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THE INJECTIVE FACTORS OF TYPE III_λ, $0 < \lambda < 1$

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Dedicated to the memory of Henry A. Dye

We give a new proof for Connes' result that an injective factor of type III_λ, $0 < \lambda < 1$ on a separable Hilbert space is isomorphic to the Powers factor R_λ. Our approach is based on lengthy, but relatively simple operations with completely positive maps together with a technical result that gives a necessary condition for that two n-tuples $(\xi_1, \dots, \xi_n)$ and $(\eta_1, \dots, \eta_n)$ of unit vectors in a Hilbert W^*-bimodule are almost unitary equivalent. As a step in the proof we obtain the following strong version of Dixmier's approximation theorem for III_λ-factors: Let N be a factor of type III_λ, $0 < \lambda < 1$, and let φ be a normal faithful state on N for which $\sigma_{t_0}^\varphi = \mathrm{id}$ ($t_0 = -2\pi/\log\lambda$); then for every $x \in N$ the norm closure of $\mathrm{conv}\{uxu^* | u \in U(M_\varphi)\}$ contains a scalar operator.

1. Introduction and preliminaries. In [**6**, §7] Connes proved that, for each $\lambda \in]0, 1[$, there is up to isomorphism only one injective factor of type III_λ (with separable predual), namely the Powers factor,

$$R_\lambda = \bigotimes_{n=1}^{\infty}(M_2, \varphi_\lambda).$$

Here M_2 is the algebra of complex 2×2-matrices and φ_λ is the state on M_2 given by

$$\varphi_\lambda \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{1}{1+\lambda}(\lambda\varphi(x_{11}) + \varphi(x_{22})).$$

(The notion R_λ was introduced by Araki and Woods in [**1**]. In Powers' original work [**19**], R_λ denoted M_α, where $\alpha = \lambda/(1+\lambda)$.)

Connes' approach for proving uniqueness of the injective factor of type III_λ ($\lambda \in]0, 1[$ fixed) is the following: By [**4**, §4] every factor N of type III_λ has an essentially unique crossed product decomposition

$$N = P \times_\theta Z$$

where P is a II_∞-factor and θ is an isomorphism of P for which $\tau \circ \theta = \lambda\tau$, where τ is a normal faithful semifinite trace on P. Moreover, N is injective if and only if P is injective. Hence, to prove the uniqueness (up to isomorphism) of N, one needs to show that

(i) There is only one injective factor P of type II_∞.

(ii) Any two automorphisms θ_1, θ_2 of the injective factor P in (i) for which

$$\tau \circ \theta_1 = \tau \circ \theta_2 = \lambda \tau$$

are outer conjugate. Note that by [**8**, Chapter 3] outer conjugacy of two automorphisms, for which $\tau \circ \theta_1 = \tau \circ \theta_2 = \lambda\tau$ implies conjugacy, i.e. there exists $\alpha \in \mathrm{Aut}(P)$ such that $\theta_2 = \alpha\theta_1\alpha^{-1}$.

The proof of (i) was established by Connes previously in the same paper [**6**, §5] by proving that "injective $\Leftrightarrow$ hyperfinite" for factors on a separable Hilbert space, and (ii) was proved one year earlier (1974) also by Connes [**5**] by developing a powerful machinery for classification of automorphisms of factors up to outer conjugacy. In [**11**] we gave a simplified proof of Connes' result "injective $\Leftrightarrow$ hyperfinite", and recently Popa [**18**] has given a third approach to this important biimplication in the type II case.

The purpose of this paper is to give an alternative proof of the uniqueness of the injective factors of type III_λ, $0 < \lambda < 1$, which still relies on the uniqueness of the injective factors of type II_1 and II_∞, but which substitutes Connes' analysis of outer conjugacy classes of automorphisms with some lengthy, but relatively simple, manipulations involving completely positive maps. The proof follows closely the ideas of our proof of "Injective $\Leftrightarrow$ hyperfinite" for II_1-factors given in [**11**, §§3, 4 and 5]. The tracial state in the II_1-factor case is substituted by a normal faithful state φ on a III_λ-factor for which $\sigma_{t_0}^\varphi = \mathrm{id}$, $(t_0 = -2\pi/\log\lambda)$.

In §5, we show that in case of an injective factor N of type III_λ with separable predual, the identity map on N has an approximate factorization through full matrix algebras (in the sense of Choi and Effros [**3**, pp. 75–76]) of a very special form:

For $m \in \mathbf{N}$, let ψ_m denote the tensor product state

$$\psi_m = \bigotimes_{i=1}^m \varphi_\lambda \quad \text{on } M_{2^m} = \bigotimes_{i=1}^m M_2.$$

Then for every finite set $x_1, \dots, x_n$ of operators in N and every $\varepsilon > 0$, there exist completely positive maps,

$$S\colon N \to M_{2^m}, \quad T\colon M_{2^m} \to N$$

such that

$$S(1) = 1, \quad T(1) = 1,$$
$$\psi_m \circ S = \varphi, \quad \varphi \circ T = \psi_m,$$
$$\sigma_t^{\psi_m} \circ S = S \circ \sigma_t^{\varphi}, \quad \sigma_t^{\varphi} \circ T = T \circ \sigma_t^{\psi_m}, \qquad t \in \mathbf{R},$$

and

$$T \circ S(x_k) \to x_k \quad \sigma\text{-strongly for } k = 1, \dots, n.$$

In §§2, 3 and 4 we prove a number of technical results, which enable us to derive from this factorization result that given $x_1, \dots, x_n \in N$ and $\varepsilon > 0$ there exists a finite dimensional subfactor F on N and $y_1, \dots, y_n \in F$, such that

$$\varphi = \varphi|_F \otimes \varphi|_{F^c} \qquad (F^c = F' \cap N),$$
$$(F, \varphi|_F) \cong (M_{2^m}, \psi_m) \quad \text{for some } m \in \mathbf{N}$$

and

$$\|x_k - y_k\|_\varphi < \varepsilon, \qquad k = 1, \dots, n,$$

where $\|a\|_\varphi = \varphi(a^*a)^{1/2}$ (cf. Lemma 6.4). From this one obtains quite easily that the factor N is isomorphic to the Powers factor R_λ and that the isomorphism can be chosen such that φ corresponds to the infinite product state

$$\omega_\lambda = \bigotimes_{i=1}^{\infty} \varphi_\lambda,$$

on R_λ. It should be noted that once the uniqueness of the injective factor of type III_λ ($\lambda \in]0, 1[$ fixed) is established, one can derive Connes' outer conjugacy result (ii) above by using [**4**, Theorem 4.4.1 (c)].

In a subsequent paper [**13**] we will apply similar techniques to give a new approach to Connes' result [7] that injective factors with trivial bicentralizers are isomorphic to the Araki-Woods factor R_∞. This result was the key to settle the uniqueness problem for injective factors of type III_1 (cf. [**12**]).

We give below some preliminaries on factors of type III_λ, $0 < \lambda < 1$, which can be extracted from Connes' paper on classification of type III-factors [**4**]:

Let M be a factor of type III_λ. By [**4**, §4], M has a normal faithful semifinite weight ω, such that $\sigma_{t_0}^{\omega} = \mathrm{id}$ ($t_0 = -2\pi/\log\lambda$), and such that the centralizer M_ω is a factor of type II_∞. Moreover, the restriction of w to M_ω is a semifinite trace. Let e be a finite projection in M_ω for which $\omega(e) = 1$. Then $\omega_e = \omega|_{eMe}$ is a normal faithful state on eMe and $\sigma_{t_0}^{\omega_e} = \mathrm{id}$. If M is σ-finite, then the projection e is equivalent to 1, and hence $eMe \cong M$. Therefore:

Every σ-finite factor N of type III_λ *admits a normal faithful state φ, such that $\sigma_{t_0}^{\varphi} = \mathrm{id}$ $(t_0 = -2\pi/\log\lambda)$.*

Let (N, φ) be as in Proposition 1.1 and let Tr be the trace on $B(H)$, where H is an infinite dimensional separable Hilbert space. Then $\varphi \otimes \mathrm{Tr}$ is a "trace generalisée" in the sense of [**4**, §4]. Hence, it follows from [**4**, Theorem 4.2.6] that:

Let N be a factor of type III_λ, *and let φ be a normal faithful state on N for which $\sigma_{t_0}^{\varphi} = \mathrm{id}$ $(t_0 = -2\pi/\log\lambda)$. Then*

(a) $\mathrm{sp}(\Delta_\varphi) = \{\lambda^n | n \in \mathbf{Z}\} \cup \{0\}$.

(b) *The centralizer M_φ of φ is a factor of type* II_1.

(c) $M_\varphi' \cap N = \mathbf{C}I$.

Note that (b) implies that φ is inner homogeneous in the sense of Takesaki [**22**]. If φ is a normal state, such that $\sigma_{t_0}^{\varphi} = \mathrm{id}$, then

$$\varepsilon(x) = \frac{1}{t_0}\int_0^{t_0} \sigma_t^{\varphi}(x)\,dx$$

defines a normal faithful φ-invariant conditional expectation of N onto M_φ. Hence, if N is injective, so is M_φ. By the equivalence of "injectivity" and "hyperfiniteness" for II_1-factors, one gets:

Let N be an injective factor of type III_λ *acting on a separable Hilbert space, and let φ be a normal faithful state for which $\sigma_{t_0}^{\varphi} = \mathrm{id}$ $(t_0 = -2\pi/\log\lambda)$. Then M_φ is isomorphic to the hyperfinite factor of type* II_1.

2. Almost unitary equivalence in Hilbert N-bimodules. In this section we will prove a technical result which generalizes [**11**, Theorem 4.2] to Hilbert W^*-bimodules.

Throughout this section N is a von Neumann algebra, and H is a normal Hilbert N-bimodule, i.e. H is a Hilbert space on which there are defined left and right actions by elements from N:

$$\left.\begin{array}{l}(x,\xi) \to x\xi \\ (x,\xi) \to \xi x\end{array}\right\}, \quad x \in N, \quad \xi \in H$$

such that the above maps $N \times H \to H$ are bilinear and

$$(x\xi)y = x(\xi y), \quad x, y \in N, \quad \xi \in H.$$

Moreover $x \to L_x$, where $L_x\xi = x\xi$, $\xi \in H$, is a normal unital *-homomorphism, and $x \to R_x$, where $R_x\xi = \xi x$, $\xi \in H$, is a normal unital *-antihomomorphism (see, e.g., [**16**, §2]).

DEFINITION 2.1. Let N be a von Neumann algebra, let (N, H) be a normal Hilbert N-bimodule, and let $\delta \in \mathbf{R}_+$. Two n-tuples $(\xi_1, \dots, \xi_n)$ and $(\eta_1, \dots, \eta_n)$ of unit vectors in H are called δ-related if there exists a family $(a_i)_{i\in I}$ of operators in N, such that

$$\sum_{i\in I} a_i^* a_i = \sum_{i\in I} a_i a_i^* = 1$$

and

$$\sum_{i\in I} \|a_i\xi_k - \eta_k a_i\|^2 < \delta, \qquad k = 1, \dots, n.$$

REMARK 2.2. Note that if $\sum_{i\in I} a_i^* a_i = \sum_{i\in I} a_i a_i^* = 1$, then for all $\xi, \eta \in H$,

$$\sum_{i\in I} \|a_i\xi - \eta a_i\|^2 = \sum_{i\in I} \|a_i^*\eta - \xi a_i^*\|^2$$

because the left side is equal to

$$\|\xi\|^2 + \|\eta\|^2 - 2\sum_{i\in I} \mathrm{Re}(a_i\xi, \eta a_i),$$

and the right side is equal to

$$\|\xi\|^2 + \|\eta\|^2 - 2\sum_{i\in I} \mathrm{Re}(\xi a_i^*, a_i^*\eta),$$

and it is clear that

$$\mathrm{Re}(a_i\xi, \eta a_i) = \mathrm{Re}(a_i\xi a_i^*, \eta) = \mathrm{Re}(\xi a_i^*, a_i^*\eta).$$

Hence, δ-relatedness is symmetric with respect to permutation of the two n-tuples.

