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AN APPLICATION OF DYE'S THEOREM ON PROJECTION LATTICES TO ORTHOGONALLY DECOMPOSABLE ISOMORPHISMS

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Dedicated to the memory of Professor Henry Abel Dye

The predual M_* of a von Neumann algebra M is an orthogonally decomposable ordered Banach space in the sense of Yamamuro. By an application of Dye's theorem on projection lattices, it is shown that the isomorphism of this structure is implemented by a Jordan automorphism of M if M is of type I without direct summands of type I_2 in the large.

1. Main results. The notion of orthogonal decomposability (o.d.) of a real Banach space E ordered by a proper closed convex cone E_+ has been investigated by Yamamuro [**14**]. The complex Banach space $E + iE$ (with the natural conjugation $(x+iy)^* = x - iy$ for $x, y \in E$) will also be called o.d. if E is o.d. The predual M_* of a von Neumann algebra M is an o.d. (complex) ordered Banach space, in which the relevant orthogonal decomposition of $\omega = \omega^* \in M_*$ is $\omega = \omega_+ - \omega_-$ with $\omega_\pm \in M_*^+$ and $(\operatorname{supp}\omega_+) \perp (\operatorname{supp}\omega_-)$ (supp denotes the support projection) and is unique (for example, Theorem 3.2.7 in [**3**]).

An order preserving continuous linear map of an o.d. Banach space E is called an o.d. homomorphism if it preserves the orthogonal decomposition. Such o.d. homomorphisms of an ordered Hilbert space have been investigated in [**5**], [**6**]. We shall call an o.d. homomorphism an o.d. isomorphism if it is an order isomorphism. In this note, we make some comments on an o.d. isomorphism of the predual M_* of a von Neumann algebra M. Our main result is the following theorem.

THEOREM 1.1. *Let M_i $(i = 1, 2)$ be W^*-algebras with one of them type* I *having no direct summands of type* I_2 *in the large. A map α is an o.d. isomorphism of $(M_1)_*$ onto $(M_2)_*$, if and only if there exist a positive central element λ_α of M_2 with a bounded inverse and a Jordan isomorphism β_α of M_2 onto M_1 such that*

$$\alpha\omega = \omega\beta_\alpha\lambda_\alpha \quad (\forall \omega \in (M_1)_*). \tag{1.1}$$

We leave the case of a general M as an open problem.

REMARK 1.2. Any Jordan isomorphism of a W^*-algebra is a direct sum of a $*$-isomorphism and $*$-antiisomorphism **[8]**. The central element λ_α will be a positive number if M_1 or M_2 (and hence both) is a factor.

REMARK 1.3. If α is implementable by a linear map φ_α between the standard representation space of M_1 and M_2 in the form

$$(\alpha\omega)(x) = \omega(\varphi_\alpha^* x \varphi_\alpha), \tag{1.2}$$

then α is of the form (1.1) by a theorem of Dang and Yamamuro (**[5]**, (2.2)). Conversely, any α of the form (1.1) is implemented in the form of (1.2) by such a *linear* o.d. isomorphism φ_α where the order is by the natural positive cone of the standard representation space (the multiplication by $\lambda_\alpha^{1/2}$ and the *unitary* implementer of β_α). Therefore the issue here is the implementability.

Specializing Theorem 1.1 to a norm preserving α, we obtain the following generalization of Wigner's theorem **[13, 1, 2, 12, 11]**.

COROLLARY 1.4. *Let M_i $(i = 1, 2)$ be W^*-algebras with one of them type* I *having no direct summands of type* I_2 *in the large. A bijective map α of $(M_1)^+_{*1}$ onto $(M_2)^+_{*1}$ is affine and preserves orthogonality of states if and only if there exists a Jordan isomorphism β_α of M_2 onto M_1 such that*

$$\alpha\omega = \omega\beta_\alpha \quad (\forall\omega \in (M_1)^+_{*1}). \tag{1.3}$$

REMARK 1.5. $(M_i)^+_{*1}$ is the set of all normal normalized positive functionals on M_i, i.e. the state space of M_i. Its affine structure is based on the physical (or probabilistic) notion of a mixture of states. The orthogonality of two states means that the two states are sharply distinguishable in the following sense: if the system is surely in one of the two states, then it is surely not in the other state. Wigner uses the condition of the preservation of the transition probability between two states. The present version uses only the zero transition probability. Wigner's theorem is for $B(H)$ and here it is generalized to a type I von Neumann algebra possibly with a nontrivial center (with type I_2 excluded).

