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AN ADDITION FORMULA FOR THE INDEX OF SEMIGROUPS OF ENDOMORPHISMS OF $B(H)$

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To the memory of Henry Dye: teacher, colleague, friend

It is shown that the numerical index $d_*(\alpha)$ of a semigroup α of $*$-endomorphisms of $B(H)$ obeys the logarithmic addition formula

$$d_*(\alpha \otimes \beta) = d_*(\alpha) + d_*(\beta).$$

The proof makes essential use of the theory of continuous product systems

1. Introduction. Let $\alpha = \{\alpha_t : t \geq 0\}$ be a semigroup of normal $*$-endomorphisms of $B(H)$ such that $\alpha_t(1) = 1$ and $\langle \alpha_t(A)\xi, \eta \rangle$ is continuous in t for fixed ξ, $\eta \in H$ and $A \in B(H)$. Following Powers [2] we shall refer to such an α as an *E_0-semigroup*. It will be convenient to rule out the degenerate case in which the α_t's are all automorphisms, and hence we require that $\alpha_t(B(H)) \neq B(H)$ for some (and therefore every) positive t. On the other hand, we will occasionally need to drop the hypothesis that $\alpha_t(1) = 1$, and will refer to such an α simply as a *$*$-semigroup*. We emphasize that it is essential for the techniques below that all Hilbert spaces be separable.

In [1], a numerical invariant $d_*(\alpha)$ was introduced for E_0-semigroups α which can be defined rather concretely as follows. Fixing α, let $\mathscr{U}_\alpha$ denote the set of all strongly continuous semigroups $U = \{U_t : t \geq 0\}$ of bounded operators on the Hilbert space H of α satisfying $U_0 = 1$ and

$$\alpha_t(A)U_t = U_t A, \qquad A \in B(H),\ t \geq 0.$$

$\mathscr{U}_\alpha$ can be empty (cf. [3]). But if it is not, then for every pair of elements U, $V \in \mathscr{U}_\alpha$, there is a unique complex number $c(U, V)$ such that

$$V_t^* U_t = e^{tc(U,V)} 1, \qquad t \geq 0.$$

The function c is self-adjoint ($\overline{c(U, V)} = c(V, U)$) and is *conditionally positive definite* in the sense that for every finite set of complex numbers $\lambda_1, \dots, \lambda_n$ with $\lambda_1 + \cdots + \lambda_n = 0$ and every set $U_1, \dots, U_n \in U_\alpha$

we have

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j c(U_i, U_j) \geq 0. \tag{1.1}$$

Using (1.1), one may construct a Hilbert space $H(\alpha)$ as follows. Letting $\mathbb{C}_0 U_\alpha$ denote the complex vector space of all finitely nonzero functions $f\colon U_\alpha \to \mathbb{C}$ satisfying

$$\sum_x f(x) = 0,$$

we define a sesquilinear form on $\mathbb{C}_0 U_\alpha$ by

$$\langle f, g \rangle = \sum_{x,y} f(x)\overline{g(y)} c(x, y).$$

$\langle \cdot, \cdot \rangle$ is positive semidefinite by (1.1), and its kernel

$$N = \{f \in \mathbb{C}_0 U_\alpha \colon \langle f, f \rangle = 0\}$$

is a linear subspace of $\mathbb{C}_0 U_\alpha \cdot \langle \cdot, \cdot \rangle$ induces an inner product on the quotient $\mathbb{C}_0 U_\alpha / N$, and the completion of the latter is a Hilbert space $H(\alpha)$ which is necessarily *separable* (cf. **[1]**, Proposition 5.2).

In case $\mathscr{U}_\alpha \neq \varnothing$, $d_*(\alpha)$ is defined as the dimension of $H(\alpha)$. Thus, $d_*(\alpha)$ belongs to $\{1, 2, \ldots, \infty\}$ where the symbol ∞ stands for the cardinal $\aleph_0$. If $\mathscr{U}_\alpha \neq \varnothing$, it will be arithmetically convenient to define $d_*(\alpha) = c$, the cardinality of the continuum. The set $\{1, 2, \ldots, \infty, c\}$ of values of d_* is an abelian semigroup under addition, where the usual addition in $\{1, 2, \ldots, \infty\}$ is extended to the set obtained from it by adjoining c according to the rules

$$x + c = c + x = c, \qquad c + c = c,$$

$x = 1, 2, \ldots, \infty$. It was shown in **[1]** that $d_*(\alpha)$ is an invariant for outer conjugacy of E_0-semigroups α, and that if α and β have the *same index* in the sense of Powers and Robinson **[4]** then $d_*(\alpha) = d_*(\beta)$. Moreover, if α is the CAR flow of rank n then $d_*(\alpha) = n$ (**[1]**, Corollary 2 of Proposition 5.3).

If α (resp. β) is an E_0-semigroup acting on $B(H)$ (resp. $B(K)$), then there is a unique E_0-semigroup $\alpha \otimes \beta$ acting on $B(H \otimes K)$ such that

$$(\alpha \otimes \beta)_t (A \otimes B) = \alpha_t(A) \otimes \beta_t(B),$$

for all $A \in B(H)$, $B \in B(K)$. It follows from the above remarks that d_* obeys the logarithmic additivity property

$$d_*(\alpha \otimes \beta) = d_*(\alpha) + d_*(\beta) \tag{1.2}$$

whenever α and β are outer conjugate to CAR flows. The purpose of this paper is to establish (1.2) in general. Equivalently, if $\mathscr{U}_{\alpha\otimes\beta} = \varnothing$ then either $\mathscr{U}_\alpha = \varnothing$ or $\mathscr{U}_\beta = \varnothing$; and if $\mathscr{U}_{\alpha\otimes\beta} \neq \varnothing$, then both $\mathscr{U}_\alpha$ and $\mathscr{U}_\beta$ are nonvoid and

$$(1.3) \qquad H(\alpha \otimes \beta) \cong H(\alpha) \oplus H(\beta).$$

The first of these two assertions is clear from the fact that if both $\mathscr{U}_\alpha$ and $\mathscr{U}_\beta$ are nonvoid and we choose $U \in \mathscr{U}_\alpha$ and $V \in \mathscr{U}_\beta$, then the semigroup $(U \otimes V)_t = U_t \otimes V_t$ belongs to $\mathscr{U}_{\alpha\otimes\beta}$ and hence $\mathscr{U}_{\alpha\otimes\beta} \neq \varnothing$. The second assertion (including (1.3)) is a consequence of Theorem 4.4 below. In particular, we show that every semigroup in $\mathscr{U}_{\alpha\otimes\beta}$ decomposes into a tensor product $U \otimes V$ where $U \in \mathscr{U}_\alpha$ and $V \in \mathscr{U}_\beta$.