THEOREM 2.3. *For every $n \in \mathbf{N}$ and every $\varepsilon > 0$, there exists a $\delta = \delta(n, \varepsilon) > 0$, such that for all von Neumann algebras N and all δ-related n-tuples $(\xi_1 \cdots \xi_n)$, $(\eta_1 \cdots \eta_n)$ of unit vectors in a normal Hilbert N-bimodule there exists a unitary $u \in N$ such that*

$$\|u\xi_k - \eta_k u\| < \varepsilon, \qquad k = 1, \dots, n.$$

The proof of Theorem 2.3 is divided into a series of lemmas:

LEMMA 2.4. *Let N be a von Neumann algebra, and ξ, η be two vectors in a normal Hilbert N-bimodule H. For $r > 0$, put*

$$g_r(t) = \begin{cases} 1, & 0 \le t \le r, \\ (r/t)^{1/2}, & t > r. \end{cases}$$

If $a \in N$, *and* $b = g_r(aa^*)a$, *then*

$$\|b\xi - \eta b\|^2 + \|b^*\eta - \xi b^*\|^2 \leq \|a\xi - \eta a\|^2 + \|a^*\eta - \xi a^*\|^2.$$

Proof. Let $M_2(H)$ be the set of 2×2-matrices $\xi = (\xi_{ij})_{i,j=1,2}$ with elements in H and with norm

$$\|\xi\|^2 = \sum_{i,j=1}^{n} \|\xi_{ij}\|^2.$$

$M_2(H)$ is a normal $M_2(H)$-bimodule, where left and right action is defined by formal matrix multiplication. Put

$$\zeta = \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \in M_2(H) \quad \text{and} \quad h = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \in M_2(N).$$

Then $h = h^*$ and for $n = 0, 1, 2, \ldots$

$$h^{2n+1} = \begin{pmatrix} 0 & a^*(aa^*)^n \\ (aa^*)^n a & 0 \end{pmatrix}.$$

Put

$$\varphi_r(t) = tg_r(t^2) = \begin{cases} -r^{1/2}, & t < -r^{1/2}, \\ t, & -r^{1/2} \leq t \leq r^{1/2}, \\ r^{1/2}, & t > r^{1/2}. \end{cases}$$

By approximating $g(t)$ uniformly with polynomials on $\mathrm{sp}(aa^*)$ we get

$$\varphi_r(h) = \begin{pmatrix} 0 & a^* g_r(aa^*) \\ g_r(aa^*)a & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix}.$$

Since

$$h\zeta - \zeta h = \begin{pmatrix} 0 & a^*\eta - \xi a^* \\ a\xi - \eta a & 0 \end{pmatrix}$$

we have

$$\|h\zeta - \zeta h\|^2 = \|a\xi - \eta a\|^2 + \|a^*\eta - \xi a^*\|^2.$$

Similarly

$$\|\varphi_r(h)\zeta - \zeta\varphi_r(h)\|^2 = \|b\xi - \eta b\|^2 + \|b^*\eta - \xi b^*\|^2.$$

Thus, we only have to prove that

$$\|\varphi_r(h)\zeta - \zeta\varphi_r(h)\|^2 \leq \|h\zeta - \zeta h\|^2.$$

Let L_h (resp. R_h) be the operator on $M_2(H)$ defined by left (resp. right) multiplication with h on $M_2(H)$. Since L_h and R_h commute, there exists a representation π of the abelian C^*-algebra $C(\mathrm{sp}(h) \times \mathrm{sp}(h))$ into $B(M_2(H))$, such that

$$\pi(f \otimes g) = L_{f(h)} R_{g(h)}, \qquad f, g \in C(\mathrm{sp}(h)).$$

Since $(\varphi(s) - \varphi(t))^2 \le (s-t)^2$, $s, t \in \mathbf{R}$ and since π is order preserving,

$$(L_{\varphi_r(h)} - R_{\varphi_r(h)})^2 \le (L_h - R_h)^2.$$

Hence

$$\|\varphi_r(h)\zeta - \zeta\varphi_{r(h)}\|^2 \le \|h\zeta - \zeta h\|^2.$$

This completes the proof of Lemma 2.4.

LEMMA 2.5. *Let N be a von Neumann algebra and let ζ, η be two δ-related unit vectors in N. Then for every $r \in \mathbf{N}$ there exist r operators $b_1, \dots, b_r \in N$, such that $\|b_i\| \le 1$, $i = 1, \dots, r$ and*

$$\left\|\left(\sum_{i=1}^r b_i^* b_i - 1\right)\xi\right\|^2 < \frac{12}{r}, \quad \left\|\left(\sum_{i=1}^r b_i b_i^* - 1\right)\eta\right\|^2 < \frac{12}{r},$$
$$\sum_{i=1}^r \|b_i\xi - \eta b_i\|^2 < 8\delta, \quad \sum_{i=1}^r \|b_i^*\eta - \xi b_i^*\|^2 < 8\delta.$$

Proof. By Definition 2.1 there exists a family $(a_i)_{i\in I}$ of operators in N, such that

$$\sum_{i\in I} a_i^* a_i = \sum_{i\in I} a_i a_i^* = 1$$

and

$$\sum_{i\in I} \|a_i\xi - \eta a_i\|^2 < \delta.$$

Moreover, by Remark 2.2 also

$$\sum_{i\in I} \|a_i^*\eta - \xi a_i^*\|^2 < \delta.$$

Therefore we can choose a finite subset $a_1, \dots, a_p$ of $(a_i)_{i\in I}$, such that

$$\sum_{i=1}^p (a_i^* a_i\xi, \xi) > 1 - \frac{1}{2r}, \quad \sum_{i=1}^p (a_i a_i^*\eta, \eta) \le \frac{1}{2r}.$$

Clearly

$$\sum_{i=1}^p a_i^* a_i \le 1, \quad \sum_{i=1}^p a_i a_i^* \le 1$$

and

$$\sum_{i=1}^p \|a_i\xi - \eta a_i\|^2 < \delta, \quad \sum_{i=1}^p \|a_i^*\eta - \xi a_i^*\|^2 < \delta.$$

Let $\Omega = \{(s_1, \dots, s_p) | s_i \in \mathbf{C}, |s_i| = 1\}$ be the p-dimensional torus and let $d\omega$ be the normalized Haar measure on Ω. For $\omega \in \Omega$ we let $s_1(\omega), \dots, s_p(\omega)$ denote the coordinate functions. Put

$$A(\omega) = \sum_{\nu=1}^{p} s_\nu(\omega) a_\nu, \qquad \omega \in \Omega.$$

As in the proof of [**11**, Lemma 4.3] one gets

$$\int_\Omega A(\omega)^* A(\omega)\, d\omega = \sum_{i=1}^{p} a_i^* a_i \leq 1,$$
$$\int_\Omega A(\omega) A(\omega)^*\, d\omega = \sum_{i=1}^{p} a_i a_i^* \leq 1,$$

and

$$\int_\Omega (A(\omega)^* A(\omega))^2\, d\omega \leq 2, \quad \int_\Omega (A(\omega) A(\omega)^*)^2\, d\omega \leq 2.$$

Let g_r be as in Lemma 2.4, and put

$$B(\omega) = g_r(A(\omega)A(\omega)^*)A(\omega), \qquad \omega \in \Omega.$$

Since $tg_r(t)^2 \leq r$, we have $B(\omega)B(\omega)^* \leq r1$; thus

$$\|B(\omega)\| \leq r^{1/2}, \qquad \omega \in \Omega.$$

Put

$$f_r(t) = tg_r(t)^2 = \begin{cases} t, & 0 \leq t \leq r, \\ r, & t > r. \end{cases}$$

Then

$$B(\omega)B(\omega)^* = f_r(A(\omega)A(\omega)^*).$$

Moreover, since

$$B(\omega) = A(\omega) g_r(A(\omega)^* A(\omega))$$

we have also

$$B(\omega)^* B(\omega) = f_r(A(\omega)^* A(\omega)).$$

Therefore,

$$(B(\omega)^* B(\omega))^\alpha \leq (A(\omega)^* A(\omega))^\alpha, \qquad \alpha > 0,$$

and

$$(B(\omega) B(\omega)^*)^\alpha \leq (A(\omega) A(\omega)^*)^\alpha, \qquad \alpha > 0$$

which implies that

$$\int_\Omega B(\omega)^* B(\omega)\, d\omega \leq 1, \quad \int_\Omega B(\omega) B(\omega)^*\, d\omega \leq 1,$$
$$\int_\Omega (B(\omega)^* B(\omega))^2\, d\omega \leq 2, \quad \int_\Omega (B(\omega) B(\omega)^*)^2\, d\omega \leq 2.$$

It is easy to check that

$$f_r(t) \geq t - t^2/4r, \qquad t \geq 0.$$

Therefore

$$\begin{aligned}\int_\Omega (B(\omega)^* B(\omega)\xi, \xi)\, d\omega &\geq \int_\Omega (A(\omega)^* A(\omega)\xi, \xi) - \frac{2}{4r}\|\xi\|^2 \\ &= \left(\sum_{i=1}^p (a_i^* a_i \xi, \xi)\right) - \frac{1}{2r} > 1 - \frac{1}{r}\end{aligned}$$

and similarly

$$\int_\Omega (B(\omega) B(\omega)^* \eta, \eta)\, d\omega > 1 - \frac{1}{r}.$$

Put $\Omega^r = \Omega \times \cdots \times \Omega$ (r factors). Then arguing as in the proof of [**11**, Lemma 4.3] one gets

$$\begin{aligned}&\int_{\Omega^r} \left\| \sum_{i=1}^r B(\omega_i)^* B(\omega_i)\xi \right\|^2 d\omega_1 \cdots d\omega_r \\ &= r \int_\Omega ((B(\omega)^* B(\omega))^2 \xi, \xi)\, d\omega + r(r-1) \left| \left(\int_\Omega (B(\omega)^* B(\omega)\xi, \xi)\, d\omega \right)^2 \right| \\ &\leq 2r + r(r-1) = r(r+1).\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\Omega^r} &\left\| \left(\frac{1}{r} \sum_{i=1}^r B(\omega_i)^* B(\omega_i) - 1 \right) \xi \right\|^2 d\omega_1 \cdots d\omega_r \\ &\leq \frac{r+1}{r} + 1 - 2 \int_\Omega (B(\omega)^* B(\omega)\xi, \xi)\, d\omega < \frac{3}{r},\end{aligned}$$

and similarly

$$\int_{\Omega^r} \left\| \left(\frac{1}{r} \sum_{i=1}^r B(\omega_i) B(\omega_i)^* - 1 \right) \eta \right\|^2 d\omega_1 \cdots d\omega_r < \frac{3}{r}.$$

Using that $(s_1, \ldots, s_p)$ are orthogonal vectors in $L^2(\Omega, d\omega)$ one gets

$$\int_\Omega \|A(\omega)\xi - \eta A(\omega)\|^2\, d\omega = \sum_{i=1}^p \|a_i \xi - \eta a_i\|^2 < \delta$$

and

$$\int_\Omega \|A(\omega)^*\eta - \xi A(\omega)^*\|^2\, d\omega = \sum_{i=1}^p \|a_i^*\eta - \xi a_i^*\|^2 < \delta.$$

Hence by Lemma 2.4

$$\int_\Omega \|B(\omega)\xi - \eta B(\omega)\|^2\, d\omega < 2\delta,$$
$$\int_\Omega \|B(\omega)^*\eta - \xi B(\omega)^*\|^2\, d\omega < 2\delta.$$

Therefore

$$\int_{\Omega^r} \frac{1}{r}\sum_{i=1}^r \|B(\omega_i)\xi - \eta B(\omega_i)\|^2\, d\omega_1 \cdots d\omega_r < 2\delta,$$
$$\int_{\Omega^r} \frac{1}{r}\sum_{i=1}^r \|B(\omega_i)^*\eta - \xi B(\omega_i)^*\|^2 d\omega_1 \cdots d\omega_r < 2\delta.$$

Put now

$$E_1 = \left\{(\omega_1,\dots,\omega_r) \in \Omega^r \Big| \left\| \left(\frac{1}{r}\sum_{i=1}^r B(\omega_i)^*B(\omega_i) - 1\right)\xi\right\|^2 \geq \frac{12}{r}\right\},$$
$$E_2 = \left\{(\omega_1,\dots,\omega_r) \in \Omega^r \Big| \left\| \left(\frac{1}{r}\sum_{i=1}^r B(\omega_i)B(\omega_i)^* - 1\right)\eta\right\|^2 \geq \frac{12}{r}\right\},$$
$$E_3 = \left\{(\omega_1,\dots,\omega_r) \in \Omega^r \Big| \frac{1}{r}\sum_{i=1}^r \|B(\omega_i)\xi - \eta B(\omega_i)\|^2 \geq 8\delta\right\},$$
$$E_4 = \left\{(\omega_1,\dots,\omega_r) \in \Omega^r \Big| \frac{1}{r}\sum_{i=1}^r \|B(\omega_i)^*\eta - \xi B(\omega_i)^*\|^2 \geq 8\delta\right\}.$$

By the inequalities proved above,

$$\int_{E_i} d\omega_1 \cdots d\omega_r < \frac{1}{4}, \qquad i = 1, 2, 3, 4.$$

Therefore $\Omega^r \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$ is non-empty.

Choose now $(\omega_1,\dots,\omega_r) \in \Omega^r \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$, and put $b_i = r^{-1/2}B(\omega_i)$. Then $\|b_i\| \leq 1$, $i = 1,\dots,r$, and the four inequalities in the lemma are satisfied.