Kadison **[9]** has given a similar theorem for a general von Neumann algebra with our assumption of the preservation of orthogonality for α replaced by the weak $*$ continuity assumption.

Theorem 1.1 is related to an isomorphism between the lattices $P(M_1)$ and $P(M_2)$ of orthogonal projections in M_1 and M_2 via the following proposition.

PROPOSITION 1.6. (1) *If α is an o.d. isomorphism of $(M_1)_*$ onto $(M_2)_*$, then there exists a unique isomorphism γ_α of the orthocomplemented lattice $P(M_1)$ onto $P(M_2)$ such that*

$$\gamma_\alpha(\operatorname{supp}\omega) = \operatorname{supp}\alpha\omega \quad (\forall\omega \in (M_1)_*^+). \tag{1.4}$$

(2) *Assume that M_1 does not have any type I_2 direct summands in the large. If γ is an isomorphism of the complete orthocomplemented lattice $P(M_1)$ onto $P(M_2)$, there exists a unique o.d. isomorphism α_γ of $(M_1)_*$ onto $(M_2)_*$ such that*

$$(\alpha_\gamma\omega)(\gamma p) = \omega(p) \quad (\forall p \in P(M_1),\ \omega \in (M_1)_*). \tag{1.5}$$

(3) *If γ and α_γ are as in (2), the γ_α determined for $\alpha = \alpha_\gamma$ as in (1) coincides with the given γ.*

(4) *If α is of the form (1.1), γ_α is determined as in (1) and α_γ is determined for $\gamma = \gamma_\alpha$ as in (2), then*

$$\alpha_\gamma\omega = \omega\beta_\alpha. \tag{1.6}$$

Now the following theorem of Dye ([7], p. 83 Corollary) is applicable to γ.

THEOREM 1.7 (DYE). *Any projection orthoisomorphism of a W^*-algebra M with no direct summands of type I_2 in the large onto a W^*-algebra N is implemented by the direct sum of a $*$-isomorphism and a $*$-antiisomorphism.*

Here a projection orthoisomorphism is a one-one mapping between the set of projections which preserves orthogonality, i.e. $P \perp Q$ if and only if their images are orthogonal. It automatically preserves order and commutativity.

2. Isomorphisms of projection lattices.

LEMMA 2.1. *If α is an o.d. isomorphism of $(M_1)_*$ onto $(M_2)_*$, then α^{-1} is an o.d. isomorphism of $(M_2)_*$ onto $(M_1)_*$.*

Proof. Let $\alpha(\omega_1) \perp \alpha(\omega_2)$ for $\omega_1, \omega_2 \in (M_1)_*^+$. We shall prove that $\omega_1 \perp \omega_2$. (The orthogonality is that of the support projections.)

Let $\omega_1 - \omega_2 = \omega_+ - \omega_-$ be the Jordan decomposition of the selfadjoint normal linear functional $\omega_1 - \omega_2$ into positive linear functionals with mutually orthogonal support:

$$\omega_+ \perp \omega_-. \tag{2.1}$$

Then, by the o.d. property of α, we have

$$\alpha(\omega_1) - \alpha(\omega_2) = \alpha(\omega_+) - \alpha(\omega_-), \qquad \alpha(\omega_+) \perp \alpha(\omega_-). \tag{2.2}$$

By assumption, $\alpha(\omega_1) \perp \alpha(\omega_2)$. By the uniqueness of the orthogonal decomposition, we obtain

$$\alpha(\omega_1) = \alpha(\omega_+), \quad \alpha(\omega_2) = \alpha(\omega_-). \tag{2.3}$$

Since α is bijective, we obtain $\omega_1 = \omega_+$ and $\omega_2 = \omega_-$. In particular, $\omega_1 \perp \omega_2$. □