We remark that while the above definition of $\mathscr{U}_\alpha$ (and therefore $H(\alpha)$) appears to differ from the definition of $\mathscr{U}_\alpha$ given in **[1]**, it is actually the same. The proof of that amounts to showing that if $\{U_t : t > 0\}$ is a weakly measurable family of bounded operators on H satisfying $U_s U_t = U_{s+t}$ for $s, t > 0$ and

$$\alpha_t(A)U_t = U_t A, \qquad A \in B(H),\ t > 0,$$

then $\{U_t : t > 0\}$ is strongly continuous and U_t tends strongly to 1 as $t \to 0+$. To see this, note that by (**[1]**, Theorem 4.1) there is a real constant a such that

$$U_t^* U_t = e^{at} 1, \qquad t > 0,$$

and hence $V_t = e^{-1/2at} U_t$ is a measurable semigroup of isometries. The assertion now follows from (**[1]**, Proposition 2.5(ii)).

2. Multipliers of $(0, \infty)$. By a *multiplier* of $(0,\infty)$ we mean a Borel-measurable function $m: (0,\infty) \times (0,\infty) \to \{|z| = 1\}$ satisfying

$$(2.1) \qquad m(x, y+z)m(y,z) = m(x+y, z)m(x,y), \qquad x, y > 0.$$

The purpose of this section is to establish that every multiplier m of $(0,\infty)$ is *trivial* in the sense that there is a measurable function $f: (0,\infty) \to \{|z| = 1\}$ satisfying

$$(2.2) \qquad m(x,y) = \frac{f(x+y)}{f(x)f(y)}, \qquad x, y > 0.$$

While this is analogous to a well-known fact about multipliers of the additive group $\mathbb{R}$ (**[5]**, Theorem 10.38), we have been unable to find the result we need in the literature.

We will deduce (2.2) from the following representation theorem. In the proof, we use a familiar theorem which asserts that every weakly continuous one-parameter group $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ of $*$-automorphisms of $B(H)$ is implemented by a strongly continuous one-parameter unitary group U:

$$\gamma_t(A) = U_t A U_t^*, \qquad t \in \mathbb{R},\ A \in B(H)$$

(for example, see [5] p. 141). Of course, the proof of that makes essential use of the fact that $\mathbb{R}$ has no nontrivial multipliers.

PROPOSITION 2.3. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be a $*$-semigroup acting on $B(H)$ such that each α_t leaves the set of compact operators invariant. Then there is a strongly continuous semigroup $\{V_t : t \geq 0\}$ of isometries in $B(H)$ such that*

$$\alpha_t(A) = V_t A V_t^*, \qquad t \geq 0,\ A \in B(H).$$

Proof. For every $t > 0$, consider the linear space of operators

$$E_t = \{T \in B(H) : \alpha_t(A)T = TA,\ A \in B(H)\}.$$

$E_t \neq \{0\}$ and is a Hilbert space relative to the inner product $[\cdot, \cdot]$ defined on it by

$$[S, T]1 = T^* S, \qquad S, T \in E_t.$$

Moreover, for each s, $t > 0$ there is a natural unitary operator which maps E_{s+t} onto $E_s \otimes E_t$ (for details, see §2 of [1]). So if $d(t)$ is the dimension of E_t, then d satisfies the functional equation

$$d(s+t) = d(s)d(t), \qquad s, t > 0. \tag{2.4}$$

The only solutions of (2.4) taking values in $\{1, 2, \ldots, \infty\}$ are $d \equiv 1$ and $d \equiv \infty$. Notice that the case $d \equiv \infty$ cannot occur. For if E_t is infinite dimensional and we choose an orthonormal basis V_1, $V_2, \ldots$ for E_t, then by ([1], Proposition 2.1) the V_n's are isometries having mutually orthogonal ranges which satisfy

$$\alpha_t(A) = \sum_{n=1}^{\infty} V_n A V_n^*, \qquad A \in B(H),$$

and this contradicts the hypothesis that α_t should map compact operators to compact operators.

In particular, we must have $d(1) = 1$. This means that $E_1 = \mathbb{C} \cdot V$ where V is an isometry which satisfies

$$\alpha_1(A) = VAV^*, \qquad A \in B(H).$$

Let U be a minimal unitary extension of V. This is to say that U is a unitary operator on a Hilbert space $\tilde{H}$ containing H which satisfies

$$\text{(2.5)} \qquad \begin{aligned} &\text{(i)} \quad U|_H = V, \\ &\text{(ii)} \quad \overline{\bigcup_{n\in\mathbb{Z}} U^n H} = \tilde{H}. \end{aligned}$$

Let $P \in B(\tilde{H})$ be the projection onto H. The map $A \in K(H) \mapsto AP \in K(\tilde{H})$ is a $*$-monomorphism which identifies the compact operators on H with the corner

$$K_0 = PK(\tilde{H})P$$

of the compact operators on $\tilde{H}$. Thus we may think of $\{\alpha_t\colon t \geq 0\}$ as a semigroup of $*$-endomorphisms of $K_0 \subseteq K(\tilde{H})$ satisfying

$$\lim_{t\to 0} \|\alpha_t(A) - A\| = 0$$

for every $A \in K_0$. Moreover, $\alpha_1(A) = UAU^*$ for $A \in K_0$.