LEMMA 2.6. *Let A be a unital C^*-algebra and let $U(A)$ be its unitary group. Let ξ, η be two unit vectors in a (unital) Hilbert A-bimodule*

H. Assume that for every $\gamma > 0$, there exists a finite set of operators $b_1, \dots, b_r \in A$, such that

$$\left\| \left(\sum_{i=1}^{r} b_i^* b_i - 1 \right) \xi \right\|^2 < \gamma, \quad \left\| \left(\sum_{i=1}^{r} b_i b_i^* - 1 \right) \eta \right\|^2 < \gamma$$

and

$$b_i \xi = \eta b_i, \quad b_i^* \eta = \xi b_i^*, \qquad i = 1, \dots, r.$$

Then

$$\inf_{u \in U(A)} \|u\xi - \eta u\| = 0.$$

Proof. The left and right actions can by standard techniques be extended (uniquely) to normal left and right actions of A^{**} on H. In this way H becomes a normal Hilbert A^{**}-bimodule. As in the proof of Lemma 2.4 we can consider the 2×2-matrices with elements in H as a normal Hilbert $M_2(A^{**})$-Hilbert bimodule. Let ζ be the unit vector in $M_2(H)$ given by

$$\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}$$

and put

$$P = \{x \in M_2(A^{**}) | x\zeta = \zeta x \text{ and } x^*\zeta = \zeta x^*\}.$$

Then P is clearly a von Neumann subalgebra of A^{**}. Let τ be the vector functional on P given by ζ. For $x, y \in P$,

$$(xy\zeta, \zeta) = (x\zeta y, \zeta) = (x\zeta, \zeta y^*) = (x\zeta, y^*\zeta) = (yx\zeta, \zeta),$$

so τ is a tracial state on P. Therefore the support projection e of τ is a central projection in P, and eP is a finite von Neumann algebra. It is clear that the two projections

$$1 \otimes e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad 1 \otimes e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in $M_2(A^{**})$ are contained P. We will prove that $e(1 \otimes e_{11})$ and $e(1 \otimes e_{22})$ are equivalent projections in P. Since τ is a faithful trace on eP it is sufficient to prove that for every central projection f in P, $f \leq e$, one has

$$\tau(f(1 \otimes e_{11})) = \tau(f(1 \otimes e_{22})).$$

Let $\gamma > 0$. By the assumptions there exist $b_1, \dots, b_r \in N$ such that

$$\left\| \left(\sum_{i=1}^{r} b_i^* b_i - 1 \right) \xi \right\|^2 < \gamma, \quad \left\| \left(\sum_{i=1}^{r} b_i b_i^* - 1 \right) \eta \right\|^2 < \gamma$$

and

$$b_i\xi = \eta b_i, \quad b_i^*\eta = \xi b_i^*, \qquad i = 1, \dots, r.$$

Put

$$c_i = \begin{pmatrix} 0 & 0 \\ b_i & 0 \end{pmatrix} \in M_2(A^{**}), \qquad i = 1, \dots, r.$$

One checks easily that $c_i\zeta = \zeta c_i$ and $c_i^*\zeta = \zeta c_i^*$, i.e. $c_i \in P$ for $i = 1, \dots, r$. Moreover

$$\sum_{i=1}^r c_i^*c_i = \left(\sum_{i=1}^r b_i^*b_i\right) \otimes e_{11}, \quad \sum_{i=1}^r c_ic_i^* = \left(\sum_{i=1}^r b_ib_i^*\right) \otimes e_{22}.$$

Therefore

$$\left\|\left(\sum_{i=1}^r c_i^*c_i - 1 \otimes e_{11}\right)\zeta\right\|^2 = \frac{1}{2}\left\|\left(\sum b_i^*b_i - 1\right)\xi\right\|^2 < \frac{1}{2}\gamma$$

and

$$\left\|\left(\sum_{i=1}^r c_ic_i^* - 1 \otimes e_{22}\right)\zeta\right\|^2 = \frac{1}{2}\left\|\left(\sum b_ib_i^* - 1\right)\eta\right\|^2 < \frac{1}{2}\gamma.$$

Hence for every central projection $f \in P$, $f \leq e$ we get

$$\tau\left(f\left(\sum_{i=1}^r c_i^*c_i - 1 \otimes e_{11}\right)\right) < \left(\frac{\gamma}{2}\right)^{1/2},$$

$$\tau\left(f\left(\sum_{i=1}^r c_ic_i^* - 1 \otimes e_{22}\right)\right) < \left(\frac{\gamma}{2}\right)^{1/2}.$$

However $\tau(f(\sum_{i=1}^r c_i^*c_i)) = \tau(f(\sum_{i=1}^r c_ic_i^*))$, because $\tau(f\cdot)$ is a trace on P. Hence

$$|\tau(f(1 \otimes e_{11})) - \tau(f(1 \otimes e_{22}))| < (2\gamma)^{1/2}.$$

Since $\gamma > 0$ was arbitrary, we get

$$\tau(f(1 \otimes e_{11})) = \tau(f(1 \otimes e_{22}))$$

which proves that $e(1 \otimes e_{11}) \sim e(1 \otimes e_{22})$ in P.

Let $w \in P$ be a partial isometry in P for which

$$w^*w = e(1 \otimes e_{11}), \quad ww^* = e(1 \otimes e_{22}).$$

Since $w^*w \leq 1 \otimes e_{11}$ and $ww^* \leq 1 \otimes e_{22}$, w is of the form

$$w = v \otimes e_{21} = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}$$

for some $v \in A^{**}$. Clearly v is a partial isometry, and since $w\zeta = \zeta w$, we have $v\xi = \eta v$. Moreover

$$((1 - v^*v)\xi, \xi) = 2((1 \otimes e_{11} - w^*w)\zeta, \zeta) = 2\tau((1 - e)(1 \otimes e_{11})) = 0.$$

Therefore

$$\|v\xi + \eta v\| = 2\|v\xi\| = 2\|\xi\| = 2.$$

Thus, by Kaplansky's density theorem,

$$\sup\{\|a\xi + \eta a\| | a \in A, \|a\| \leq 1\} = 2.$$

By the Russo-Dye-theorem **[20]** the unitball of A is the norm closed convex hull of $U(A)$. Hence also

$$\sup\{\|u\xi + \eta u\| | u \in U(A)\} = 2.$$

By the parallellogram identity

$$\|u\xi - \eta u\|^2 + \|u\xi + \eta u\|^2 = 4$$

for all $u \in U(A)$. Therefore

$$\inf_{u \in U(A)} \|u\xi - \eta u\| = 0.$$

Proof of Theorem 2.3. Let us first treat the case $n = 1$: Assume that Theorem 2.3 is false for $n = 1$. Then there exists an $\varepsilon_0 > 0$, such that for any $\gamma > 0$ there exists a von Neumann algebra N, a normal Hilbert N-bimodule H and two γ-related unit vectors ξ, η, such that

$$\inf_{u \in U(N)} \|u\xi - \eta u\| \geq \varepsilon_0.$$

Hence we can choose a sequence $(N_m)_{m \in \mathbf{N}}$ of von Neumann algebras, a sequence $(H_m)_{m \in \mathbf{N}}$ of normal Hilbert N_m bimodules and two sequences $(\xi_m)_{m \in \mathbf{N}}$ of unit vectors such that for each $m \in \mathbf{N}$, ξ_m and η_m are $(1/m)$-related unit vectors in H_m, and such that for all $m \in \mathbf{N}$

$$\inf_{u \in U(N_m)} \|u\xi_m - \eta_m u\| \geq \varepsilon_0.$$

Choose now a free ultrafilter ω on $\mathbf{N}$, and let H_ω be the ultraproduct of the Hilbert spaces $(H_m)_{m \in \mathbf{N}}$ along ω (cf. **[15]**), i.e. H_ω is the quotient Banach space

$$H_\omega = \mathscr{H} / I_\omega$$

where

$$\mathscr{H} = \left\{ (\xi'_m)_{m \in \mathbf{N}} | \xi'_m \in H_m, \sup_{m \in \mathbf{N}} \|\xi'_m\| < \infty \right\}$$

with norm

$$\|(\xi'_m)_{m\in\mathbf{N}}\| = \sup_{m\in\mathbf{N}} \|\xi'_m\|$$

and I_ω is the closed subspace,

$$I_\omega = \left\{(\xi'_m)_{m\in\mathbf{N}} \in \mathscr{H} \mid \lim_{m\to\omega} \|\xi'_m\| = 0\right\}.$$

The quotient $H_\omega = \mathscr{H}/I_\omega$ is a Banach space with norm

$$\|\xi'\| = \lim_{m\to\omega} \|\xi'_m\|,$$

where $(x_m)_{m\in\mathbf{N}}$ is any representing sequence for x. Moreover H_ω is a Hilbert space with inner product

$$(\xi', \eta') = \lim_{m\to\omega} (\xi'_m, \eta'_m).$$

Put $A = \bigoplus_{m=1}^{\infty} N_m$ in the von Neumann algebra sense. Then H_ω is a Hilbert A-bimodule with the following definition of left and right action:

If $x \in A$, $x = (x_m)_{m\in\mathbf{N}}$ and $\xi' \in H_\omega$ has representing sequence $(\xi'_m)_{m\in\mathbf{N}}$, then $x\xi'$ has representing sequence $(x_m\xi'_m)_{m\in\mathbf{N}}$ and $\xi' x$ has representing sequence $(\xi'_m x_m)_{m\in\mathbf{N}}$. The bimodule will in general not be normal, so therefore we will only consider A as a C^*-algebra.

Let $\gamma > 0$, and choose $r \in \mathbf{N}$, such that $12/r < \gamma$. By Lemma 2.5 we can for each $m \in \mathbf{N}$ find r operators $b_i^{(m)}, \dots, b_r^{(m)}$, such that $\|b_i^{(m)}\| \le 1$, $i = 1, \dots, r$, and

$$\left\|\left(\sum_{i=1}^{r} (b_i^{(m)})^* b_i^{(m)} - 1\right)\xi_m\right\|^2 < \frac{12}{r},$$

$$\left\|\left(\sum_{i=1}^{r} b_i^{(m)} (b_i^{(m)})^* - 1\right)\eta_m\right\|^2 < \frac{12}{r},$$

$$\sum_{i=1}^{r} \|b_i^{(m)}\xi_m - \eta_m b_i^{(m)}\|^2 < \frac{8}{m},$$

$$\sum_{i=1}^{r} \|(b_i^{(m)})^*\eta_m - \xi_m (b_i^{(m)})^*\|^2 < \frac{8}{m}.$$

Let $\xi, \eta \in H_\omega$ be the two unit vectors in H_ω with representing sequences $(\xi_m)_{m\in\mathbf{N}}$ and $(\eta_m)_{m\in\mathbf{N}}$, and let $b_1, \dots, b_r$ be the elements in A defined by the sequences $(b_i^{(m)})_{n\in\mathbf{N}}$, $i = 1, \dots, r$. Then $\|b_i\| \le 1$,

$i = 1, \dots, r$, and

$$\left\| \left(\sum_{i=1}^{r} b_i^* b_i - 1 \right) \xi \right\|^2 \le \frac{12}{r} < \gamma, \quad \left\| \left(\sum_{i=1}^{r} b_i b_i^* - 1 \right) \eta \right\|^2 \le \frac{12}{r} < \gamma,$$

$$b_i \xi = \eta b_i, \quad b_i^* \eta = \xi b_i^*, \qquad i = 1, \dots, r.$$

Hence by Lemma 2.6, there exists a unitary $u \in A$ such that

$$\|u\xi - \eta u\| < \varepsilon_0.$$

The operator u is of the form $u = (u_m)_{m\in\mathbf{N}}$, where $u_m \in U(N_m)$. Since

$$\lim_{m\to\omega} \|u_m \xi_m - \eta_m u_m\| = \|u\xi - \eta u\| < \varepsilon_0$$

we must have

$$\|u_m \xi_m - \eta_m u_m\| < \varepsilon_0$$

for some $m \in \mathbf{N}$, contradicting that

$$\inf_{u\in U(N_m)} \|u\xi_m - \eta_m u\| \ge \varepsilon_0 \quad \text{for all } m \in \mathbf{N}.$$

This proves Theorem 2.3 for $n = 1$. Let now $n \ge 2$. The Hilbert space $H^n = H \oplus \cdots \oplus H$ (n terms) is a normal Hilbert N-bimodule, where the left and right action is defined by

$$x(\xi_i)_{i=1}^n = (x\xi_i)_{i=1}^n, \qquad (\xi_i)_{i=1}^n x = (\xi_i x)_{i=1}^n$$

for $x \in \mathbf{N}$ and $(\xi_1, \dots, \xi_n) \in H^n$. Let $(\xi_1, \dots, \xi_n)$ and $(\eta_1, \dots, \eta_n)$ be two δ-related n-tuples of unit vectors in H. Then

$$\xi = \frac{1}{\sqrt{n}}(\xi_1, \dots, \xi_n), \qquad \eta = \frac{1}{\sqrt{n}}(\eta_1, \dots, \eta_n)$$

are two unit vectors in H^n. Moreover, ξ and η are δ-related, because for any set $(a_i)_{i\in 1}$ of operators in N

$$\begin{aligned} \sum_{i\in I} \|a_i\xi - \eta a_i\|^2 &= \frac{1}{n} \sum_{i\in I} \sum_{k=1}^{n} \|a_i\xi - \eta_k a_i\|^2 \\ &\le \max_{1\le k\le n} \left(\sum_{i=I} \|a_i\xi_k - \eta_k a_i\|^2 \right). \end{aligned}$$

Since Theorem 2.3 is valid for $n = 1$, we can for every $\varepsilon > 0$ choose a $\delta > 0$, such that when $\xi, \eta \in H^n$ comes from two δ-related n-tuples as above, then there exists $u \in u(N)$, such that $\|u\xi - \eta u\| < \varepsilon/\sqrt{n}$ or equivalently

$$\frac{1}{n} \sum_{i=1}^{n} \|u\xi_k - \eta_k u\|^2 < \frac{\varepsilon^2}{n}.$$

Hence

$$\|u\xi_k - \eta_k u\| < \varepsilon, \qquad k = 1, \ldots, n.$$

This completes the proof of Theorem 2.3.