LEMMA 2.2. *For any projection $p \in M$,*

$$p = 1 - \bigvee_{\omega} \{\operatorname{supp} \omega : \operatorname{supp} \omega \perp p\} \tag{2.4a}$$

$$= \bigvee_{\omega} \{\operatorname{supp} \omega : \operatorname{supp} \omega \leq p\}. \tag{2.4b}$$

Proof. Let H be a standard representation space of M and let p be a projection in M. Then

$$1 - p = \bigvee_{\xi} \{\operatorname{supp} \omega_\xi : \xi \in (1-p)H\} \tag{2.5}$$

where $\omega_\xi(x) = (x\xi, \xi)$. This proves (2.4a). Writing $1-p$ as p in (2.4a) and using the equivalence of $\operatorname{supp} \omega \perp (1-p)$ and $\operatorname{supp} \omega \leq p$, we obtain (2.4b). □

LEMMA 2.3. *Let α be an o.d. isomorphism of $(M_1)_*$ onto $(M_2)_*$. For $\omega_1, \omega_2 \in (M_1)_*^+$, the following are equivalent.*

$$\operatorname{supp} \omega_1 = \operatorname{supp} \omega_2, \tag{2.6a}$$

$$\operatorname{supp} \alpha(\omega_1) = \operatorname{supp} \alpha(\omega_2). \tag{2.6b}$$

Proof. By the definition of an o.d. isomorphism and Lemma 2.1, $\omega' \perp \omega$ and $\alpha(\omega') \perp \alpha(\omega)$ are equivalent. Hence

$$\{\omega'' : \omega'' \perp \alpha(\omega)\} = \{\alpha(\omega') : \omega' \perp \omega\} \tag{2.7}$$

for any $\omega \in (M_1)_*^+$. Since the right-hand side is determined by $\operatorname{supp} \omega$, it coincides for $\omega = \omega_1$ and $\omega = \omega_2$ if (2.6a) holds. Then the equality of the left-hand side together with (2.4a) implies (2.6b). Since α^{-1} is an o.d. isomorphism by Lemma 2.1, the converse also holds. □

LEMMA 2.4. *For any projection $p \in M_1$, the following holds*:

$$1 - \bigvee_{\omega}\{\operatorname{supp}\alpha(\omega) : \operatorname{supp}\omega \perp p\} \tag{2.8a}$$

$$= \bigvee_{\omega'}\{\operatorname{supp}\alpha(\omega') : \operatorname{supp}\omega' \le p\} \qquad (\equiv \gamma_\alpha(p)). \tag{2.8b}$$

Proof. If $\operatorname{supp}\omega \perp p$ and $\operatorname{supp}\omega' \le p$, then $\operatorname{supp}\omega \perp \operatorname{supp}\omega'$ and hence $\operatorname{supp}\alpha(\omega) \perp \operatorname{supp}\alpha(\omega')$. Hence the inclusion (2.8a) $\supseteq$ (2.8b) holds. Let (2.8a) be p'. For any $\omega'' \in (M_2)_*^+$ satisfying $\operatorname{supp}\omega'' \le p'$, let $\omega' = \alpha^{-1}\omega''$. Then $\operatorname{supp}\alpha(\omega') \perp \operatorname{supp}\alpha(\omega)$ whenever $\operatorname{supp}\omega \perp p$. This implies $\operatorname{supp}\omega' \perp \operatorname{supp}\omega$ by Lemma 2.1 and hence $\operatorname{supp}\omega' \le p$ by (2.4a). Therefore $\operatorname{supp}\alpha(\omega') = \operatorname{supp}\omega''$ is contained in (2.8b) and hence the inclusion (2.8a) $\subseteq$ (2.8b) holds by (2.4b). □

Proof of Proposition 1.6(1). We define γ_α by (2.8). If $p = \operatorname{supp}\omega'$, then $\gamma_\alpha(p) = \operatorname{supp}\alpha(\omega')$ by Lemma 2.1 and (2.4a). Hence (1.4) holds.

