscr{R}, \mathscr{R}^*)$$

is given by

$$\begin{aligned}
&(\Delta\phi)(r_1, \ldots, r_k) \\
&\quad = r_1 \phi(r_2, \ldots, r_k) + \sum_{i=1}^{k-1} (-1)^i \phi(r_1, \ldots, r_i r_{i+1}, \ldots, r_k) \\
&\qquad + (-1)^k \phi(r_1, \ldots, r_{k-1}) r_k.
\end{aligned}$$

LEMMA 2.4. *Let* $\phi \in C^{k-1}(\mathscr{R},\mathscr{R}^*)$; *then* $\Delta(\phi^n) = (\Delta\phi)^n$.

Proof. When looking at forms instead of operators and comparing corresponding parts in the Hochschild development we find that we have to show that the two forms γ and η defined below agree. Fix an $i \in 1,\dots,k$. Define $\gamma \in C_\sigma^{k+1}(\mathscr{R},\mathbf{C})$ by

$$\gamma(r_0,\dots,r_k) = \omega^n(r_0,\dots,r_ir_{i+1},\dots,r_k).$$

Define $\mu \in C^{k+1}(\mathscr{R},\mathbf{C})$ by

$$\mu(r_0,\dots,r_k) = \omega(r_0,\dots,r_ir_{i+1},\dots,r_k)$$

and define $\eta \in C_\sigma^{k+1}(\mathscr{R},\mathbf{C})$ by $\eta = \mu^n$. The $k+1$ linear form ν on $\mathscr{R}^{**}$ given by

$$\nu(x_0,\dots,x_k) = \overline{\omega}(x_0,\dots,x_ix_{i+1},\dots,x_k)$$

is seperately ultraweakly continuous on $\mathscr{R}^{**}$ and agrees with μ on $\mathscr{R}$, hence $\bar{\mu} = \nu$ so

$$\begin{aligned}\gamma(r_0,\dots,r_k) &= \nu(p\pi(r_0),\dots,p\pi(r_k))\\ &= \bar{\mu}(p\pi(r_0),\dots,p\pi(r_k)) = \eta(r_0,\dots,r_k).\end{aligned}$$

2.5. LEMMA. *Let* $\mathscr{A}$ *be a* C^**-algebra and* $\phi \in C^{k-1}(\mathscr{A},\mathscr{A}^*)$; *then* $\Delta\bar{\phi} = \overline{\Delta\phi}$ *in* $C_\sigma^k(\mathscr{A}^{**},\mathscr{A}^*)$.

Proof. As in the previous proof we have to compare $k+1$ linear forms on $\mathscr{A}^{**}$ which are separately ultraweakly continuous and agree on the ultraweakly dense subalgebra $\pi(\mathscr{A})$ of $\mathscr{A}^{**}$.

3. Cohomology for $\mathscr{R}$ with coefficients in $\mathscr{R}^*$ or $\mathscr{R}_*$. The results in §2 serve the purpose of making the averaging techniques used by Johnson, Kadison and Ringrose in **[10]** available in this context although $\mathscr{R}^*$ is not a normal $\mathscr{R}$ module and $\mathscr{R}_*$ may not be a dual module. We follow the notation from **[13]** and define

$$\begin{aligned}Z^k(\mathscr{R},\mathscr{R}^*) &= \ker(\Delta : C^k(\mathscr{R},\mathscr{R}^*) \to C^{k+1}(\mathscr{R},\mathscr{R}^*)),\\ B^k(\mathscr{R},\mathscr{R}^*) &= \operatorname{im}(\Delta : C^{k-1}(\mathscr{R},\mathscr{R}^*) \to C^k(\mathscr{R},\mathscr{R}^*)),\\ H^k(\mathscr{R},\mathscr{R}^*) &= Z^k(\mathscr{R},\mathscr{R}^*)/B^k(\mathscr{R},\mathscr{R}^*).\end{aligned}$$

In this section we are especially interested in separately ultraweakly continuous mappings and following the notation in §2 we will denote these spaces with a σ as a subscript like C_σ^k, Z_σ^k, B_σ^k, H_σ^k.