3. A relative Dixmier property for factors of type III_λ. Let N be a (σ-finite) factor of type III_λ, and let φ be a normal faithful state on N for which $\sigma^\varphi_{t_0} = \mathrm{id}$ ($t_0 = -2\pi/\log\lambda$). Then the centralizer M_φ of φ has trivial relative commutant

$$M'_\varphi \cap N = \mathbf{C}I,$$

(cf. §1). Since the unitary group $U(M_\varphi)$ leaves the faithful state φ invariant, it follows from [**17**, §2, Theorem 1] that for every $x \in M$ the σ-weak closure of

$$\mathrm{conv}\{uxu^* | u \in U(M_\varphi)\}$$

contains a scalar operator. We prove below that already the norm closure of the convex set contains a scalar operator. It is not known whether the same holds if ψ is an unbounded (normal semifinite faithful) weight on M with $\sigma^\psi_{t_0} = \mathrm{id}$, i.e. ψ is a "trace généralisée" in the sense of Connes [**4**, §4.3]. By a result of Halpern, Kaftal and Weiss [**14**, Theorem 4.6 and §5] one has in this case that the norm closure of $\mathrm{conv}\{uxu^* | u \in U(M_\psi)\}$ contains a scalar operator for all $x \in M$ for which $t \to \sigma^\psi_t(x)$ is norm-continous.

THEOREM 3.1. *Let N be a factor of type III_λ, and let φ be a normal faithful state on N for which $\sigma^\varphi_{t_0} = \mathrm{id}$ ($t_0 = -2\pi/\log\lambda$). Then for every $x \in N$*

$$\varphi(x)1 \in \overline{\mathrm{conv}}\{uxu^* | u \in U(M_\varphi)\}$$

(*norm closure*).

Following [**22**] we put

$$N_n = \{x \in N | \sigma^\varphi_t(x) = \lambda^{\mathrm{int}}, t \in \mathbf{R}\}$$

and we let ε_n be the projection of norm 1 of N onto N_n given by

$$\varepsilon_n(x) = \frac{1}{t_0}\int_0^{t_0} \sigma^\varphi_t(x)\lambda^{-\mathrm{int}}\,dt.$$

Note that $N_0 = N_\varphi$ and that:

$$\begin{aligned} \varepsilon_0(1) &= 1, & \varphi \circ \varepsilon_n &= 1 \quad \text{for } n \neq 0, \\ \varphi \circ \varepsilon_0 &= \varphi, & \varphi \circ \varepsilon_n &= 0 \quad \text{for } n \neq 0. \end{aligned}$$

Every $x \in N$ has a formal expansion

$$x \sim \sum_{n=-\infty}^{\infty} \varepsilon_n(x).$$

The sum is in general not σ-strongly convergent. By standard Fourier analysis one gets that for $x, y \in N$,

$$x = y \Leftrightarrow \varepsilon_n(x) = \varepsilon_n(y) \quad \text{for all } n \in \mathbf{Z}.$$

We prove first

LEMMA 3.2. *If $x \in N_n$, $n \neq 0$, then*

$$0 \in \overline{\operatorname{conv}}\{uxu^* | u \in U(M_\varphi)\}$$

(*norm closure*).

Proof. If $x \in N$ then $x^* \in N_{-n}$, so it is sufficient to consider the case $x \in N_n$, $n > 0$: let $x = u|x|$ be the polar decomposition of x. Since $x^*x \in N_0 = N_\varphi$, and since

$$u = \lim_{\varepsilon \to 0} x(x^*x + \varepsilon)^{-1/2} \quad (\sigma\text{-strongly}),$$

it follows that $u \in N_n$. Hence by [22, Lemma 1.6],

$$\varphi(uu^*) = \lambda^n \varphi(u^*u) \leq \lambda^n.$$

Choose an integer $m \in N$, such that

$$1/m < 1 - \lambda^n.$$

Since N_φ is a II_1-factor with trace φ, we can choose projection $q \leq 1 - uu^*$, such that

$$\varphi(q) = 1/m.$$

Note that

$$qx = q(1 - uu^*)u|x| = 0.$$

By comparison theory there exists m orthogonal projections $q_1, \ldots, q_m$ in N_φ with sum 1, such that $q = q_m$ and

$$\varphi(q_i) = 1/m, \qquad i = 1, \ldots, m.$$

Moreover we can choose a unitary $u \in M_\varphi$, such that

$$\begin{aligned} uq_iu^* &= q_{i+1}, \qquad i = 1, \ldots, m-1, \\ uq_mu^* &= q_1. \end{aligned}$$

Put $x_j = u^j x(u^j)^*$, $j = 1, \dots, m$. Since $q_m x = qx = 0$, we have

$$q_j x_j = (u^j q_m (u^j)^*)(u^j x(u^j)^*) = 0$$

for $j = 1, \dots, m$. Moreover $\|x_j\| = \|x\|$.

Let $\xi \in H$ (the Hilbert space on which N acts), then

$$\begin{aligned}\sum_{j=1}^m \|(1-q_j)\xi\|^2 &= \sum_{i=1}^m ((1-q_j)\xi, \xi) \\ &= m\|\xi\|^2 - \sum_{i=1}^m (q_j\xi, \xi) = (m-1)\|\xi\|^2.\end{aligned}$$

Since $x_j^* q_j = 0$, $j = 1, \dots, m$, it follows that

$$\begin{aligned}\left\|\sum_{j=1}^m x_j^*\xi\right\| &= \left\|\sum_{j=1}^m x_j^*(1-q_j)\xi\right\| \\ &\leq \left(\sum_j^m \|x_j\|^2\right)^{1/2} \left(\sum_{j=1}^m \|(1-q_j)\xi\|^2\right)^{1/2} \\ &\leq m^{1/2}\|x\|(m-1)^{1/2}\|\xi\|,\end{aligned}$$

which shows that

$$\left\|\sum_{j=1}^m x_j\right\| = \left\|\sum_{j=1}^m x_j^*\right\| \leq m^{1/2}(m-1)^{1/2}\|x\|.$$

Put

$$y = \frac{1}{m}\sum_{j=1}^m u_j x u_j^* = \frac{1}{m}\sum_{j=1}^m x_j.$$

Then

$$\|y\| \leq \left(1 - \frac{1}{m}\right)^{1/2} x.$$

Since $u_j \in M_\varphi$, $y \in N_n$, so we can iterate the argument and get that for every $l \in \mathbf{N}$, there exist

$$\alpha_1, \dots, \alpha_l \in \operatorname{conv}\{\operatorname{ad}(u) | u \in U(M_\varphi)\}$$

such that

$$\|\alpha_l \circ \alpha_{l-1} \circ \cdots \circ \alpha_1(x)\| \leq \left(1 - \frac{1}{m}\right)^{l/2} \|x\|.$$

Since $(1 - 1/m)^{l/2} \to 0$ for $l \to \infty$, Lemma 3.2 is proved.

LEMMA 3.3. *Let $n > 0$ be an integer for which*

$$4\lambda^n \le 1 - \lambda.$$

If $x \in N$ and $\varepsilon_k(x) = 0$ for $|k| < n$, then

$$0 \in \overline{\operatorname{conv}}\{uxu^* | u \in M_\varphi\}$$

(norm closure).

Proof. Let r_k (resp. s_k) be the support projection (resp. range projection) of $\varepsilon_k(x)$, $k \in \mathbf{Z}$. By the proof of Lemma 3.2

$$\varphi(r_k) \le \lambda^k, \qquad k > 0,$$

and since $\varepsilon_{-k}(x)^* = \varepsilon(x^*)$, we have also

$$\varphi(s_{-k}) \le \lambda^k, \qquad k > 0.$$

Moreover, $r_k, s_k \in M_\varphi$ for all $k \in \mathbf{Z}$. Put

$$q = \left(\bigvee_{k=n}^{\infty} r_k\right) \vee \left(\bigvee_{k=n}^{\infty} s_k\right).$$

Since φ is a trace on M_φ,

$$\varphi(q) \le \sum_{k=n}^{\infty} (\varphi(r_k) + \varphi(s_k)) \le \frac{2\lambda^n}{1-\lambda} \le \frac{1}{2}.$$

Hence we can choose a projection $p \in M_\varphi$, $p \le 1 - q$, such that

$$\varphi(p) = 1/2.$$

Clearly

$$\begin{aligned} p\varepsilon_k(x) &= 0, \qquad k \ge n, \\ \varepsilon_k(x)p &= 0, \qquad k \le -n. \end{aligned}$$

Since $\varepsilon_k(x) = 0$ for $|k| < n$, we get

$$p\varepsilon_k(x)p = 0 \quad \text{for all } k \in \mathbf{Z}.$$

Moreover, since $p \in M_\varphi = N_0$,

$$\varepsilon_k(pxp) = p\varepsilon_k(x)p \quad \text{for all } k \in \mathbf{Z},$$

which implies that $pxp = 0$.

Since $\varphi(p) = \varphi(1-p) = \frac{1}{2}$, there exists a selfadjoint unitary $u \in M_\varphi$, such that

$$upu^* = 1 - p.$$

Put

$$y = \tfrac{1}{2}(x + uxu^*).$$

Then

$$\|pyp\| = \tfrac{1}{2}\|puxu^*p\| \le \tfrac{1}{2}\|x\|,$$

and since $(1-p)u = up$, we get also

$$\|(1-p)y(1-p)\| = \tfrac{1}{2}\|(1-p)x(1-p)\| \le \tfrac{1}{2}\|x\|.$$

Put next $v = 2p - 1$, and put

$$z = \tfrac{1}{2}(y + vyv^*) = pyp + (1-p)y(1-p).$$

Then

$$\|z\| = \max\{\|pyp\|, \|(1-p)y(1-p)\|\} \le \tfrac{1}{2}\|x\|$$

and

$$z \in \mathrm{conv}\{uxu^* | u \in U(M_\varphi)\}.$$

It is clear that

$$\varepsilon_k(z) = 0 \quad \text{for } |k| < n,$$

so by iterating the argument as in Lemma 3.2, we get

$$0 \in \overline{\mathrm{conv}}\{uxu^* | u \in U(M_\varphi)\}.$$

Proof of Theorem 3.1. Let $x \in N$, and let n be as in Lemma 3.3. Then

$$x = \varepsilon_0(x) + \sum_{0<|k|<n} \varepsilon_k(x) + x'$$

where

$$\varepsilon_k(x') = 0 \quad \text{for } |k| < n.$$

Let $\varepsilon > 0$ and put $\sigma = \varepsilon/2n$. Since $\varepsilon_0(x) \in N_0 = M_\varphi$, and since $\varphi(\varepsilon_0(x)) = \varphi(x)$, it follows by the Dixmier approximation theorem for II_1-factors (cf. [**9**, Chapter III, §5]) that there exists

$$\alpha_0 \in \mathrm{conv}\{\mathrm{ad}(u) | u \in M_\varphi\}$$

such that

$$\|\alpha_0(\varepsilon_0(x)) - \varphi(x)1\| < \sigma.$$

Using that every

$$\alpha \in \mathrm{conv}\{\mathrm{ad}(u) | u \in M_\varphi\}$$

commutes with every ε_k, $k \in \mathbf{Z}$, we can by Lemma 3.2 find

$$\alpha_1, \dots, \alpha_{2n-2} \in \mathrm{conv}\{\mathrm{ad}(u) | u \in M_\varphi\}$$

such that

$$\begin{aligned}
&\|\alpha_1\alpha_0\varepsilon_1(x)\| < \sigma,\\
&\|\alpha_2\alpha_1\alpha_0\varepsilon_2(x)\| < \sigma,\\
&\vdots\\
&\|\alpha_{n-1}\alpha_{n-2}\cdots\alpha_0(\varepsilon_{n-1}(x))\| < \sigma,\\
&\|\alpha_n\alpha_{n-1}\cdots\alpha_0(\varepsilon_{-1}(x))\| < \sigma,\\
&\|\alpha_{n+1}\alpha_n\cdots\alpha_0(\varepsilon_{-2}(x))\| < \sigma,\\
&\vdots\\
&\|\alpha_{2n-2}\alpha_{2n-3}\cdots\alpha_0(\varepsilon_{-n+1}(x))\| < \sigma.
\end{aligned}$$

Since for $|k| < n$,

$$\begin{aligned}
&\varepsilon_k(\alpha_{2n-2}\alpha_{2n-3}\cdots\alpha_0(x'))\\
&\qquad = \alpha_{2n-2}\alpha_{2n-3}\cdots\alpha_0(\varepsilon_k(x')) = 0
\end{aligned}$$

we can by Lemma 3.3 find

$$\alpha_{2n-1} \in \operatorname{conv}\{\operatorname{ad}(u) | u \in M_\varphi\}$$

such that

$$\|\alpha_{2n-1}\alpha_{2n-2}\cdots\alpha_0(x')\| < \sigma.$$

Put $\beta = \alpha_{2n-1}\alpha_{2n-2}\cdots\alpha_0$. Then

$$\begin{aligned}
&\|\beta(\varepsilon_0(x)) - \varphi(x)1\| < \sigma,\\
&\|\beta(\varepsilon_k(x))\| < \sigma \quad \text{for } 0 < |k| < n,\\
&\|\beta(x')\| < \sigma.
\end{aligned}$$

Hence

$$\|\beta(x) - \varphi(x)1\| < 2n\sigma = \varepsilon.$$

This completes the proof of Theorem 3.1.