If $p_1 \le p_2$, then (2.8) implies

$$\gamma_\alpha(p_1) \le \gamma_\alpha(p_2). \tag{2.9}$$

By using $1 - p$ for p in (2.8b) and comparing it with (2.8a) for p, we obtain

$$\gamma_\alpha(1 - p) = 1 - \gamma_\alpha(p). \tag{2.10}$$

By (2.8b), $\operatorname{supp}\omega' \le p$ implies $\operatorname{supp}\alpha(\omega') \le \gamma_\alpha(p)$. Conversely, if $\operatorname{supp}\alpha(\omega') \le \gamma_\alpha(p)$, then $\alpha(\omega') \perp \alpha(\omega)$ whenever $\operatorname{supp}\omega \perp p$. This implies $\omega' \perp \omega$ by Lemma 2.1 and hence $\operatorname{supp}\omega' \le p$ by (2.4a). Therefore the following are equivalent.

$$\operatorname{supp}\alpha(\omega') \le \gamma_\alpha(p), \tag{2.11a}$$

$$\operatorname{supp}\omega' \le p. \tag{2.11b}$$

If we denote $\alpha' = \alpha^{-1}$, then this equivalence and (2.4) imply

$$\gamma_{\alpha'}\gamma_\alpha(p) = p, \quad \gamma_\alpha\gamma_{\alpha'}(p') = p'. \tag{2.12}$$

(2.9) and (2.12) imply that γ_α is a lattice isomorphism. Together with (2.10), we see that γ_α is an isomorphism of orthocomplemented lattices.

The mapping of $\operatorname{supp}\omega$ specified by (1.4) together with the structure of the orthocomplemented lattice completely determines the mapping $\gamma_\alpha(p)$ by (2.4). Hence the uniqueness of γ_α follows. □

We note that if M is σ-finite (e.g. M_* is separable), then any p is of the form $p = \operatorname{supp}\omega$ for some $\omega \in M_*^+$ and the proof is much shorter.

3. Proof of Theorem 1.1. The right-hand side of (1.1) is linear and continuous in the norm topology. Since the multiplication of a central element λ_α does not change the support of states, we have

$$\operatorname{supp}(\omega\beta_\alpha\lambda_\alpha) = \beta_\alpha^{-1}(\operatorname{supp}\omega). \tag{3.1}$$

Since a Jordan isomorphism β_α^{-1} preserves the orthogonality of projections, the "if" part of Theorem 1.1 follows.

To prove the converse, let α be an o.d. isomorphism of $(M_1)_*$ onto $(M_2)_*$ and γ_α be the isomorphism of $P(M_1)$ onto $P(M_2)$ given by Proposition 1.6 (1). Because of the o.d. property of α and Lemma 2.1, γ_α is an orthoisomorphism and hence can be extended to a Jordan isomorphism of M_1 onto M_2 by Dye's Theorem (Theorem 1.7). Let β_α be the inverse of this Jordan isomorphism and let

$$\alpha'\omega = (\alpha\omega)\beta_\alpha^{-1}. \tag{3.2}$$

Then α' is an o.d. automorphism of $(M_1)_*$ and

$$\operatorname{supp}(\alpha'\omega) = \beta_\alpha(\operatorname{supp}\alpha\omega) = \operatorname{supp}\omega, \tag{3.3}$$

namely α' preserves the support of any $\omega \in (M_1)_*^+$. Thus we can restrict our attention to the case of $M = M_1 = M_2$ and an o.d. automorphism α of M_* which preserves the support of any $\omega \in M_*^+$.

We now limit ourselves to the case of a type I von Neumann algebra M.

LEMMA 3.1. *If α is an o.d. automorphism of M_* preserving the support of all $\omega \in M_*^+$ and if ω is such that* $\operatorname{supp}\omega$ *is an abelian projection of M, then there exists a central element λ_ω with a bounded inverse such that*

$$\alpha\omega = \omega\lambda_\omega. \tag{3.4}$$

Proof. Let $e = \operatorname{supp}\omega$ and $\bar{e}$ be its central support. Then $eMe = eZ$ because e is assumed to be an abelian projection. Here Z is the center of M and eZ is isomorphic to $\bar{e}Z$. Let $\overline{\omega}$ and $\overline{\alpha\omega}$ be the restriction of ω and $\alpha\omega$ to Z. They have a common support $\bar{e}$. Let

$$\lambda_\omega = (1-\bar{e}) + d(\overline{\alpha\omega})/d\overline{\omega} \tag{3.5}$$

where the last term is the Radon-Nikodym derivative with support $\bar{e}$.