3.1. Proposition. *Let $\rho \in Z^k_\sigma(\mathscr{R}, \mathscr{R}_*)$ and let $\mathscr{F}$ be an injective von Neumann subalgebra of $\mathscr{R}$. Then there exists $\xi \in C^{k-1}_\sigma(\mathscr{R}, \mathscr{R}_*)$ such that $\|\xi\| \le 2^{k+1}\|\rho\|$ and $(\rho - \Delta\xi)$ is an $\mathscr{F}$ module map (cf.* [3, §5]).

Proof. Suppose $\mathscr{A}$ is a finite dimensional C^*-subalgebra of $\mathscr{R}$. Then the unitary group in $\mathscr{A}$ is compact, and it is possible to obtain Proposition 3.1 with respect to $\mathscr{A}$ instead of $\mathscr{F}$ through the techniques described in [13, §4, proof of Theorem 4.3]. The $\mathscr{A}$ module property is not written down explicitly in the formulation of [13, Theorem 4.3] but the relations (4.3.2) and (4.3.3) in the proof are precisely the $\mathscr{A}$ module properties for the reduced cocycle. By Connes result [4] any injective von Neumann algebra with separable predual is the ultraweak closure of an increasing sequence of finite dimensional C^*-subalgebras $(\mathscr{A}_n)$. To any n we find ξ_n with $\|\xi_n\| \le 2^{k+1}\|\rho\|$ and $(\rho - \Delta\xi_n)$ is an $\mathscr{A}_n$ module map which vanishes whenever any of the arguments are in $\mathscr{A}_n$. Since the sequence (ξ_n) is bounded we can use a Banach limiting process and obtain a $\xi_0 \in C^{k-1}(\mathscr{R}, \mathscr{R}^*)$ such that $(\rho - \Delta\xi_0)$ is a module map with respect to the algebra $\mathscr{A}_0 = \bigcup \mathscr{A}_n$. By Lemma 2.3 $(\rho - \Delta\xi_0)^n$ is an $\mathscr{F}$ module map and by Lemma 2.4

$$(\rho - \Delta\xi_0)^n = \rho - (\Delta\xi_0)^n = \rho - \Delta(\xi_0^n).$$

The proposition follows if $\mathscr{F}$ has separable predual. The problem of generalizing this result to the case without separability conditions is dealt with in [9, p. 309] in a similar context. The arguments there are based on [7] and carry over word for word.

Remark. The problems under consideration above are very much of the same nature as those considered in the proof of Theorem 2.1 in [9] and it is also possible to use that theorem in the averaging process above, but the real trouble is not so much the averaging method, but rather the establishment of the fact, that the normal parts of the coboundaries used also have the right properties.

3.2. Theorem. *Let $\mathscr{R}$ be a properly infinite or an injective von Neumann algebra. Then $H^k_\sigma(\mathscr{R}, \mathscr{R}_*) = 0$ for any $k \in \mathbf{N}$.*

Proof. Suppose first that $\mathscr{R}$ is injective and let $\rho \in Z^k_\sigma(\mathscr{R}, \mathscr{R}_*)$. By Proposition 3.1 there exists a $\xi \in C^{k-1}_\sigma(\mathscr{R}, \mathscr{R}_*)$ such that $(\rho - \Delta\xi)$ is an $\mathscr{R}$ module map. This means that there exists an ultraweakly continuous functional $\phi \in \mathscr{R}_*$ (given by $\phi = (\rho - \Delta\xi)\ (I, \dots, I)$), such that

$$(\rho - \Delta\xi)(r_1, \dots, r_k) = (r_1 \cdots r_k\phi)(\cdot) = (\phi r_1 \cdots r_k)(\cdot).$$