It is clear that by repeated use of Theorem 3.1one gets the following "Relative Dixmier averaging process" (cf. [**9**, Chapter III, §3, proof of Lemma 5]):

COROLLARY 3.4. *Let N and φ be as in Theorem* 3.1. *Then for every finite set $x_1, \ldots, x_n$ of operators in N and every $\varepsilon > 0$ there exists a convex combination α of inner automorphisms implemented by unitaries* *from M_φ, such that*

$$\|\alpha(x_k) - \varphi(x_k)1\| < \varepsilon$$

for $k = 1, \ldots, n$.

4. A result on σ^φ-invariant completely positive maps. Let N be a von Neumann algebra with a faithful normal state φ, and let $F \subseteq N$ be a von Neumann subalgebra for which

$$\sigma_t^\varphi(F) = F, \qquad t \in \mathbf{R}.$$

We say that a linear map $T: F \to N$ is *σ^φ-invariant* if

$$T(\sigma_t^\varphi(x)) = \sigma_t^\varphi(T(x)), \qquad x \in F.$$

Note that if F is a finite dimensional subfactor of N, then $N \cong F \otimes F^c$, where F^c is the relative commutant of F in N. In this case the condition

(i)

$$\sigma_t^\varphi(F) = F, \qquad t \in \mathbf{R},$$

is equivalent to

(ii)

$$\varphi = \varphi|_F \otimes \varphi|_{F^c}.$$

Indeed, the implication (ii) $\Rightarrow$ (i) is obvious, and if F satisfies (i), then by **[21]** there is a (unique) normal faithful conditional expectation $\varepsilon: N \to F$ for which

$$\varphi \circ \varepsilon = \varphi.$$

For $x \in F$ and $y \in F^c$,

$$\varphi(xy) = \varphi \circ \varepsilon(xy) = \varphi(x\varepsilon(y)).$$

But $\varepsilon(y)$ must commute with every element in F, and so $\varepsilon(y) = \lambda 1$ for some $\lambda \in \mathbf{C}$. Moreover

$$\lambda = \varphi \circ \varepsilon(y) = \varphi(y).$$

Hence

$$\varphi(xy) = \varphi(x)\varphi(y)$$

which shows that (i) $\Rightarrow$ (ii).

The main result of this section is the following generalization of **[11**, Proposition 5.2] to factors of type III_λ:

THEOREM 4.1. *Let N be a factor of type* III_λ, *let φ be a normal faithful state on N, such that $\sigma_{t_0}^\varphi = \mathrm{id}$ $(t_0 = -2\pi/\log\lambda)$. Let F be a finite dimensional subfactor of N for which*

$$\varphi = \varphi|_F \otimes \varphi|_{F^c}$$

and let $T\colon F \to N$ be a σ^φ-invariant completely positive map, satisfying

$$T(1) = 1 \quad \textit{and} \quad \varphi \circ T = \varphi|_F.$$

Then for every $\delta > 0$ there exists a sequence $(a_i)_{i=1}^\infty$ of operators in the centralizer M_φ of φ, such that

$$\sum_{i=1}^\infty a_i^* a_i = \sum_{i=1}^\infty a_i a_i^* = 1$$

and

$$\left\| T(x) - \sum_{i=1}^\infty a_i^* x a_i \right\| \le \delta \|x\| \quad \textit{for all } x \in F.$$

LEMMA 4.2. *Let N be a III_λ-factor and φ a normal faithful state on N for which $\sigma_{t_0}^\varphi = \mathrm{id}$ $(t_0 = -2\pi/\log\lambda)$. If e, f are two projections in the centralizer M_φ, such that*

$$\varphi(f) = \lambda^n \varphi(e)$$

for some $n \in \mathbf{Z}$, then there is a partial isometry $u \in N_n$, i.e.

$$\sigma_t^\varphi(u) = \lambda^{int} u, \qquad t \in \mathbf{R},$$

such that $e = u^*u$, $f = uu^*$.

Proof. Let ω be the functional on M_2 given by

$$\omega = \mathrm{Tr}(h\cdot), \quad \text{where } h = \begin{pmatrix} \lambda^n & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $(e_{rs})_{j=1,2}$ be the matrix units in M_2. Then $\sigma_{t_0}^\omega = \mathrm{id}$. Let $\chi = \varphi \otimes \omega$. Since $\sigma_{t_0}^\chi = \mathrm{id}$ and since $N \otimes M_2 \cong N$ is of type III_λ, the centralizer M_χ is a II_1-factor. Put

$$\tilde{e} = e \otimes e_{11}, \quad \tilde{f} = f \otimes e_{22}.$$

Then

$$\tilde{e}, \tilde{f} \in M_\chi, \quad \text{and} \quad \chi(\tilde{e}) = \lambda^n \tau(e) = \tau(f) = \chi(\tilde{f}).$$

Since χ is a scalar multiple of the unique tracial state on M_χ, we have $\tilde{e} \sim \tilde{f}$ in M_χ. Hence, there exists $v \in M_\chi$, such that

$$\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = v^* v, \quad \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} = vv^*.$$

Since $v^*v \le 1 \otimes e_{11}$ and $vv^* \le 1 \otimes e_{22}$, v is of the form $v = u \otimes e_{21}$ for some $u \in M$. Clearly

$$u^*u = e \quad \text{and} \quad uu^* = f.$$

Moreover, since $v \in M_\chi$,

$$\sigma_t^\varphi(u) \otimes \sigma_t^\omega(e_{21}) = u \otimes e_{21}, \qquad t \in \mathbf{R}.$$

But $\sigma_t^\omega(e_{21}) = h^{it} e_{21} h^{-it} = \lambda^{-\,\mathrm{int}} e_{21}$. Hence

$$\sigma_t^\varphi(u) = \lambda^{\mathrm{int}} u, \qquad t \in \mathbf{R}.$$

LEMMA 4.3. *Let N be a factor of type* III_λ, *let φ be a normal faithful state on N such that $\sigma_{t_0}^\varphi = \mathrm{id}$. For each $n \in \mathbf{Z}$ there exists a finite set $v_1, \dots, v_p$ partial isometries in*

$$N_n = \{x \in N \,|\, \sigma_t^\varphi(x) = \lambda^{\mathrm{int}} x, t \in \mathbf{R}\}$$

such that

$$\sum_{i=1}^{p} v_i^* v_i = 1.$$

Proof. The case $n = 0$ is trivial (take $p = 1$ and $v_1 = 1$). Assume next $n > 0$. Since M_φ is a II_1-factor with trace φ, we can choose a projection $f \in M_\varphi$, such that $\varphi(f) = \lambda^n$. By Lemma 4.2 there exists an isometry $v \in N_n$, for which $v^*v = 1$ and $vv^* = f$, so $p = 1$ and $v_1 = v$ can be used. Let now $n < 0$. Then $\lambda^n > 1$. Let q (resp. r) be the integer part (resp. fractional part) of λ^n:

$$\lambda^n = q + r, \quad q \in \mathbf{N}, \quad 0 \le r < 1.$$

Choose q orthogonal projections $e_1, \dots, e_q$ in M_φ with $\varphi(e_j) = \lambda^{-n}$, and put

$$e_{q+1} = 1 - \sum_{j=1}^{q} e_j.$$

Then $\varphi(e_{q+1}) = \lambda^{-n} r$. Let $f \in M_\varphi$ be a projection for which $\varphi(e) = r$. By Lemma 4.2 there exist partial isometries $v_1, \dots, v_{q+1} \in N_n$, such that

$$v_j^* v_j = e_j, \qquad j = 1, \dots, q+1$$

and

$$v_j v_j^* = \begin{cases} 1, & j = 1, \dots, q, \\ f, & j = q+1. \end{cases}$$

Clearly

$$\sum_{j=1}^{q+1} v_j^* v = 1.$$

LEMMA 4.4. *Let N be a factor of type* III_λ, *let φ be a normal faithful state on N, for which $\sigma_{t_0}^{\varphi} = \mathrm{id}$. Let $F \subseteq N$ be a finite dimensional subfactor, such that*

$$\varphi = \varphi|_F \otimes \varphi|_{F^c}$$

and let $T\colon F \to N$ be a σ^{φ}-invariant completely positive map. Then there exists a finite set $a_1, \dots, a_l \in M_\varphi$, such that

$$T(x) = \sum_{j=1}^{l} a_j^* x a_j, \qquad x \in M_\varphi.$$

Proof. Let h be the Radon-Nikodym derivative of $\varphi = \varphi|_F$ with respect to the trace Tr on F. We can choose a system $(e_{rs})_{r,s=1,\dots,m}$ of matrix units for F, such that

$$h = \sum_{r=1}^{m} \lambda_r e_{rr}, \qquad \lambda_1, \dots, \lambda_m \in \mathbf{R}_+.$$

Since $\sigma_{t_0}^{\varphi} = \mathrm{id}$, it follows that

$$\lambda_r/\lambda_s \in \{\lambda^n | n \in \mathbf{Z}\}.$$

Let $n_r \in \mathbf{Z}$ be the integer for which

$$\lambda_r/\lambda_1 = \lambda^{n_r}.$$

Note that for $r, s \in \{1, \dots, m\}$,

$$\sigma_t^{\varphi}(e_{rs}) = \sigma_t^{\psi}(e_{rs}) = h^{it} e_{rs} h^{-it} = \lambda^{it(n_r - n_s)} e_{rs}.$$

Since T is completely positive, the operator

$$a = \sum_{r,s=1}^{m} T(e_{rs}) \otimes e_{rs}$$

in $N \otimes F$ is positive (cf. [3, Lemma 2.1]). Let $b = a^{1/2}$. Then b is of the form

$$b = \sum_{r,s=1}^{m} b_{rs} \otimes e_{rs}, \qquad b_{rs} \in N$$

and

$$T(e_{rs}) = \sum_{k=1}^{m} b_{kr}^* b_{ks}, \qquad r, s = 1, \dots, m.$$

Put

$$c_{kl} = \sum_{r=1}^{m} e_{rl} b_{kr}, \qquad k, l = 1, \dots, m.$$

Then a simple calculation (cf. [**11**, proof of Proposition 2.1]) shows that

$$\sum_{k,l}^{m} c_{kl}^* e_{rs} c_{kl} = T(e_{rs}).$$

Hence,

$$T(y) = \sum_{k,l=1}^{m} c_{kl}^* y c_{kl}, \qquad y \in F.$$

Let ω be the positive functional on F given by

$$\omega(y) = \mathrm{Tr}(h^{-1}y), \qquad y \in F.$$

Then

$$\sigma_t^{\omega}(e_{rs}) = \lambda^{-it(n_r - n_s)} e_{rs}.$$

Since T is σ^{φ}-invariant,

$$\sigma_t^{\varphi}(T(e_{rs})) = T(\sigma_t^{\varphi}(e_{rs})) = \lambda^{it(n_r - n_s)} e_{rs}.$$

Hence

$$\sigma_t^{\varphi\otimes\omega}(T(e_{rs}) \otimes e_{rs}) = T(e_{rs}) \otimes e_{rs},$$

and therefore $a \in M_{\varphi\otimes\omega}$. Thus also $b = a^{1/2} \in M_{\varphi\otimes\omega}$, which implies that

$$\sigma_t^{\varphi}(b_{rs}) = \lambda^{it(n_r - n_s)}, \qquad r, s = 1, \dots, m.$$