Notice that there is a natural way to extend $\{\alpha_t\}$ to a semigroup $\{\beta_t\}$ of $*$-endomorphisms of the C^*-algebra $K(\tilde{H})$ of compact operators on $\tilde{H}$. To see this, let

$$K_n = U^n K_0 U^{*n}, \qquad n \in \mathbb{Z}.$$

We have $K_{n+1} \subset K_n$, and as n decreases to $-\infty$ the C^*-algebras K_n increase to a dense $*$-subalgebra of $K(\tilde{H})$. For each $n \leq 0$ we can define a semigroup $\{\beta_t\colon t \geq 0\}$ of $*$-endomorphisms of K_n by

$$\beta_t(A) = U^n \alpha_t(U^{-n} A U^n) U^{-n},$$

$A \in K_n$, $t \geq 0$. $\{\beta_t\}$ is clearly conjugate to $\{\alpha_t\}$. Moreover, the restriction of β_t to K_0 is α_t since for $A \in K_0$ we have

$$\begin{aligned} \beta_t(A) &= U^n \alpha_t(\alpha_{-n}(A)) U^{-n} \\ &= U^n \alpha_{t-n}(A) U^{-n} = U^n \alpha_{-n}(\alpha_t(A)) U^{-n} \\ &= U^n U^{-n}(\alpha_t(A)) U^n U^{-n} = \alpha_t(A). \end{aligned}$$

Similarly, one checks that the defintion of β_t on K_{n-1} agrees with the definition of β_t on the smaller algebra K_n for every $n \leq 0$. Hence β_t is well-defined on the dense $*$-subalgebra $K_0 \cup K_{-1} \cup K_{-2} \cup \cdots$ of $K(\tilde{H})$. So $\{\beta_t\colon t \geq 0\}$ extends uniquely to a semigroup of $*$-endomorphisms of $K(\tilde{H})$ satisfying

$$\text{(2.6)} \qquad \begin{aligned} &\text{(i)} \quad \lim_{t\to 0} \|\beta_t(A) - A\| = 0, \\ &\text{(ii)} \quad \beta_t|_{K_0} = \alpha_t, \quad \text{and} \\ &\text{(iii)} \quad \beta_t(UBU^{-1}) = U\beta_t(B)U^{-1}, \qquad B \in K(\tilde{H}),\ t \geq 0. \end{aligned}$$

We claim that $\beta_1(K(\tilde{H})) = K(\tilde{H})$. Indeed, $\beta_1(K(\tilde{H}))$ is a C^*-subalgebra of $K(\tilde{H})$ which is invariant under the automorphism $B \mapsto U^{-1}BU$ and which contains

$$\beta_1(K_0) = \alpha_1(K_0) = VK_0V^* = UK_0U^{-1} = K_1.$$

Hence $\beta_1(K(\tilde{H}))$ contains $K_1 \cup K_0 \cup K_{-1} \cup \cdots$, and the claim follows since the latter is dense in $K(\tilde{H})$.

By the semigroup property we conclude that $\beta_t(K(\tilde{H})) = K(\tilde{H})$ for every $t \geq 0$ and this implies that every β_t is a $*$-automorphism of $K(\tilde{H})$. Extending β_t to negative values of t by $\beta_t = \beta_{|t|}$, we obtain a C^*-dynamical system $(K(\tilde{H}), \mathbb{R}, \beta)$ which extends naturally to a W^*-dynamical system $(B(\tilde{H}), \mathbb{R}, \beta)$. By the preceding remarks there is a strongly continuous one-parameter unitary group $\{W_t \colon t \in \mathbb{R}\}$ acting on $\tilde{H}$ such that

$$\beta_t(B) = W_t B W_t^{-1}, \qquad t \geq 0,\ B \in B(H).$$

For $t \geq 0$, β_t leaves the corner $K_0 = PK(\tilde{H})P$ invariant and hence $W_t P W_t^{-1} \leq P$. It follows that the subspace $H = P\tilde{H}$ is invariant under $\{W_t \colon t \geq 0\}$, and we obtain the desired semigroup of isometries $\{V_t \colon t \geq 0\}$ by setting $V_t = W_t|_H$. □

COROLLARY. *Let* $m\colon (0,\infty)\times(0,\infty) \to \{|z| = 1\}$ *be a Borel-measurable function satisfying the multiplier equation*

$$m(x, y+z)m(y,z) = m(x+y, z)m(x,y), \qquad x, y, z > 0.$$

Then there is a measurable function $f\colon (0,\infty) \to \{|z| = 1\}$ *such that*

$$m(x,y) = \frac{f(x+y)}{f(x)f(y)}, \qquad x, y > 0.$$

Proof. For every $t > 0$, define an operator U_t on $L^2(0,\infty)$ by

$$U_t f(x) = \begin{cases} m(t, x-t)f(x-t), & x > t, \\ 0, & 0 \leq x \leq t. \end{cases}$$

$\{U_t \colon t > 0\}$ is a measurable family of isometries which, because of the multiplier equation for m, satisfies

$$U_s U_t = m(s,t)U_{s+t}, \qquad s, t > 0.$$

Therefore, $\alpha_t(A) = U_t A U_t^*$ defines a semigroup $\{\alpha_t \colon t > 0\}$ of $*$-endomorphisms of $B(L^2(0,\infty))$ such that

(2.7) $$t \in (0,\infty) \mapsto \langle \alpha_t(A)f, g\rangle$$

is measurable for fixed f, $g \in L^2(0,\infty)$ and $A \in B(L^2(0,\infty))$. By ([**1**], Proposition 2.5(i)) the functions (2.7) are continuous and $\alpha_t(A) \to A$ weakly as $t \to 0+$ for every bounded operator A. So if we define $\alpha_0(A) = A$ for all A, then $\{\alpha_t : t \geq 0\}$ satisfies the hypothesis of Proposition 2.3.