Let e_n be the spectral projection of λ_ω for the interval $[n, n+1)$. Since α preserves the support of any $\omega \in M_*^+$, we obtain

$$\operatorname{supp}\alpha(\omega e_n) = \operatorname{supp}(\omega e_n) = e_n e,$$
$$\operatorname{supp}\alpha(\omega\cdot(1-e_n)) = \operatorname{supp}(\omega\cdot(1-e_n)) = (1-e_n)e.$$

Applying these support properties to

$$\alpha\omega = \alpha(\omega e_n + \omega \cdot (1 - e_n)) = \alpha(\omega e_n) + \alpha(\omega \cdot (1 - e_n)),$$

we obtain

$$\alpha(\omega e_n) = (\alpha\omega) e_n \tag{3.6}$$

and hence

$$\|\alpha(\omega e_n)\| = (\alpha\omega)(e_n) = \omega(\lambda_\omega e_n) \geq n\omega(e_n) = n\|\omega e_n\|. \tag{3.7}$$

Therefore $\omega(e_n) = \|\omega e_n\|$ must vanish for $n > \|\alpha\|$. Since $e_n \leq \overline{e} = \operatorname{supp} \overline{\omega}$ for $n \neq 1$, we have $e_n = 0$ for $n > \|\alpha\|$ and $n \neq 1$. Hence λ_ω is bounded. The same argument for α^{-1} proves that λ_ω has a bounded inverse. □

We will show that λ_ω can be taken independent of ω.

LEMMA 3.2. *Let* $\omega_1, \omega_2 \in M_*^+$ *be such that* $\operatorname{supp} \omega_1$ *and* $\operatorname{supp} \omega_2$ *are abelian projections. Let* $\overline{e}$ *be the product* (inf) *of their central support. Then*

$$\lambda_{\omega_1} \overline{e} = \lambda_{\omega_2} \overline{e}. \tag{3.8}$$

Proof. Let $\omega_i' = \omega_i \overline{e}$ $(i = 1, 2)$. Then $\operatorname{supp} \omega_i' = \overline{e}$ and

$$\alpha\omega_i' = (\alpha\omega_i)\overline{e} = \omega_i \lambda_{\omega_i} \overline{e} = \omega_i' \lambda_{\omega_i} \overline{e}, \tag{3.9}$$

where the first equality is by (3.6). Hence $\lambda_{\omega_i} \overline{e} = \lambda_{\omega_i'}$ and we have reduced the problem to the case where the equality

$$\lambda_{\omega_1} = \lambda_{\omega_2} \tag{3.10}$$

is to be proved for ω_1 and ω_2 with the same central support.

(1) *The case where* $\operatorname{supp} \omega_1 = \operatorname{supp} \omega_2$. For $A = \sum_{i=1}^N c_i e_i$ with $N < \infty$, positive numbers c_i and central projections e_i, we obtain by (3.6)

$$\alpha(\omega_1 A) = \sum c_i \alpha(\omega_1 e_i) = \sum c_i \alpha(\omega_1) e_i = \alpha(\omega_1) A \tag{3.11a}$$

$$= \omega_1 A \lambda_{\omega_1}. \tag{3.11b}$$

Since the $\omega_1 A$ with such central elements A are norm dense in the set of $\omega \in M_*^+$ with $\operatorname{supp} \omega \leq \operatorname{supp} \omega_1$ (because $\operatorname{supp} \omega_1$ is assumed to be an abelian projection of M), we have $\alpha\omega = \omega\lambda_{\omega_1}$, i.e. $\lambda_\omega = \lambda_{\omega_1}$, for all such ω and in particular (3.10) holds.