Therefore

$$\sigma_t^{\varphi}(c_{kl}) = \lambda^{it(n_k - n_l)}, \qquad k, l = 1, \dots, m.$$

Hence we have shown that there exist $d = m^2$ operators $c_1, \dots, c_d$ in N, such that

$$T(y) = \sum_{j=1}^{d} c_j^* y c_j, \qquad y \in F,$$

and integers $n_1, \dots, n_d$, such that

$$\sigma_t^{\varphi}(c_j) = \lambda^{in_j t} c_j, \qquad t \in \mathbf{R}, j = 1, \dots, d.$$

Since $N = F \otimes F^c \cong M_m \otimes F^c$, F^c is also a factor of type III_λ. Moreover, $\sigma_t^{\varphi|_{F^c}}$ is just the restriction of σ_t^{φ} to F^c. Particularly $\sigma_{t_0}^{\varphi|_{F^c}} = \mathrm{id}_{F^c}$. Therefore we can apply Lemma 4.3 to the pair $(F^c, \varphi|_{F^c})$ and obtain:

For $j = 1, \dots, d$ there exists a finite set $v_{j,1}, \dots, v_{j,p(j)}$ of operators in F^c for which

$$\sigma_t^{\varphi}(v_{jl}) = \lambda^{-in_j t}$$

and

$$\sum_{l=1}^{p(j)} v_{jl}^* v_{jl} = 1.$$

Put

$$a_{jl} = v_{jl} c_j, \qquad j = 1, \dots, d, l = 1, \dots, p(j).$$

Then $\sigma_t^\varphi(a_{jl}) = a_{jl}$ for all j and l. Moreover, since $v_{jl} \in F^c$, we get for $x \in F$:

$$\begin{aligned} T(x) &= \sum_{j=1}^{d} c_j^* x c_j = \sum_{j=1}^{d} c_j^* \left(\sum_{l=1}^{p(j)} v_{jl}^* x v_{jl} \right) c_j \\ &= \sum_{j=1}^{d} \left(\sum_{l=1}^{p(j)} a_{jl}^* x a_{jl} \right). \end{aligned}$$

This proves Lemma 4.4.

LEMMA 4.5. *Let N, φ and F be as in Lemma* 4.4 *and let ε be the* (*unique*) *φ-invariant conditional expectation of N onto F. Then for every $a \in N$*

$$\varepsilon(a) \in \overline{\operatorname{conv}}\{uau^* | u \in U(F^c \cap M_\varphi)\}$$

(*norm closure*).

Proof. Using that $\varphi = \varphi|_F \otimes \varphi|_{F^c}$ it is easily seen that

$$\varepsilon(xy) = x\varphi(y), \qquad x \in F, y \in F^c.$$

Let $(e_{ij})_{i,j=1,\dots,m}$ be a system of matrix units for F. Then

$$a = \sum_{i,j=1}^{m} e_{ij} a_{ij}$$

where $a_{ij} \in F^c$. Hence

$$\varepsilon(a) = \sum_{i,j=1}^{m} \varphi(a_{ij}) e_{ij}.$$

Let $\delta > 0$. By Corollary 3.4 there exists a convex combination α of inner automorphisms of F^c given by unitaries in $F^c \cap M_\varphi|_{F^c} = F^c \cap M_\varphi$ such that

$$\|\alpha(a_{ij}) - \varphi(a_{ij})1\| < \delta/m^2$$

for $i, j = 1, \dots, m$. Let $\beta = \mathrm{id}_F \otimes \alpha$. Then

$$\beta \in \operatorname{conv}\{\mathrm{ad}_N(u) | u \in U(F^c)\}$$

and

$$\beta(a) = \sum_{i,j=1}^{m} e_{ij}\alpha(a_{ij}).$$

Hence

$$\|\beta(a) - \varepsilon(a)\| < \sum_{i,j}^{m} \|\alpha(a_{ij}) - \varphi(a_{ij})1\| < \delta.$$

This proves Lemma 4.5.

Proof of Theorem 4.1. At this stage we can almost copy the proof of [**11**, Proposition 5.2]:

By Lemma 4.4 there exists a finite set $b_1, \dots, b_d$ of operators in M_φ, such that

$$T(x) = \sum_{i=1}^{d} b_i^* x b_i, \qquad x \in F.$$

Particularly

$$\sum_{i=1}^{d} b_i^* b_i = T(1) = 1.$$

Let $\varepsilon\colon N \to F$ be as in Lemma 4.5. Since $\varphi \circ T = \varphi|_F$, we get for $x \in F$:

$$\begin{aligned}\varphi\left(x\varepsilon\left(\sum_{i=1}^{d} b_i b_i^*\right)\right) &= \varphi\circ\varepsilon\left(x\left(\sum_{i=1}^{d} b_i b_i^*\right)\right)\\ &= \varphi\left(x\left(\sum_{i=1}^{d} b_i b_i^*\right)\right) = \varphi\left(\sum_{i=1}^{d} b_i^* x b_i\right)\\ &= \varphi\circ T(x) = \varphi(x).\end{aligned}$$

Since $\varepsilon(\sum_{i=1}^{d} b_i b_i^*) \in F$, and since φ is faithful, this equality implies that

$$\varepsilon\left(\sum_{i=1}^{d} b_i b_i^*\right) = 1.$$

By Lemma 4.5 there exists a convex combination α of inner automorphisms

$$\alpha = \sum_{j=1}^{r} \lambda_j \operatorname{ad}(u_j),$$

where $u_j \in U(F^c \cap M_\varphi)$, such that

$$\left\| \alpha\left(\sum_{i=1}^{d} b_i b_i^*\right) - 1 \right\| < \delta/2.$$

Put $b_{ij} = \lambda_j^{1/2} u_j b_i$, $i = 1, \ldots, d$, $j = 1, \ldots, r$.
Then as in [**11**, p. 194]

$$\sum_{i,j} b_{ij}^* b_{ij} = 1, \quad \left\| \sum_{i,j} b_{ij} b_{ij}^* - 1 \right\| < \delta/2$$

and

$$T(x) = \sum_{i,j} b_{ij}^* x b_{ij}, \qquad x \in F.$$

Let us reindex the b_{ij}-operators to

$$b_1, \ldots, b_p, \quad \text{where } p = dr.$$

Put next

$$a_i = (1 - \delta/2)^{1/2} b_i.$$

Then

$$\sum_{i=1}^{p} a_i^* a_i \le 1 - \delta/2 \quad \text{and} \quad \sum_{i=1}^{p} a_i^* a_i \le (1 - \delta/2)(1 + \delta/2) \le 1.$$

Since $a_i \in M_\varphi$ which is a II_1-factor we can by [**11**, Lemma 5.1] find operators $(a_i)_{i=p+1}^{\infty}$ in M_φ, such that

$$\sum_{i=1}^{\infty} a_i^* a_i = \sum_{i=1}^{\infty} a_i a_i^* = 1.$$

Since $\sum_{i=p+1}^{\infty} a_i^* a_i = \delta/2$, we get as in [**11**, p. 195] that

$$\left\| T(x) - \sum_{i=1}^{\infty} a_i^* x a_i \right\| \le \delta \|x\|, \qquad x \in F.$$

For the applications of Theorem 4.1 in §6 we shall need the following:

PROPOSITION 4.6. *Let N be a factor of type* III_λ, $0 < \lambda < 1$, *and let φ be a normal faithful state on N for which $\sigma_{t_0}^{\varphi} = \mathrm{id}$. Let $m \in \mathbf{N}$ and let $\psi = \mathrm{Tr}(h\cdot)$ be a normal faithful state on M_m for which $\sigma_{t_0}^{\psi} = \mathrm{id}$. Then there exists an isomorphism α of M_m onto a subfactor F of N, such that*

$$\varphi = \varphi|_F \otimes \varphi|_{F^c} \quad \text{and} \quad \psi = \varphi \circ \alpha.$$

Proof. We can assume that h is a diagonal matrix

$$h = \sum_{j=1}^{m} \lambda_j e_{jj}, \qquad \lambda_1, \ldots, \lambda_m \in \mathbf{R}_+.$$

The condition $\sigma_{t_0}^{\psi} = \mathrm{id}$ implies that $\lambda_i/\lambda_j \in \{\lambda^n | n \in \mathbf{Z}\}$. Clearly

$$\sum_{j=1}^{m} \lambda_j = \psi(1) = 1,$$

and since M_φ is a II_1-factor we can choose orthogonal projections $f_1, \ldots, f_m \in M_\varphi$ with sum 1, such that

$$\varphi(f_j) = \lambda_j, \qquad j = 1, \ldots, m.$$

Moreover, by Lemma 4.2 there exist partial isometries $v_1, \ldots, v_n \in M$, such that

$$v_j^* v_j = f_1, \quad v_j v_j^* = f_j \quad \text{and} \quad \sigma_t^{\varphi}(v_j) = \lambda^{in_j t} v_j, \qquad t \in \mathbf{R},$$

where $n_j \in \mathbf{Z}$ is given by $\lambda^{n_j} = \lambda_j/\lambda_1$. Put now

$$f_{rs} = v_r v_s^*, \qquad r, s = 1, \ldots, m.$$

Then $\{f_{rs} | r, s = 1, \ldots, n\}$ form a system of matrix units, and

$$\sum_{r=1}^{m} f_{rr} = \sum_{r=1}^{m} f_r = 1.$$

Moreover,

$$\sigma_t^{\varphi}(f_{rs}) = \lambda^{i(n_r - n_s)t} f_{rs}, \qquad t \in \mathbf{R},$$

so σ_t^{φ} leaves the factor

$$F = \operatorname{span}\{f_{rs} | r, s = 1, \ldots, m\}$$

globally invariant. Hence by the remarks in the beginning of §4

$$\varphi = \varphi|_F \otimes \varphi|_{F^c}.$$

Since $\sigma_t^{\varphi}(v_j) = \lambda^{in_j t} v_j$, we have by [22, Lemma 1.6] that for $r \neq s$

$$\varphi(f_{rs}) = \varphi(v_r v_s^*) = \lambda^{n_r} \varphi(v_s^* v_r).$$

Thus $\varphi(f_{rs}) = 0$, because v_r and v_s have orthogonal range projections. If $r = s$

$$\varphi(f_{rs}) = \varphi(f_r) = \lambda_r.$$

This shows that if $\alpha\colon M_m \to F$ is the isomorphism given by $\alpha(e_{rs}) = f_{rs}$, then

$$\varphi \circ \alpha = \operatorname{Tr}(h \cdot) = \psi.$$

5. Completely positive factorizations through matrix algebras. In [3, pp. 75–76] Choi and Effris proved that a von Neumann algebra N is semidiscrete (= injective by [24]) if, and only if, the identity map on N has an approximate factorization through full matrix algebras in the sense that there exists a net (m_α) of integers, and two nets of σ-weakly completely positive maps

$$S_\alpha : N \to M_m, \quad T_\alpha : M_m \to N,$$

such that $S_\alpha(1) = 1$, $T_\alpha(1) = 1$ and $T_\alpha \circ S_\alpha$ converges pointwise σ-weakly to the identity map on N. In this section we shall show that in case of an injective factor N of type III_λ, the approximate factorization can be chosen in a special form, which takes the modular automorphism group of a fixed state on N into account. For any faithful state φ on a von Neumann algebra N, we put

$$\|a\|_\varphi = \varphi(a^*a)^{1/2}, \qquad a \in N.$$

THEOREM 5.1. *Let N be an injective factor of type* III_λ, $0 < \lambda < 1$ *with separable predual and let φ be a normal faithful state on N for which $\sigma_{t_0}^\varphi = \mathrm{id}$ ($t_0 = -2\pi/\log\lambda$). Let φ_λ be the state on the 2×2-matrices M_2 given by*

$$\varphi_\lambda \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{1}{1+\lambda}(\lambda x_{11} + x_{22})$$

and put $\psi_m = \varphi_\lambda \otimes \cdots \otimes \varphi_\lambda$ (m times). Then for every finite set $x_1, \ldots, x_n$ of operators in N and every $\varepsilon > 0$, there exists $m \in \mathbf{N}$ and completely positive maps,

$$S : N \to M_{2^m}, \quad T : M_{2^m} \to N,$$

such that

$$\begin{aligned} S(1) &= 1, & T(1) &= 1, \\ \psi_m \circ S &= \varphi, & \varphi \circ T &= \psi_m, \\ \sigma_t^{\psi_m} \circ S &= S \circ \sigma_t^\varphi, & \sigma_t^\varphi \circ T &= T \circ \sigma_t^{\psi_m} \end{aligned}$$

for $t \in \mathbf{R}$, and

$$\|T \circ S(x_k) - x_k\|_\varphi < \varepsilon, \qquad k = 1, \ldots, n.$$

Let (N, φ) be as in Theorem 5.1. As in the preceding sections

$$N_n = \{x \in N | \sigma_t^\varphi(x) = \lambda^{\mathrm{int}} x, t \in \mathbf{R}\}$$

and $\varepsilon_n : N \to N_n$ is given by

$$\varepsilon_n(x) = \frac{1}{t_0} \int_0^{t_0} \lambda^{-\mathrm{int}} \sigma_t^\varphi(x)\, dt.$$