Hence there is a strongly continuous semigroup $\{V_t : t \geq 0\}$ of isometries on $L^2(0,\infty)$ such that $\alpha_t(A) = V_t A V_t^*$, i.e.,

$$U_t A U_t^* = V_t A V_t^*, \qquad A \in B(L^2(0,\infty)),\ t > 0. \tag{2.8}$$

Fix $t > 0$. (2.8) implies that $U_t^* V_t$ commutes with every bounded operator and hence there is a scalar $f(t)$ such tht $U_t^* V_t = f(t)1$. f is measurable because of the measurability of U and V. Taking $A = 1$ in (2.8) we obtain $U_t U_t^* = V_t V_t^*$ and hence $V_t = U_t U_t^* V_t = f(t) U_t$ for every $t > 0$. Thus, $|f(t)| = 1$ and, for every s, $t > 0$ we have

$$\begin{aligned} f(s+t) U_{s+t} = V_{s+t} = V_s V_t &= f(s) f(t) U_s U_t \\ &= f(s) f(t) m(s,t) U_{s+t}. \end{aligned}$$

The required formula follows by multiplying the latter equation on the left by U_{s+t}^*. □

3. Compact morphisms of product systems. We begin by recalling the definition of a (continuous) product system. This is a measurable family of separable infinite dimensional Hilbert spaces

$$p : E \to (0,\infty) \tag{3.1}$$

which is endowed with a measurable associative multiplication

$$x, y \in E \times E \mapsto xy \in E$$

which acts like tensoring in the following sense. Letting $E_t = p^{-1}(t)$ be the Hilbert space over $t \in (0,\infty)$, we require that

$$\begin{aligned} &\text{(i)} \quad E_{s+t} = \overline{\text{span}} E_s E_t, \qquad s,t > 0, \quad \text{and} \\ &\text{(ii)} \quad \text{for all } x, x' \in E_s \quad \text{and} \quad y, y' \in E_t, \quad \text{one has} \\ &\qquad \langle xy, x'y' \rangle = \langle x, x' \rangle \langle y, y' \rangle. \end{aligned} \tag{3.2}$$

In more detail, the symbol E in (3.1) denotes a standard Borel space, p denotes a measurable surjection such that each fiber $E_t = p^{-1}(t)$ is endowed with the structure of a complex Hilbert space, in such a way that there is a separable infinite dimensional Hilbert space H_0 and a Borel isomorphism

$$\theta : E \to (0,\infty) \times H_0$$

of E onto the indicated trivial family which commutes the diagram

$$\begin{array}{ccc} E & \xrightarrow{\theta} & (0,\infty)\times H_0 \\ & {}_p\searrow \quad \swarrow_p & \\ & (0,\infty) & \end{array} \tag{3.3}$$

and is unitary on fiber spaces.

By a *unit* of E we mean a measurable section

$$t \in (0,\infty) \mapsto u_t \in E_t$$

such that $u_{s+t} = u_s u_t$ for all s, $t > 0$, and which is not the zero section. $\mathscr{U}_E$ will denote the set of units of E. Finally, a *morphism* of product systems is a Borel map of product systems $\theta \colon E \to F$ such that $\theta(xy) = \theta(x)\theta(y)$ for all x, $y \in E$ and such that the restriction $\theta_t = \theta|_{E_t}$ of θ to each fiber E_t is a bounded linear operator from E_t to F_t for every $t > 0$. A morphism θ is called *compact* if each θ_t is a compact operator.

If one is given a pair of units $u \in \mathscr{U}_E$ and $v \in \mathscr{U}_F$, then one can define a morphism $\theta \colon E \to F$ as follows:

$$\theta(x) = \langle x, u_t\rangle v_t, \qquad x \in E_t,\ t > 0.$$

θ is compact because each θ_t is of rank at most one. The purpose of this section is to establish the following result, which asserts that these are the only compact morphisms. This will allow us to identify the units of a tensor product $E \otimes F$ of product systems in Corollary 3.9.

THEOREM 3.4. *Let E and F be product systems and let $\theta \colon E \to F$ be a compact morphism such that $\theta_{t_0} \neq 0$ for some $t_0 > 0$. Then there exist units $u \in \mathscr{U}_E$, $v \in \mathscr{U}_F$ such that*

$$\theta_t(x) = \langle x, u_t\rangle v_t, \quad x \in E_t,\ t > 0.$$

Proof. We first consider the case in which $F = E$ and each θ_t is a *positive* compact operator in $B(E_t)$, $t > 0$. We will show that there is a unit u in $\mathscr{U}_E$ such that

$$\theta_t(x) = \langle x, u_t\rangle u_t, \qquad x \in E_t,\ t > 0.$$

Note first that $\|\theta_{s+t}\| = \|\theta_s\| \cdot \|\theta_t\|$ for every s, $t > 0$. Indeed, (3.2) implies that there is a unitary operator

$$W_{s,t} \colon E_{s+t} \to E_s \otimes E_t$$

which implements a unitary equivalence of the maps $\theta_{s+t} \in B(E_{s+t})$ and $\theta_s \otimes \theta_t \in B(E_s \otimes E_t)$, from which the assertion is evident. The function $t \in (0, \infty) \mapsto \|\theta_t\|$ is clearly measurable and is nonzero at $t_0 > 0$. It follows from a simple argument (see the proof of Theorem 4.1 of **[1]**) that there is a real constant a such that

$$\|\theta_t\| = e^{at}, \qquad t > 0.$$

So by replacing θ_t with $e^{-at}\theta_t$ if necessary, we can assume that $\|\theta_t\| = 1$ for every positive t.

For each $t > 0$ let e_t be the projection of E_t onto the nonzero finite dimensional subspace

$$\{\xi \in E_t : \theta_t \xi = \xi\}.$$

Since the sequence θ_t, θ_t^2, $\theta_t^3, \dots$ converges strongly to e_t for every positive t, it follows that $t \mapsto e_t$ is a measurable family of operators.

We claim that each e_t is one-dimensional. To see this, note that for every s, $t > 0$ the unitary equivalence $\theta_{s+t} \cong \theta_s \otimes \theta_t$ implies that $\theta_{s+t}^n \cong \theta_s^n \otimes \theta_t^n$ for every $n \geq 1$, and hence $e_{s+t} \cong e_s \otimes e_t$. So the dimension $d(t)$ of e_t satisfies the functional equation

$$d(s+t) = d(s)d(t), \qquad s, t > 0.$$

The only solution of the latter, taking values in $\{1, 2, \dots\}$, is the function $d(t) = 1$, $t > 0$, and the claim is proved.