(2) *The case where* $\operatorname{supp}\omega_1 \perp \operatorname{supp}\omega_2$. We keep the assumption that ω_1 and ω_2 have the same central support and $\operatorname{supp}\omega_i$ are abelian projections. There exists a partial unitary $u \in M$ such that $u^*u = \operatorname{supp}\omega_1$ and $uu^* = \operatorname{supp}\omega_2$. Let

$$\omega_2' = \omega_1 \cdot (\operatorname{Ad} u^*) \quad (\text{i.e. } \omega_2'(A) = \omega_1(u^*Au)). \tag{3.12}$$

Then $\operatorname{supp}\omega_2' = uu^* = \operatorname{supp}\omega_2$ and hence $\lambda_{\omega_2'} = \lambda_{\omega_2}$ by (1). Let

$$\omega(A) = \omega_1((1+u^*)A(1+u))/2. \tag{3.13}$$

For $\lambda_1, \lambda_2, \lambda \in \bar{e}Z^+$, $Z \equiv M \cap M'$, the condition

$$\omega_1\lambda_1 + \omega_2'\lambda_2 \geq \omega\lambda \tag{3.14}$$

is equivalent to

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \geq (\lambda/2)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3.15}$$

i.e.

$$\lambda \leq (\lambda_1 + \lambda_2)^{-1}(2\lambda_1\lambda_2) \tag{3.16}$$

where inverse is taken on $\operatorname{supp}\lambda_1\lambda_2$. (Because of the support properties, we can restrict our attention to

$$(u^*u + uu^*)M(u^*u + uu^*) \approx Z\bar{e} \otimes M_2$$

where M_2 is 2×2 matrices.) For given λ_1, λ_2, there exists the supremum of λ in $\bar{e}Z$ given by the right-hand side of (3.16).

Since α is a linear order isomorphism, (3.14) is equivalent to

$$\alpha(\omega_1\lambda_1) + \alpha(\omega_2'\lambda_2) \geq \alpha(\omega\lambda). \tag{3.17}$$

We note that

$$\operatorname{supp}\omega = (1+u)e(1+u^*)/2 \tag{3.18}$$

is immediately seen to be an abelian projection with the same central support as e. By (3.11a), we obtain the equivalence of (3.17) with

$$\alpha(\omega_1)\lambda_1 + \alpha(\omega_2')\lambda_2 \geq \alpha(\omega)\lambda, \tag{3.19}$$

or equivalently with

$$\omega_1\lambda_{\omega_1}\lambda_1 + \omega_2'\lambda_{\omega_2'}\lambda_2 \geq \omega\lambda_\omega\lambda. \tag{3.20}$$

For given λ_1 and λ_2, the infimum of λ satisfying this inequality is

$$\lambda_{\inf} = (\lambda_{\omega_1}\lambda_1 + \lambda_{\omega_2'}\lambda_2)^{-1}\lambda_\omega^{-1}(2\lambda_{\omega_1}\lambda_1\lambda_{\omega_2'}\lambda_2). \tag{3.21}$$

(Note that λ_ω^{-1} is bounded on $\operatorname{supp}\omega = \overline{e} \geq \operatorname{supp}(\lambda_{\omega_1}\lambda_1\lambda_{\omega_2'}\lambda_2)$.) By equivalence of (3.20) and (3.14), this must coincide with

$$\lambda_{\mathrm{inf}} = (\lambda_1 + \lambda_2)^{-1}(2\lambda_1\lambda_2). \tag{3.22}$$

We now obtain

$$2(\lambda_1 + \lambda_2)\lambda_{\omega_1}\lambda_1\lambda_{\omega_2'}\lambda_2 = 2\lambda_1\lambda_2\lambda_\omega(\lambda_{\omega_1}\lambda_1 + \lambda_{\omega_2'}\lambda_2)$$

for any $\lambda_1, \lambda_2 \in \overline{e}Z^+$ where $\operatorname{supp}\lambda_{\omega_1} = \operatorname{supp}\lambda_{\omega_2'} = \operatorname{supp}\lambda_\omega = \overline{e}$. Therefore

$$\lambda_{\omega_1}\lambda_{\omega_2'} = \lambda_\omega\lambda_{\omega_1} = \lambda_\omega\lambda_{\omega_2'}$$

and hence

$$\lambda_{\omega_1} = \lambda_{\omega_2'} \quad (= \lambda_\omega). \tag{3.23}$$

Since $\lambda_{\omega_2'} = \lambda_{\omega_2}$, we obtain (3.10) for the present case.