In particular ϕ is a trace, but we will not make use of that. Define $\eta \in C_\sigma^{k-1}(\mathscr{R},\mathscr{R}_*)$ by $\eta(r_1,\dots,r_{k-1}) = (r_1\cdots r_{k-1})\phi$. If k is even, then an easy computation shows that $\Delta\eta = (\rho - \Delta\xi)$ so ρ is a coboundary. If k is odd then the condition $\Delta(\rho - \Delta\xi) = 0$ yields $\phi = 0$ and ρ is a coboundary in this case too. Let $\mathscr{F}$ denote a copy of $B(l^2(\mathbf{N}))$ in $\mathscr{R}$ in the properly infinite case such that $I_{\mathscr{R}} \in \mathscr{F} \subset \mathscr{R}$. Let $\rho \in Z_\sigma^k(\mathscr{R},\mathscr{R}_*)$; by Proposition 3.1 there exists a $\xi \in C_\sigma^{k-1}(\mathscr{R},\mathscr{R}_*)$ such that $(\rho - \Delta\xi)$ is an $\mathscr{F}$ module map. Choose two isometries v and w in $\mathscr{F}$ such that $vv^* + ww^* = I = v^*v = w^*w$. Now to any $\varepsilon > 0$ we choose $(k+1)$ operators $r_0,\dots,r_k$ in the unit ball $\mathscr{R}_1$ of $\mathscr{R}$ such that

$$|\langle r_0, (\rho - \Delta\xi)(r_1,\dots,r_k)\rangle| \geq \|\rho - \Delta\xi\| - \varepsilon.$$

For $s = 0,\dots,k$ define operators x_s by $x_s = vr_sv^* + wr_sw^*$. Then $\|x_s\| \leq 1$ and an easy application of the module property shows, that

$$\langle x_0, (\rho - \Delta\xi)(x_1,\dots,x_k)\rangle = 2\langle r_0, (\rho - \Delta\xi)(r_1,\dots,r_k)\rangle.$$

Hence

$$\|\rho - \Delta\xi\| \geq 2(\|\rho - \Delta\xi\| - \varepsilon)$$

so $\rho = \Delta\xi$, and the theorem follows.

COROLLARY 3.3. *Let $\mathscr{A}$ be a nuclear C^*-algebra or a C^*-algebra without bounded traces. Then the bounded Hochschild cohomology for $\mathscr{A}$ with coefficients in $\mathscr{A}^*$ vanishes.*

Proof. If $\mathscr{A}$ is nuclear, then $\mathscr{A}^{**}$ is injective **[2]**. If $\mathscr{A}$ has no bounded traces, then $\mathscr{A}^{**}$ is a properly infinite von Neumann algebra. Let $\phi \in Z^k(\mathscr{A},\mathscr{A}^*)$; then following the notation in §2 ϕ extends to $\bar\phi \in Z_\sigma^k(\mathscr{A}^{**},\mathscr{A}^*)$. By the theorem there exists $\eta \in C_\sigma^{k-1}(\mathscr{A}^{**},\mathscr{A}^*)$ such that $\bar\phi = \Delta\eta$. Letting $\xi \in C^{k-1}(\mathscr{A},\mathscr{A}^*)$ denote the restriction of η to $\mathscr{A}$ we get $\phi = \Delta\xi$.

REMARK. In the case $k = 1$ the corollary is known to hold for all C^*-algebras; this was proved for $\mathscr{R}$ semifinite in **[1]** and in full generality in **[9]**.