We claim next that there is a measurable section $t \in (0, \infty) \mapsto \xi_t \in E_t$ of unit vectors such that $e_t(\xi_t) = \xi_t$, $t > 0$. To prove this, we may assume by (3.3) that E is the trivial family $(0, \infty) \times H_0$ and that e_t is a one-dimensional projection in $B(H_0)$ for every $t > 0$. Choose an orthonormal basis ζ_1, $\zeta_2, \dots$ for H_0. For each $t > 0$, define $n(t)$ to be the smallest positive integer k such that $e_t(\zeta_k) \neq 0$. The function $n\colon (0, \infty) \to \mathbb{N}$ is measurable, and therefore

$$\xi_t = \frac{e_{n(t)}(\zeta_{n(t)})}{\|e_{n(t)}(\zeta_{n(t)})\|}, \qquad t > 0,$$

defines a measurable section with the asserted properties.

We now show that ξ_t has the form

$$\xi_t = f(t) u_t,$$

where u is a unit of E and $f\colon (0, \infty) \to \mathbb{C}$ is a measurable function satisfying $|f(t)| = 1$ for every $t > 0$. To prove this, we claim first that $\xi_s \xi_t$ is proportional to ξ_{s+t} for every s, $t > 0$. Indeed,

$$\theta_{s+t}(\xi_s \xi_t) = \theta_s(\xi_s)\theta_t(\xi_t) = \xi_s \xi_t,$$

so that $\xi_s \xi_t$ is a unit vector in the range of e_{s+t}. The claim follows from the fact that e_{s+t} is one-dimensional.

Thus, there is a unique function $m\colon (0,\infty) \times (0,\infty) \to \{|z| = 1\}$ such that

$$\xi_s \xi_t = m(s,t)\xi_{s+t}. \tag{3.6}$$

m is clearly measurable. Note that m must satisfy the multiplier equation

$$m(r, s+t)m(s,t) = m(r+s,t)m(r,s), \qquad r,s,t > 0. \tag{3.6}$$

Indeed, fixing $r, s, t > 0$ and using associativity of the multiplication in E we have

$$m(r, s+t)m(s,t)\xi_r \xi_s \xi_t = m(r,s+t)\xi_r \xi_{s+t} = \xi_{r+s+t},$$

whereas

$$m(r+s,t)m(r,s)\xi_r \xi_s \xi_t = m(r+s,t)\xi_{r+s}\xi_t = \xi_{r+s+t},$$

and (3.6) follows.

By Proposition 2.3 there is a measurable function $f\colon (0,\infty) \to \{|z| = 1\}$ such that

$$m(s,t) = f(s+t)/f(s)f(t), \qquad s,t > 0.$$

If we define $u_t = f(t)\xi_t$, then $\|u_t\| = 1$ for all $t > 0$ and (3.5) implies that u is a unit.

It follows that θ_t can be decomposed as an orthogonal sum of operators

$$\theta_t = e_t + \rho_t \tag{3.7}$$

where e_t is the one-dimensional projection $e_t(x) = \langle x, u_t \rangle u_t$ and where $\{\rho_t\colon t > 0\}$ is a measurable family of positive compact operators satisfying $\|\rho_t\| < 1$ and $\rho_t e_t = e_t \rho_t = 0$, $t > 0$.

It remains to show that each ρ_t is zero. Fix s, $t > 0$. We have seen that θ_{s+t} is unitarily equivalent to $\theta_s \otimes \theta_t$, and hence $e_{s+t} + \rho_{s+t}$ is unitarily equivalent to the direct sum

$$(e_s \otimes e_t) \oplus (e_s \otimes \rho_t) \oplus (\rho_s \otimes e_t) \oplus (\rho_s \otimes \rho_t).$$

Because $u_{s+t} = u_s u_t$, e_{s+t} is identified with $e_s \otimes e_t$ in the above unitary equivalence, and hence ρ_{s+t} is unitarily equivalent to the direct sum

$$(e_s \otimes \rho_t) \oplus (\rho_s \otimes e_t) \oplus (\rho_s \otimes \rho_t).$$

Taking $s = t$ and noting that $\|\rho_t\| < 1$ for all $t > 0$, we conclude that if $\rho_{2t} \neq 0$, then necessarily $\rho_t \neq 0$ and $\|\rho_{2t}\| = \|\rho_t\|$. Moreover, if $d(s)$ is the dimension of the eigenspace

$$\{\xi \in E_s: \rho_s(\xi) = \|\rho_s\|\xi\},$$

then we may also conclude that $d(2t) = 2d(t)$ whenever $\rho_{2t} \neq 0$.

Now suppose there is a $t_0 > 0$ such that $\rho_{t_0} \neq 0$. The preceding paragraph implies that

$$d(t_0) = 2d(t_0/2) = 4d(t_0/4) = \cdots = 2^n d(t_0/2^n)$$

for every $n \geq 1$. Since $d(t_0/2^n)$ is a positive integer we conclude that 2^n divides $d(t_0)$ for every $n \geq 1$, which is absurd.

This completes the proof in the case where $F = E$ and each θ_t is a positive operator. Now suppose more generally that $\theta: E \to F$ is an arbitrary morphism such that $\theta_t: E_t \to F_t$ is compact for every $t > 0$. The adjoint $\theta^* = \{\theta_t^*: t > 0\}$ defines a measurable family of compact operators from F to E. We claim that θ^* is a morphism, i.e.,

$$\theta_{s+t}^*(xy) = \theta_s^*(x)\theta_t^*(y), \qquad x \in F_s,\ y \in F_t, \tag{3.8}$$

for every $s, t > 0$. To see this, fix s and t and choose $x \in F_s$, $y \in F_t$, $x' \in E_s$, $y' \in E_t$. We have

$$\begin{aligned}\langle \theta_{s+t}^*(xy), x'y'\rangle &= \langle xy, \theta_{s+t}(x'y')\rangle \\ &= \langle xy, \theta_s(x')\theta_t(y')\rangle = \langle x, \theta_s(x')\rangle\langle y, \theta_t(y')\rangle \\ &= \langle \theta_s^*(x), x'\rangle\langle\theta_t^*(y), y'\rangle = \langle\theta_s^*(x)\theta_t^*(y), x'y'\rangle.\end{aligned}$$

(3.8) follows because E_sE_t spans E_{s+t}.