(3) *The general case.* Since M is assumed to be type I without type I_2 direct summands in the large, there exists a central projection E such that ME is abelian and $M(1-E)$ is without abelian or type I_2 direct summands. As before, we may deal with ME and $M(1-E)$ separately, splitting $\omega \in M_*^+$ as $\omega = \omega E + \omega(1-E)$. For ME, the case (1) already proves the lemma.

Now we are left with the case of M without abelian or type I_2 direct summands. If ω_1 and ω_2 are any two elements of M_*^+ with a common central support such that $\operatorname{supp}\omega_1$ and $\operatorname{supp}\omega_2$ are both abelian projections, then there exists $\omega \in M_*^+$ with the same central support as ω_1 and ω_2 such that $\operatorname{supp}\omega$ is an abelian projection orthogonal to ω_1 and ω_2. We can apply the case (2) for the pair (ω_1, ω) and for (ω, ω_2) to obtain

$$\lambda_{\omega_1} = \lambda_\omega = \lambda_{\omega_2}.$$

□

LEMMA 3.3. *λ_ω can be taken to be independent of ω.*

Proof. One can find (by a transfinite induction) a net of $\omega_\nu \in M_*^+$ such that the central support $\overline{e}_\nu$ of ω_ν are mutually orthogonal and their sum is 1. We define

$$\lambda_\alpha = \sum_\nu \lambda_{\omega_\nu}\overline{e}_\nu. \tag{3.24}$$

By the argument in the proof of the boundedness of λ_ω, we have

$$\|\alpha\|\overline{e}_\nu \geq \lambda_{\omega_\nu}\overline{e}_\nu \geq \|\alpha^{-1}\|^{-1}\overline{e}_\nu. \tag{3.25}$$

Since $\sum \bar{e}_\nu = 1$, λ_α is bounded with a bounded inverse. It belongs to the center Z.

For any $\omega \in M_*^+$, let $\bar{e}$ be its central support. We have

$$\lambda_\omega \bar{e}\,\bar{e}_\nu = \lambda_{\omega_\nu} \bar{e}\,\bar{e}_\nu = \lambda_\alpha \bar{e}\,\bar{e}_\nu \tag{3.26}$$

for all ν by Lemma 3.3. Therefore, summing up over ν, we obtain

$$\lambda_\omega \bar{e} = \lambda_\alpha \bar{e}, \quad \alpha\omega = \omega\lambda_\omega = \omega\bar{e}\lambda_\omega = \omega\bar{e}\lambda_\alpha = \omega\lambda_\alpha. \tag{3.27}$$ □

Combining (3.2) with (3.27) for α', we obtain

$$\alpha\omega = \omega\beta_\alpha\lambda_\alpha \tag{3.28}$$

when $\operatorname{supp}\omega$ is an abelian projection. The linear hull of such ω is norm-dense in M_*. Since α is linear and norm continuous, we obtain (3.28) for all $\omega \in M_*$.

4. Proof of Corollary 1.4. Clearly (1.3) satisfies the required condition for α.

To prove the converse, let α be given. Let $(M_i)_*^h$ be the set of all selfadjoint elements of $(M_i)_*$. Each $\omega \in (M_i)_*^h$ has a unique decomposition

$$\omega = c_+\omega_+ - c_-\omega_- \tag{4.1}$$

where $\omega_\pm \in (M_i)_{*1}^+$, $\operatorname{supp}\omega_+ \perp \operatorname{supp}\omega_-$ and $c_\pm$ are real positive numbers. We extend α to a bijective map of $(M_1)_*^h$ onto $(M_2)_*^h$ by

$$\alpha\omega = c_+\alpha(\omega_+) - c_-\alpha(\omega_-). \tag{4.2}$$

We now prove that α is real linear, norm preserving, order preserving and orthogonality preserving.