LEMMA 5.2. *Let (N, φ) be as in Theorem* 5.1.
For $p \in \mathbf{N}$, put

$$\gamma_p(x) = \sum_{n=-(p-1)}^{p-1} \left(1 - \frac{|n|}{p}\right) \varepsilon_n(x), \qquad x \in \mathbf{R}.$$

Then γ_p is a completely positive map on N,

$$\gamma_p(1) = 1, \quad \varphi \circ \gamma_p = \varphi$$

and

$$\lim_{p \to \infty} \|\gamma_p(x) - x\|_\varphi = 0 \quad \forall x \in M.$$

Proof. It is easily seen that $\varepsilon_0(1) = 1$, $\varphi \circ \varepsilon_0 = \varphi$ and for $n \neq 0$, $\varepsilon_n(1) = 0$, $\varphi \circ \varepsilon_n = 0$. Hence

$$\gamma_p(1) = 1 \quad \text{and} \quad \varphi \circ \gamma_p = \varphi.$$

Put

$$g_p(u) = \sum_{|n|<p} \left(1 - \frac{|n|}{p}\right) e^{inu}, \qquad p \in \mathbf{N}, u \in \mathbf{R}.$$

Then g_p is the Fejér kernel from the theory of Fourier series (cf. [**10**, p. 79]):

$$g_p(u) = \frac{1}{p} \frac{\sin^2(pu/2)}{\sin^2(u/2)}, \qquad u \notin 2\pi\mathbf{Z}.$$

Note that $g_p(u) \geq 0$ for all $u \in \mathbf{R}$, g is periodic with period 2π, and

$$\frac{1}{2\pi} \int_0^{2\pi} g_p(u)\, du = 1.$$

Since $\lambda = e^{-t_0}$,

$$\sum_{|n| \leq p-1} \left(1 - \frac{|n|}{p}\right) \lambda^{-\mathrm{int}} = g_p(2\pi t/t_0).$$

Hence

$$\gamma_p(x) = \frac{1}{t_0} \int_0^{t_0} g_p(2\pi t/t_0) \sigma_t^\varphi(x)\, dt, \qquad x \in N.$$

The complete positivity of γ_p follows now from the positivity of the function g_p. For $x \in N$

$$\begin{aligned}
\|\gamma_p(x) - x\|_\varphi &= \left\| \frac{1}{t_0} \int_0^{t_0} g_p(2\pi t/t_0)(\sigma_t^\varphi(x) - x)\, dt \right\|_\varphi \\
&\leq \frac{1}{t_0} \int_0^{t_0} g_p(2\pi t/t_0) \|\sigma_t^\varphi(x) - x\|_\varphi\, dt \\
&\to 0 \quad \text{for } p \to \infty
\end{aligned}$$

because $(g_p)_{p \in \mathbf{N}}$ form an approximate unit in the sense that

$$\lim_{p \to \infty} \frac{1}{2\pi} \int_0^{2\pi} g_p(u) f(u)\, du = f(0)$$

for every continuous function f on $\mathbf{R}$ with period 2π.

LEMMA 5.3. *Let (N, φ) be as in Theorem 5.1, and let (R, τ) be the hyperfinite II_1-factor with tracial state τ. For every finite set $x_1, \ldots, x_n \in N$ and every $\varepsilon > 0$, there exist completely positive maps*

$$S: N \to R \quad \textit{and} \quad T: R \to N$$

and a normal faithful state ψ on R, such that $h = d\psi/d\tau$ has finite spectrum and

$$\lambda_1/\lambda_2 \in \{\lambda^n | n \in \mathbf{Z}\} \quad \textit{for all } \lambda_1, \lambda_2 \in \mathrm{sp}(h).$$

Moreover

$$\begin{aligned} S(1) &= 1, & T(1) &= 1, \\ \psi \circ S &= \varphi, & \varphi \circ T &= \psi, \\ \sigma_t^\psi \circ S &= S \circ \sigma_t^\varphi, & \sigma_t^\varphi \circ T &= T \circ \sigma_t^\psi, \end{aligned}$$

for $t \in \mathbf{R}$, and

$$\|T \circ S(x_k) - x_k\|_\varphi < \varepsilon, \qquad k = 1, \ldots, n.$$

Proof. By Lemma 5.2, we can choose $p \in \mathbf{N}$, such that

$$\|\gamma_p(x_k) - x_k\|_\varphi < \varepsilon, \qquad k = 1, \ldots, n.$$

Let M_p be the algebra of $p \times p$ complex matrices with matrix units $(e_{rs})_{r,s=1,\ldots,p}$ and let ω be the state on M_p given by

$$\omega = \mathrm{Tr}(h_0 \cdot), \quad \text{where } h_0 = c \cdot \sum_{r=1}^{p} \lambda^r e_{rr}$$

and c is the normalization constant $c = (\sum_{r=1}^p \lambda^r)^{-1}$.

Note that for $r, s = 1, \ldots, p$:

$$\sigma_t^\omega(e_{rs}) = h_0^{it} e_{rs} h_0^{-it} = \lambda^{i(r-s)t} e_{rs}.$$

Particularly $\sigma_{t_0}^\omega = \mathrm{id}$. Put $\chi = \varphi \otimes w$ on $N \otimes M_p$. Then also $\sigma_{t_0}^\chi = \mathrm{id}$.

Since $N \otimes M_p \cong N$ is a factor of type III$_\lambda$, the centralizer M_χ is a II_1-factor. Moreover, since $N \otimes M_p$ is injective with separable predual, M_χ is injective with separable predual. Hence $M_\chi \cong R$.

For $x \in N$, and $r, s \in \{1, \ldots, n\}$:

$$\sigma_t^\chi(x \otimes e_{rs}) = \lambda^{i(r-s)} \sigma_t^\varphi(x) \otimes e_{rs}.$$

Therefore

$$M_\chi = \left\{ \sum_{r,s=1}^{p} x_{rs} \otimes e_{rs} \in N \otimes M_p | x_{rs} \in N_{s-r} \right\}$$

where $(N_n)_{n \in \mathbf{Z}}$ are the subspaces of N defined in the beginning of §3. Moreover, since $\sigma_{t_0}^\chi = \mathrm{id}$, the χ-invariant conditional expectation of $N \otimes M_p$ onto R is given by

$$\varepsilon_\chi(x) = \frac{1}{t_0} \int_0^{t_0} \sigma_t^\chi(x)\, dt.$$

Hence for $x_{rs} \in N$,

$$\varepsilon_\chi \left(\sum_{r,s=1}^{p} x_{rs} \otimes e_{rs} \right) = \sum_{r,s=1}^{p} \varepsilon_{s-r}(x_{rs}) \otimes e_{rs}.$$

Define now linear maps $S \colon N \to M_\chi$ and $T \colon M_\chi \to N$ by

$$S(x) = \sum_{r,s=1}^{p} \varepsilon_{s-r}(x) \otimes e_{rs}, \qquad x \in N,$$

$$T \left(\sum_{r,s=1}^{p} y_{rs} \otimes e_{rs} \right) = \frac{1}{p} \sum_{r,s=1}^{p} y_{rs}, \qquad y_{rs} \in N_{s-r}.$$

Clearly $S(1) = 1$ and $T(1) = 1$. We show next that S and T are completely positive:

Let $e_0 \in M_m$ be the orthogonal projection on the 1-dimensional subspace of $\mathbf{C}^n$ spanned by the vector $(1, 1, \ldots, 1)$. Then

$$e_0 = \frac{1}{p} \sum_{r,s=1}^{p} e_{rs}.$$

Hence

$$S(x) = p\varepsilon_\chi(x \otimes e_0), \qquad x \in N,$$

which shows that S is completely positive. Let ω_0 be the pure state on M_m given by

$$\omega_0(z) = \mathrm{Tr}(e_0 z), \qquad z \in M_m.$$

Then $\omega_0(e_{rs}) = 1/p$ for $r, s = 1, \ldots, p$. Therefore

$$T(y) = (\mathrm{id}_N \otimes \omega_0)(y)$$

for all $y \in M_\chi \subseteq N \otimes M_p$. This shows that T is completely positive.

The number of pairs $(r, s) \in \{1, \dots, p\}^2$ for which $s - r = k$ is $p - |k|$ when $|k| < p$, and 0 when $|k| \geq p$. Hence for $x \in N$,

$$T \circ S(x) = \frac{1}{p} \sum_{r,s=1}^{p} \varepsilon_{s-r}(x) = \sum_{|k|<p} \left(1 - \frac{|k|}{p}\right) \varepsilon_k(x) = \gamma_p(x)$$

where $\gamma_p : N \to N$ is the map defined in Lemma 5.2. Hence

$$\|T \circ S(x_k) - x_k\|_\varphi < \varepsilon \quad \text{for } k = 1, \dots, n.$$

Put $\psi = \varphi \circ T$. Then $\psi \circ S = \varphi \circ \gamma_p = \varphi$. For $y_{rs} \in N_{s-r}$,

$$\gamma\left(\sum_{r,s=1}^{p} y_{rs} \otimes e_{rs}\right) = \frac{1}{p} \sum_{r,s=1}^{p} \varphi(y_{rr})$$

because φ vanishes on N_k when $k \neq 0$. Hence

$$\psi(y) = \frac{1}{p}(\varphi \otimes \mathrm{Tr})(y), \qquad y \in M_\chi \subseteq N \otimes M_p.$$

Let τ be the tracial state on M_χ, i.e. τ is the restriction of χ to M_χ. Then

$$\tau(y) = (\varphi \otimes \omega)(y) = (\varphi \otimes \mathrm{Tr})((1 \otimes h_0)(y)).$$

Since $h_0 \in M_\chi$ it follows that

$$\frac{d\psi}{d\tau} = \left(\frac{d\tau}{d\psi}\right)^{-1} = \frac{1}{p}(1 \otimes h_0^{-1}).$$

By definition

$$\mathrm{sp}(h_0) = \{c\lambda, c\lambda^2, \dots, c\lambda^p\}$$

for some $c > 0$. Hence $h = d\psi/d\tau$ has finite spectrum, and

$$\lambda_1/\lambda_2 \in \{\lambda^n | n \in \mathbf{Z}\}$$

for all $\lambda_1, \lambda_2 \in \mathrm{sp}(h)$.

It is clear that $\sigma^{\varphi \otimes \mathrm{Tr}} = \sigma^\varphi \otimes \mathrm{id}$ leaves M_χ globally invariant. Since ψ is the restriction of $(1/p)(\varphi \otimes \mathrm{Tr})$ to M_χ it follows that

$$\sigma_t^\psi(y) = (\sigma_t^\varphi \otimes \mathrm{id})(y), \qquad y \in M_\chi.$$

Hence, by the definition of S and T

$$\sigma_t^\psi \circ \sigma(x) = S \circ \sigma_t^\varphi(x), \qquad x \in M,$$
$$\sigma_t^\psi \circ T(y) = T \circ \sigma_t^\psi(y), \qquad y \in M_\chi.$$

Since $M_\chi \cong R$ we have proved Lemma 5.3.

LEMMA 5.4. *Let R_λ be the Powers factor with infinite tensor product state ω_λ. If ψ is a normal faithful state on the hyperfinite* II_1*-factor R of the form*

$$\psi = \tau(h\cdot),$$

where τ is the tracial state on R, and h is a positive self-adjoint operator with finite spectrum for which

$$\lambda_1/\lambda_2 \in \{\lambda^n | n \in \mathbf{Z}\} \quad \text{for all } \lambda_1/\lambda_2 \in \mathrm{sp}(h),$$

then there exists a σ^{ω_λ}-invariant subfactor P of R_λ, such that

$$(R, \psi) \cong (P, \omega_\lambda|_P).$$

Proof. By the assumptions on h,

$$h = \sum_{i=1}^{r} \lambda_i e_i$$

where $e_i, \dots, e_r$ are orthogonal projections in R with sum 1, and λ_i/λ_j is of the form λ^n, $n \in \mathbf{Z}$ for $i, j = 1, \dots, n$. Put $\alpha_i = \lambda_i \tau(e_i)$. Then

$$\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \psi(e_i) = 1.$$

Since R_λ is of type III_λ, the centralizer of ω_λ is a II_1-factor, so we can choose orthogonal projections $f_1, \dots, f_r$ in M_{w_λ}, such that $\sum_{i=1}^{r} f_i = 1$, and

$$\omega_\lambda(f_i) = \alpha_i, \qquad i = 1, \dots, r.$$

Put $= \sum_{i=1}^{r} \lambda_i f_i$, and put $\chi(x) = \omega_\lambda(k^{-1}x)$, $x \in R_\lambda$.