Therefore $\omega_t = \theta_t^*\theta_t$, $t > 0$, defines a morphism of E consisting of positive compact operators, not all of which are zero. By the above argument, there is a unit $u \in \mathscr{U}_E$ satisfying $\|u_t\| = 1$ and a real number a such that

$$\omega_t(x) = e^{at}\langle x, u_t\rangle u_t, \qquad x \in E_t,\ t > 0.$$

It follows that the initial space of θ_t is the one-dimensional space spanned by u_t, $t > 0$. Put $v_t = \theta_t(u_t)$. v is a unit of F because u is a unit of E and θ is a morphism. It follows that for every $x \in E_1$ we have

$$\theta_t(x) = \theta_t(\langle x, u_t\rangle u_t) = \langle x, u_t\rangle v_t,$$

as required. □

Now let E, F be two product systems and let

$$E \otimes F = \{E_t \otimes F_t: t > 0\}$$

be the tensor product of product systems (cf. [**1**], §3). The multiplication in $E \otimes F$ is defined uniquely by requiring

$$(x \otimes y)(u \otimes v) = (xu) \otimes (yv),$$

for $x \in E_s$, $y \in F_s$, $u \in E_t$, $v \in F_t$, $s, t > 0$. If $u \in \mathcal{U}_E$ and $v \in \mathcal{U}_F$ are units then one can define a unit $u \otimes v$ of $E \otimes F$ by $(u \otimes v)_t = u_t \otimes v_t$, $t > 0$.

COROLLARY 3.9. *Let E and F be product systems. Then every unit of $E \otimes F$ decomposes as a tensor product $u \otimes v$ where u and v are units of E and F respectively.*

Proof. Let $\bar{F}$ be the *conjugate* of the product system F, i.e., $\bar{F}$ consists of the same family of Hilbert spaces $p: F \to (0, \infty)$ except that scalar multiplication in the fibers of $\bar{F}$ is conjugated: thus for $\lambda \in \mathbb{C}$ and $x \in \bar{F}_t$, $\lambda \cdot x$ means $\bar{\lambda} x$ rather than λx. The multiplications in $\bar{F}$ and F are the same. The identity map of F can be considered a Borel isomorphism of F on $\bar{F}$ which we denote by $x \mapsto \bar{x}$. This map preserves multiplication and is anti-unitary on the fiber spaces. The inner product in $\bar{F}_t$ is given by $\langle \bar{x}, \bar{y} \rangle = \langle y, x \rangle$, x, $y \in F_t$.

Now let w be a unit of $E \otimes F$. For each $t > 0$, the bounded bilinear map

$$x, y \in E_t \times F_t \mapsto \langle x \otimes y, w_t \rangle$$

can be viewed as a sesquilinear map on $E_t \times \bar{F}_t$. Thus there is a unique bounded linear operator $\theta_t: E_t \to \bar{F}_t$ such that

$$\langle \theta_t(x), \bar{y} \rangle = \langle x \otimes y, w_t \rangle, \qquad x \in E_t, \ y \in F_t. \tag{3.10}$$

Notice that $\theta: E \to \bar{F}$ is a morphism. Indeed θ is clearly a measurable family of bounded linear operators and it is multiplicative because if $x \in E_s$, $x' \in E_t$ then for every vector in F_{s+t} of the form yy' with $y \in F_s$ and $y' \in F_t$ we have

$$\begin{aligned}\langle \theta_{s+t}(xx'), \overline{yy'} \rangle &= \langle xx' \otimes yy', w_{s+t} \rangle \\ &= \langle (x \otimes y)(x' \otimes y'), w_s w_t \rangle \\ &= \langle x \otimes y, w_s \rangle \langle x' \otimes y', w_t \rangle = \langle \theta_s(x), \bar{y} \rangle \langle \theta_t(x'), \overline{y'} \rangle \\ &= \langle \theta_s(x)\theta_t(x'), \bar{y}\overline{y'} \rangle = \langle \theta_s(x)\theta_t(x'), \overline{yy'} \rangle.\end{aligned}$$

The assertion follows because F_{s+t} is spanned by $F_s F_t$.

We claim that each θ_t is a Hilbert-Schmidt operator. Indeed, if ξ_1, $\xi_2, \dots$ (resp. $\eta_1, \eta_2, \dots$) is an orthonormal basis for E_t (resp. F_t) then

$$\sum_m \|\theta_t(\xi_m)\|^2 = \sum_{m,n} |\langle \theta_t(\xi_m), \bar{\eta}_n\rangle|^2$$
$$= \sum_{m,n} |\langle \xi_m \otimes \eta_n, w_t\rangle|^2 = \|w_t\|^2 < \infty,$$

because $\{\xi_m \otimes \eta_n : m, n \geq 1\}$ is an orthonormal basis for $E_t \otimes F_t$.

Thus, θ is a compact morphism. By Theorem 3.4 and the fact that every unit v of $\bar{F}$ has the form $v_t = \bar{w}_t$ for some unit of w of F, we conclude that there are units $u \in \mathscr{U}_E$ and $v \in \mathscr{U}_F$ such that

$$\theta_t(x) = \langle x, u_t\rangle \overline{v_t}, \qquad x \in E_t,\ t > 0.$$

Substitution of the latter in (3.10) gives

$$\langle \langle x, u_t\rangle \overline{v_t}, \bar{y}\rangle = \langle x \otimes y, w_t\rangle, \qquad x \in E_t,\ y \in F_t.$$

The left side can be written

$$\langle x, u_t\rangle \langle \overline{v_t}, \bar{y}\rangle = \langle x, u_t\rangle \langle y, v_t\rangle = \langle x \otimes y, u_t \otimes v_t\rangle.$$

It follows that $w_t = u_t \otimes v_t$, as asserted. □

REMARK. Referring back to the context of the introduction, let $\{\alpha_t : t \geq 0\}$ and $\{\beta_t : t \geq 0\}$ be E_0-semigroups acting on $B(H)$ and $B(K)$ respectively, and let $\{W_t : t \geq 0\}$ be a semigroup of isometries in $B(H \otimes K)$ such that

$$(\alpha \otimes \beta)_t(A) W_t = W_t A, \qquad A \in B(H \otimes K),\ t \geq 0.$$

Then there are semigroups of isometries U in $\mathscr{U}_\alpha$ and V in $\mathscr{U}_\beta$ such that $W_t = U_t \otimes V_t$ for every $t \geq 0$. This follows from Corollary 3.9 together with the basic results on the relation between an E_0-semigroup $\gamma = \{\gamma_t : t \geq 0\}$ and its associated product system ([**1**], §2).