(1) *Linearity.* Let $\omega_1, \omega_2 \in (M_1)_{*1}^+$ and

$$\omega = c_1\omega_1 - c_2\omega_2, \qquad c_1 > 0,\ c_2 > 0. \tag{4.3}$$

With the decomposition (4.1), we have

$$c_1\omega_1 + c_-\omega_- = c_2\omega_2 + c_+\omega_+ = c\omega' \tag{4.4}$$

with $c = c_1 + c_- = c_2 + c_+$ and

$$\omega' \equiv (c_1/c_+)\omega_1 + (c_-/c)\omega_- = (c_2/c)\omega_2 + (c_+/c)\omega_+ \in (M_1)_{*1}^+.$$

By the affine property of α on $(M_1)_{*1}^+$, we obtain

$$c_1\alpha(\omega_1) + c_-\alpha(\omega_-) = c_2\alpha(\omega_2) + c_+\alpha(\omega_+) \quad (= c\alpha(\omega')). \tag{4.5}$$

Hence (4.2) implies

$$\alpha\omega = c_1\alpha(\omega_1) - c_2\alpha(\omega_2). \tag{4.6}$$

Combining (4.6) with affine property of α, we now obtain real linearity:

$$\alpha\left(\sum_{i=1}^{N} c_i\omega_i\right) = \sum_{i=1}^{N} c_i\alpha(\omega_i)$$

for any real c's and $\omega_i \in (M_1)^+_{*1}$ and hence for any $\omega_i \in (M_1)^h_*$.

(2) *Order.* Since α is bijective between $(M_1)^+_*$ and $(M_2)^+_*$ by definition, the linearity implies that α is an order isomorphism.

(3) *Orthogonality.* This is immediate from the assumption that α on $(M_1)^+_{*1}$ preserves the orthogonality.

(4) *Norm.* From (4.1)

$$\|\omega\| = c_+\|\omega_+\| + c_-\|\omega_-\| = c_+ + c_-.$$

Since $\alpha(\omega_+)$ is orthogonal to $\alpha(\omega_-)$ (due to the orthogonality of $\omega_\pm$), (4.2) implies

$$\|\alpha\omega\| = c_+\|\alpha(\omega_+)\| + c_-\|\alpha(\omega_-)\| = c_+ + c_-.$$

Therefore $\|\alpha\omega\| = \|\omega\|$ for any $\omega \in (M_1)^h_*$.

By Theorem 1.1, we have

$$\alpha\omega = \omega\beta_\alpha\lambda_\alpha.$$

Due to $\|\alpha\| = 1$, (3.25) implies $\lambda_{\omega_\nu} = 1$ and hence $\lambda_\alpha = 1$. Therefore there exists a Jordan isomorphism β_α of M_1 onto M_2 satisfying $\alpha\omega = \omega\beta_\alpha$ for all $\omega \in (M_1)^h_*$ and in particular for $\omega \in (M_1)^+_{*1}$.

5. Proof of Proposition 1.6. (1) is already proved in §2.

Proof of (2) *and* (3). By a generalized Gleason theorem (**[4]**, **[15]**, **[16]**, **[10]**) there exists a unique $\alpha_\gamma\omega \in M_*$ for any ω satisfying

$$(\alpha_\gamma\omega)(p) = \omega(\gamma^{-1}(p)). \tag{5.1}$$

(Here the continuity in p is immediate from the right hand side, cf. **[16]**.) From this definition, α_γ is linear, maps M_*^+ into M_*^+ and satisfies

$$\operatorname{supp}\alpha_\gamma(\omega) = \gamma(\operatorname{supp}\omega). \tag{5.2}$$

Since $\alpha_{(\gamma^{-1})}$ is an inverse of α_γ with the same properties, α_γ is an o.d. isomorphism. Furthermore, (5.2) implies $\gamma = \gamma_\alpha$ for $\alpha = \alpha_\gamma$ due to (1.4) and due to the uniqueness of the isomorphism γ_α satisfying (1.4). □

Proof of (4). If α is of the form (1.1), let γ be defined by $\gamma^{-1}(p) = \beta_\alpha(p)$. Then γ is an isomorphism of the orthocomplemented lattice $P(M_1)$ onto $P(M_2)$. Since the multiplication of the invertible central element λ_α does not change the support of the functional ω, we also have

$$\gamma \operatorname{supp} \omega = \operatorname{supp} \alpha(\omega),$$

namely $\gamma = \gamma_\alpha$. Since $\gamma^{-1}(p) = \beta_\alpha(p)$, we obtain

$$(\alpha_\gamma \omega)(p) = \omega(\gamma^{-1}(p)) = \omega(\beta_\alpha(p))$$

by (1.5). This proves (1.6). □

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