4. Cyclic cohomology. In this article we have shifted back and forth between $(k+1)$-linear forms on a C^*-algebra $\mathscr{A}$ and k-linear mappings from $\mathscr{A}$ to $\mathscr{A}^*$. Suppose $\phi \in Z^k(\mathscr{A},\mathscr{A}^*)$; then by definition $\Delta\phi = 0$. This relation implies the following identity for the corresponding form ω_ϕ

$$\begin{aligned} 0 = \omega_{\Delta\phi}(a_0, \dots, a_{k+1}) &= \langle a_0, \Delta\phi(a_1, \dots, a_{k+1})\rangle \\ &= \sum_{i=0}^{k}(-1)^i \omega_\phi(a_0, \dots, a_i a_{i+1}, \dots, a_{k+1}) \\ &\quad + (-1)^{k+1}\omega_\phi(a_{k+1}a_0, \dots, a_k). \end{aligned}$$

This relation involves the first variable place twice and the rest once. Since forms in general do not have a natural way of distinguishing the role played by different entries it seems natural from the point of view of the forms that the condition should be symmetric in some sense. We have no idea of whether such considerations have been in the mind of A. Connes **[6]**, but it is a fact, that he has established the cyclic cohomology theory, which is symmetric with respect to the role played by the different entries in the forms. Following **[6]** we say that a form $\omega \in C^{k+1}(\mathscr{A}, \mathbf{C})$ is cyclic iff

$$\omega(a_1, \dots, a_k, a_0) = (-1)^k \omega(a_0, a_1, \dots, a_k).$$

A cochain in $C^k(\mathscr{A}, \mathscr{A}^*)$ is said to be a cyclic cochain if the corresponding $(k+1)$ linear form is cyclic in the sense defined above. From **[6]** it follows that the cyclic cochains form a sub-complex of the (norm continuous) Hochschild complex. The sets of cyclic cocycles, cyclic coboundaries and cyclic cohomology groups are denoted by $Z^k_\lambda(\mathscr{A})$, $B^k_\lambda(\mathscr{A})$ and $H^k_\lambda(\mathscr{A})$. Moreover we have a long exact sequence established in **[6]** which relates the Hochschild and the cyclic cohomology:

$$\begin{aligned} 0 &\to H^0_\lambda(A) \to H^0(\mathscr{A}, \mathscr{A}^*) \to 0 \to H^1_\lambda(\mathscr{A}) \\ &\to H^1(\mathscr{A}, \mathscr{A}^*) \to H^0_\lambda(\mathscr{A}) \to H^2_\lambda(\mathscr{A}) \to H^2(\mathscr{A}, \mathscr{A}^*) \\ &\to \cdots \to H^n(\mathscr{A}, \mathscr{A}^*) \to H^{n-1}_\lambda(\mathscr{A}) \to H^{n+1}_\lambda(\mathscr{A}) \to H^{n+1}(\mathscr{A}, \mathscr{A}^*). \end{aligned}$$

4.1. THEOREM. *If $\mathscr{A}$ is a C^*-algebra without bounded traces, then the norm continuous cyclic cohomology vanishes.*

Proof. If $\mathscr{A}$ has no bounded traces then this is exactly the same as saying that $H^0_\lambda(A) = 0$. By Theoem 3.2 $H^n(\mathscr{A}, \mathscr{A}^*) = 0$ for $n \in \mathbf{N}$. When this is put into the long exact sequence we get first that $H^1_\lambda(\mathscr{A}) = 0$ and secondly that $H^{n-1}_\lambda(\mathscr{A}) = H^{n+1}_\lambda(\mathscr{A})$, $n \in \mathbf{N}$. The theorem follows.

4.2. COROLLARY. *If $\mathscr{A}$ is a nuclear C^*-algebra, then the odd norm continuous cyclic cohomology groups vanish and the even ones are all isomorphic to the space of all bounded traces on $\mathscr{A}$.*

Proof. By Corollary 3.3 the Hochschild cohomology vanishes, and we get as above that the odd groups all vanish, since

$$H_\lambda^1(\mathscr{A}) = 0.$$

Furthermore the even groups are all isomorphic to

$$H_\lambda^0(\mathscr{A}),$$

which in turn is just the space of bounded traces.

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