4. Dimension and index. We now apply the results of §3 to prove an addition theorem for the dimension of product systems and the index of E_0-semigroups.

Let E be a product system and let $\mathscr{U}_E$ be its set of units. Theorem 4.1 of [**1**] asserts that for every pair of units u, $v \in \mathscr{U}_E$ there is a complex number $c(u, v)$ such that

$$\langle u_t, v_t\rangle = e^{tc(u,v)}, \qquad t > 0.$$

$c : \mathscr{U}_E \times \mathscr{U}_E \to \mathbb{C}$ is called the covariance function of E and it is self-adjoint and conditionally positive definite. As in the introduction, one can use c to construct a (necessarily separable) Hilbert space H_E. The *dimension* of E is defined as follows

$$\dim E = \begin{cases} \dim H_E, & \text{if } \mathcal{U}_E \neq \emptyset, \\ c, & \text{if } \mathcal{U}_E = \emptyset. \end{cases} \tag{4.1}$$

Notice that (4.1) differs slightly from the definition given in §5 of **[1]**; in the former we define $\dim E$ to be 0 in the exceptional case where $\mathcal{U}_E = \emptyset$. The present definition leads to somewhat more attractive algebraic formulas.

Before presenting the main results we prove a simple lemma about abstract covariance functions which will facilitate the computation of $\dim E$. By an (abstract) covariance function we mean a pair (X, c) consisting of a nonvoid set X and a function $c: X \times X \to \mathbb{C}$ satisfying

$$\begin{array}{ll} \text{(i)} & \overline{c(x,y)} = c(y,x), \quad \text{and} \\ \text{(ii)} & \sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} c(x_i, x_j) \geq 0 \end{array} \tag{4.2}$$

for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ satisfying $\lambda_1 + \cdots + \lambda_n = 0$, all $x_1, \dots, x_n \in X$, and all $n = 1, 2, \dots$. Starting with the vector space $\mathbb{C}_0 X$ of all finitely nonzero functions $f: X \to \mathbb{C}$ satisfying $\sum_x f(x) = 0$, we can construct a Hilbert space $H(X, c)$ by the same method sketched in the introduction (for more detail, in §5 of **[1]**).

We define the *direct sum* of two covariance functions (X, a) and (Y, b) to be the covariance function $(X \times Y, c)$ where $c: (X \times Y) \times (X \times Y) \to \mathbb{C}$ is defined by

$$c((x,y),(x',y')) = a(x,x') + b(y,y').$$

LEMMA 4.3. *Let (X, a), (Y, b) be covariance functions.*

(i) *If there exists a surjective function $\theta: X \to Y$ such that $b(\theta x, \theta y) = a(x, y)$ for all x, $y \in X$, then*

$$\dim H(X, a) = \dim H(Y, b).$$

(ii) *If $(X \times Y, c)$ is the direct sum of (X, a) and (Y, b), then*

$$\dim H(X \times Y, c) = \dim H(X, a) + \dim H(Y, b).$$

Proof. To prove (i), we will construct a unitary operator from $H(X, a)$ to $H(Y, b)$. Define sesquilinear forms $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ on $\mathbb{C}_0 X$ and $\mathbb{C}_0 Y$ respectively by

$$\langle f, g \rangle = \sum_{x,y \in X} f(x) \overline{g(y)} a(x,y),$$

$$\langle h, k \rangle' = \sum_{u,v \in Y} h(u) \overline{k(v)} b(u,v).$$

Both are positive semidefinite, and

$$N = \{f \in \mathbb{C}_0X : \langle f, f \rangle = 0\},$$
$$N' = \{g \in \mathbb{C}_0Y : \langle g, g \rangle' = 0\}$$

are subspaces of $\mathbb{C}_0X$ and $\mathbb{C}_0Y$ respectively such that $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ induce inner products on $\mathbb{C}_0X/N$ and $\mathbb{C}_0Y/N'$. $H(X, a)$ and $H(Y, b)$ are the respective completions of these inner product spaces.

We can define a linear transformation $W_0 : \mathbb{C}_0X \to \mathbb{C}_0Y$ by

$$W_0(f) = \sum_{x \in X} f(x)\delta_{\theta x},$$

δ_y denoting the function with value 1 at $y \in Y$ and 0 elsewhere. Noting that

$$\sum_{y \in Y} W_0(f)(y) = \sum_{y \in Y} \left(\sum_{\theta x = y} f(x) \right) = \sum_x f(x) = 0,$$

we see that W_0 does not indeed map $\mathbb{C}_0X$ into $\mathbb{C}_0Y$. We claim that W_0 is surjective. Indeed, if g is any nonzero element of $\mathbb{C}_0Y$ and $\{y_1, \ldots, y_n\}$ is the set of points where $g \neq 0$, then we may find $x_1, \ldots, x_n$ in X such that $\theta x_i = y_i$, $1 \leq i \leq n$. Putting

$$f = \sum_{i=1}^{n} g(y_i)\delta_{x_i},$$

we have $f \in \mathbb{C}_0X$ and $W_0(f) = g$.

Notice next that $\langle W_0(f), W_0(g) \rangle' = \langle f, g \rangle$ for all f, $g \in \mathbb{C}_0X$. For we can write

$$\begin{aligned} \langle W_0(f), W_0(g) \rangle &= \sum_{y, y' \in Y} \left(\sum_{\substack{\theta x = y \\ \theta x' = y'}} f(x)\overline{g(x')}b(\theta x, \theta x') \right) \\ &= \sum_{y, y'} \left(\sum_{\substack{\theta x = y \\ \theta x' = y'}} f(x)\overline{g(x')}a(x, x') \right) \\ &= \sum_{x, x'} f(x)\overline{g(x')}a(x, x') = \langle f, g \rangle, \end{aligned}$$

as asserted. It follows that $W_0(N) \subseteq N'$ and that W_0 induces a surjective isometry

$$W : \mathbb{C}_0X/N \to \mathbb{C}_0Y/N'.$$

The closure of W is the required unitary operator from $H(x, a)$ to $H(Y, b)$.

To prove (ii), we exhibit a unitary operator V from $H(X \times Y, c)$ to $H(X, a) \oplus H(Y, b)$. First, we define a linear transformation V_0 from $\mathbb{C}_0(X \times Y)$ into the direct sum of vector spaces $\mathbb{C}_0X + \mathbb{C}_0Y$ by $V_0(f) = (f_1, f_2)$ where

$$f_1(x) = \sum_{y \in Y} f(x, y), \quad f_2(y) = \sum_{x \in X} f(x, y).$$

Note that V_0 is surjective. For if $f \in \mathbb{C}_0X$ and $g \in \mathbb{C}_0(Y)$ and we choose any points $x_0 \in X$, $y_0 \in Y$, then $(f, g) = V_0(h)$ where h is the function in $\mathbb{C}_0(X \times Y)$ defined by

$$h(x, y) = \delta_{x_0}(x)g(y) + f(x)\delta_{y_0}(y).$$

Let $\langle \cdot, \cdot \rangle$ be the sesquilinear form defined on $\mathbb{C}_0(X \times Y)$ by c:

$$\langle f, g \rangle = \sum f(x, y)\overline{g(x', y')}c((x, y), (x', y')),$$

the sum extended over all x, $x' \in X$ and all y, $y' \in Y$. Using

$$c((x, y), (x', y')) = a(x, x') + b(y, y'),$$

we find that

$$\begin{aligned}\langle f, g \rangle &= \sum_{x, x'} f_1(x)\overline{g_1(x')}a(x, x') + \sum_{y, y'} f_2(y)\overline{g_2(y')}b(y, y') \\ &= \langle f_1, g_1 \rangle_1 + \langle f_2, g_2 \rangle_2\end{aligned}$$

where $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are the sesquilinear forms defined on $\mathbb{C}_0X$ and $\mathbb{C}_0Y$ by a and b respectively. This formula implies that V_0 induces a surjective isometry of inner product spaces

$$V : \mathbb{C}_0(X \times Y)/N \to \mathbb{C}_0(X)/N_1 \oplus \mathbb{C}_0Y/N_2$$

and so the closure of V is a unitary operator from $H(X \times Y, c)$ to $H(X, a) \oplus H(Y, b)$. □

THEOREM 4.4. *For any two product systems E, F we have*

$$\dim(E \otimes F) = \dim E + \dim F.$$

Proof. Assume first that at least one of the two sets $\mathcal{U}_E$, $\mathcal{U}_F$ is void. Then at least one of the two cardinals $\dim E$, $\dim F$ is c, and hence their sum is c. On the other hand, Corollary 3.9 implies that $\mathcal{U}_{E \otimes F}$ must be void and so $\dim(E \otimes F)$ is also c.

Thus we can assume that both E and F possess units. Let $(\mathscr{U}_E, a)$, $(\mathscr{U}_F, b)$, and $(\mathscr{U}_{E\otimes F}, c)$ be the covariance functions of E, F, and $E\otimes F$ respectively and let $(\mathscr{U}_E \times \mathscr{U}_F, d)$ be the sum of $(\mathscr{U}_E, a)$ and $(\mathscr{U}_F, b)$:

$$d((u,v),(u',v')) = a(u,u') + b(v,v').$$

We claim that

$$\dim H(\mathscr{U}_E \times \mathscr{U}_F, d) = \dim H(\mathscr{U}_{E\otimes F}, c) = \dim(E \otimes F). \tag{4.5}$$

By Lemma 4.3(i), it is enough to observe that the map $\theta\colon \mathscr{U}_E \times \mathscr{U}_F \to \mathscr{U}_{E\otimes F}$ defined by $\theta(u,v) = u\otimes v$ is surjective and satisfies

$$c(\theta(u,v),\theta(u',v')) = a(u,u') + b(v,v'). \tag{4.6}$$

The surjectivity of θ is immediate from (3.9). If u, $u' \in \mathscr{U}_E$ and v, $v' \in \mathscr{U}_F$ then for every $t > 0$ we have

$$\begin{aligned} e^{tc(u\otimes v, u'\otimes v')} &= \langle (u\otimes v)_t, (u'\otimes v')_t\rangle \\ &= \langle u_t \otimes v_t, u'_t \otimes v'_t\rangle = \langle u_t, u'_t\rangle\langle v_t, v'_t\rangle \\ &= e^{ta(u,u')}e^{tb(v,v')} = e^{t(a(u,u')+b(v,v'))}, \end{aligned}$$

and (4.6) follows from this.

Finally, Lemma 4.3(ii) implies that

$$\begin{aligned} \dim H(\mathscr{U}_E \times \mathscr{U}_F, d) &= \dim H(\mathscr{U}_E, a) + \dim H(\mathscr{U}_F, b) \\ &= \dim E + \dim F, \end{aligned}$$

and we are done. □

The main result on additivity of the index of E_0-semigroups is now a simple consequence of (4.4).

COROLLARY 4.7. *Let α, β be E_0-semigroups. Then we have*

$$d_*(\alpha\otimes\beta) = d_*(\alpha) + d_*(\beta).$$

Proof. Let E_α and E_β be the product systems associated to α and β as in §2 of [**1**]. By ([**1**], Proposition 3.15 et seq.), $E_{\alpha\otimes\beta}$ is isomorphic to the tensor product of product systems $E_\alpha \otimes E_\beta$. Hence

$$d_*(\alpha\otimes\beta) = \dim E_{\alpha\otimes\beta} = \dim(E_\alpha \otimes E_\beta).$$

By Theorem 4.4 the right side is

$$\dim E_\alpha + \dim E_\beta = d_*(\alpha) + d_*(\beta).$$ □

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