Then χ is a positive normal faithful functional on R_λ. In fact χ is a state, because

$$\chi(f_i) = \lambda_i^{-1}\alpha_i = \tau(e_i), \qquad i = 1, \dots, r,$$

which implies that $\chi(1) = 1$. Moreover,

$$\sigma_t^\chi(x) = k^{-it}\sigma^{\omega_\lambda}(x)k^{+it}, \qquad x \in R_\lambda.$$

By the assumption on λ_i/λ_j, k^{it_0} is a scalar operator ($t_0 = -2\pi/\log\lambda$), and therefore $\sigma_{t_0}^\chi = \mathrm{id}$. Since R_λ is an injective factor of type III_λ, the centralizer $P = M_\chi$ is isomorphic to the hyperfinite factor of type II_1. Let α be a *-isomorphism of R onto P, and put $e_i' = \alpha(e_i)$, $i = 1, \dots, r$. Clearly $\xi \circ \alpha = \tau$ by uniqueness of the trace. Hence

$$\chi(e_i') = \tau(e_i) = \chi(f_i), \qquad i = 1, \dots, r,$$

so $e_i' \sim f_i$ (equivalence in P). Choose partial isometries $v_i \in P$ for which

$$e_i' = v_i^* v_i, \quad f_i = v_i v_i^*$$

and put $u = \sum_{i=1}^r v_i$. Then $u \in U(P)$ and

$$u e_i' u^* = f_i, \qquad i = 1, \dots, r.$$

Hence $\beta = \mathrm{ad}(u) \circ \alpha$ is an isomorphism of R onto P for which $\beta(e_i) = f_i$ and therefore also $\beta(h) = k$. Thus

$$w_\lambda \circ \beta = \chi(k\beta(\cdot)) = \tau(h\cdot) = \psi$$

which proves that

$$(R, \psi) \cong (P, \omega_\lambda|_P).$$

Finally $k \in M_{w_\lambda}$ implies that χ is σ^{ω_λ} -invariant, and thus P is (globally) σ^{ω_λ}-invariant.

Proof of Theorem 5.1. Let (N, φ) be as in Theorem 5.1, let $x, \dots, x_n \in N$ and let $\varepsilon > 0$. Choose

$$S \colon N \to R, \qquad T \colon R \to N$$

and $\psi \in R_*^+$, such that the conditions in Lemma 5.3 are satisfied. By Lemma 5.4 we may realize R as a σ^{ω_λ}-invariant subfactor of N, such that $\psi = \omega_\lambda|_R$. Let ε be the (unique) ω_λ-invariant conditional expectation of R_λ onto R (cf. [**21**]). Let S' be the map S considered as map from N to R_λ and put $T' = T \circ \varepsilon$ (from R_λ to N). Then

$$\begin{gathered} S'(1) = 1, \quad T'(1) = 1, \\ \omega_\lambda \circ S' = \varphi, \quad \varphi \circ T' = \omega_\lambda, \\ \sigma_t^{\omega_\lambda} \circ S' = S' \circ \sigma_t^\varphi, \qquad \sigma_t^\varphi \circ T' = T' \circ \sigma_t^{\omega_\lambda} \end{gathered}$$

for $t \in \mathbf{R}$. Moreover

$$\|T' \circ S'(x_k) - x_k\|_\varphi < \varepsilon, \qquad k = 1, \dots, n.$$

For $m \in \mathbf{N}$, let F_m be the subfactor of R_λ given by the tensor product of the first m copies of M_2 in $R_\lambda = \bigotimes_{n=1}^\infty (M_2, \varphi_\lambda)$. Then the infinite tensor product state ω_λ satisfies

$$\omega_\lambda = \omega_\lambda|_{F_m} \otimes \omega_\lambda|_{F_m^c}.$$

Let ε_m be the ω_λ-invariant conditional expectation of R_λ onto F_m. Then

$$\|\varepsilon_m(x) - x\|_{\omega_\lambda} \to 0 \quad \text{for all } x \in R_\lambda.$$

Put now

$$S_m = \varepsilon_m \circ S', \qquad T_m = T'|_{F_m}$$

and let ψ_m be the restriction of ω_λ to F_m. Then

$$S_m\colon N \to F_m, \qquad T_m\colon F_m \to N$$

are completely positive.

$$\begin{gathered} S_m(1) = 1, \qquad T_m(1) = 1, \\ \psi_m \circ S_m = \varphi, \qquad \varphi \circ T_m = \psi_m, \\ \sigma_t^{\psi^m} \circ S_m = S_m \circ \sigma_t^\varphi, \qquad \sigma_t^\varphi \circ T = T \circ \sigma_t^{\psi_m} \end{gathered}$$

for $t \in \mathbf{R}$. By the Schwarz inequality for completely positive maps $S'(x)^* S'(x) \le S'(x^*x)$, $x \in N$. Thus

$$\|S'x\|_{\omega_\lambda} \le \|x\|_\varphi, \qquad x \in N$$

and similarly,

$$\|T'y\|_\varphi \le \|y\|_{\omega_\lambda}, \qquad y \in R_\lambda.$$

Hence

$$\begin{aligned} \|T_m \circ S_m(x_k) - T' \circ S'(x_k)\|_\varphi &= \|T'(\varepsilon_m \circ S'(x_k) - S'(x_k))\|_\varphi \\ &\le \|\varepsilon_m \circ S(x_k) - S(x_k)\|_{\omega_\lambda} \to 0 \quad \text{for } m \to \infty. \end{aligned}$$

Therefore we can choose $m \in \mathbf{N}$, such that

$$\|S_m \circ T_m(x_k) - x_k\|_\varphi < \varepsilon, \qquad k = 1, \dots, n.$$

This completes the proof of Theorem 5.1, because

$$(F_m, \psi_m) = \bigotimes_{k=1}^{m} (M_2, \varphi_\lambda).$$

6. Injective factors of type III$_\lambda$, $0 < \lambda < 1$, are Powers factors. Throughout this section R_λ denotes the Powers factor of type III$_\lambda$, i.e.:

$$R_\lambda = \bigotimes_{n=1}^{\infty} (M_2, \varphi_\lambda)$$

where φ_λ is the state on the 2×2 complex matrices given by

$$\varphi_\lambda \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{1}{1+\lambda}(\lambda x_{11} + x_{22}).$$

We let ω_λ denote the infinite tensor product state $\omega_\lambda = \bigotimes_{n=1}^\infty \varphi_\lambda$ on R_λ. Note that $\sigma_{t_0}^{\omega_\lambda} = \mathrm{id}$ for $t_0 = -2\pi/\log\lambda$.

THEOREM 6.1. *Let N be an injective factor of type* III_λ *with separable predual, and let φ be a normal faithful state on N for which $\sigma_{t_0}^{\varphi} =$* id. *Then N is isomorphic to the Powers factor R_λ. Moreover, the isomorphism α of N onto R_λ can be chosen such that $\varphi = \omega_\lambda \circ \alpha$.*

We prove first three lemmas:

LEMMA 6.2. *Let (N, φ) be as in Theorem 6.1. For every finite set $u_1, \dots, u_n \in U(N)$ and every $\delta > 0$ there exists $m \in \mathbf{N}$, a completely positive map T from $M_{2^m} = \bigotimes_{i=1}^m M_2$ to M and unitary operators $v_1, \dots, v_n$ in M_{2^m}, such that $\psi = \varphi \circ T$ is equal to $\bigotimes_{i=1}^m \varphi_\lambda$*

$$\sigma_t^{\varphi} \circ T = T \circ \sigma_t^{\psi}, \qquad t \in \mathbf{R}$$

and

$$\|T(v_k) - u_k\|_\varphi < \delta, \qquad k = 1, \dots, n.$$

Proof. Let $\varepsilon > 0$ be such that $\varepsilon + (2\varepsilon)^{1/2} < \delta$. Choose $m \in \mathbf{N}$, and

$$S\colon N \to M_{2^m}, \quad T\colon M_{2^m} \to N$$

satisfying the conditions in Theorem 5.1 with respect to $(u_1, \dots, u_n, \varepsilon)$, and put

$$y_k = S(u_k), \qquad k = 1, \dots, n.$$

Then $\|y_k\| \le 1$ and

$$\|T(y_k) - u_k\|_\varphi < \varepsilon, \qquad k = 1, \dots, n.$$

Using the Schwarz inequality for completely positive maps, we have

$$\|T(y_k)\|_\varphi \le \|y_k\|_\psi$$

(cf. proof of Theorem 5.1). Therefore

$$\|y_k\|_\psi \ge \|u_k\|_\varphi - \varepsilon = 1 - \varepsilon.$$

We can find unitary operators $v_1, \dots, v_n \in M_{2^m}$, such that

$$y_k = v_k h_k, \qquad k = 1, \dots, n$$

where $h_k = (y_k^* y_k)^{1/2}$. Note that $\|h_k\|_\psi = \|y_k\|_\psi$. Since $0 \le h_k \le 1$,

$$h_k^2 + (1 - h_k)^2 \le 1.$$

Hence

$$\begin{aligned}\|v_k - y_k\|_\psi^2 &= \|1 - h_k\|_\psi^2 \le 1 - \|h_k\|_\psi^2 \\ &< 1 - (1 - \varepsilon)^2 < 2\varepsilon.\end{aligned}$$

Using again the Schwarz inequality for T, it follows that

$$\begin{aligned}\|T(v_k)-u_k\|_\varphi &\leq \|T(v_k-y_k)\|_\varphi+\|T(y_k)-u_k\|_\varphi\\ &< (2\varepsilon)^{1/2}+\varepsilon<\delta\end{aligned}$$

for $k=1,\dots,n$. This proves Lemma 6.2.

If N_1, N_2 are von Neumann algebras with states φ_1, φ_2 on N_1 and N_2, respectively, we write

$$(N_1,\varphi_1)\cong(N_2,\varphi_2)$$

if there is an isomorphism α of N_1 onto N_2 for which $\varphi_1=\alpha\circ\varphi_2$.

LEMMA 6.3. *Let N be an injective factor of type* III_λ, $0<\lambda<1$, *with separable predual, and let φ be a normal faithful state on N for which $\sigma^\varphi_{t_0}=\mathrm{id}$ $(t_0=-2\pi/\log\lambda)$. Let $u_1,\dots,u_n\in U(N)$ and let $\delta>0$. Then there exists a finite dimensional subfactor $F\subseteq N$, unitary operators $v_1,\dots,v_n\in U(F)$ and a sequence $(a_i)_{i=1}^\infty$ of operators in M_φ such that*

(i) $\varphi=\varphi|_F\otimes\varphi|_{F^c}$,

(ii) $(F,\varphi|_F)\cong\bigotimes_{j=1}^m(M_2,\varphi_\lambda)$ *for some* $m\in\mathbf{N}$,

(iii) $\sum_{i=1}^\infty a_i^*a_i=\sum_{i=1}^\infty a_ia_i^*=1$,

(iv) $\sum_{i=1}^\infty\|v_ka_i-a_iu_k\|_\varphi^2<\delta$ *for* $k=1,\dots,n$.

Proof. By combining Lemma 6.2 and Proposition 4.6 we get that there exists a finite dimensional subfactor F of N, such that (i) and (ii) hold, and a completely positive map $T\colon F\to N$ for which

$$T(1)=1,\quad \varphi\circ T=\varphi|_F,$$

and there exist $v_1,\dots,v_n\in U(F)$, such that

$$\|T(v_k)-u_k\|_\varphi<\delta/4.$$

By Theorem 4.1 there exist $(a_i)_{i=1}^\infty$ in M_φ, such that

$$\sum_{i=1}^\infty a_i^*a_i=\sum_{i=1}^\infty a_ia_i^*=1$$

and

$$\left\|T(x)-\sum_{i=1}^\infty a_i^*xa_i\right\|\leq\frac{\delta}{4}\|x\|,\qquad x\in F.$$

Since $\|x\|_\varphi\leq\|x\|$ for all $x\in M$ we get in particular

$$\left\|T(v_k)-\sum_{i=1}^\infty a_i^*v_ka_i\right\|_\varphi\leq\delta/4.$$

Hence

$$\left\| u_k - \sum_{i=1}^{\infty} a_i^* v_k a_i \right\|_\varphi < \delta/2.$$

Moreover,

$$\sum_{i=1}^{\infty} \|a_i u_k\|_\varphi^2 = \sum_{i=1}^{\infty} \varphi(u_k^* a_i^* a_i u_k) = \varphi(u_k^* u_k) = 1$$

and since $a_i \in M_\varphi$ we have also

$$\sum_{i=1}^{\infty} \|v_k a_i\|_\varphi^2 = \sum_{i=1}^{\infty} \varphi(a_i^* v_k^* v_k a_i) = \sum_{i=1}^{\infty} \varphi(a_i a_i^* v_k^* v_k) = \varphi(v_k^* v_k) = 1.$$

Therefore

$$\begin{aligned} \sum_{i=1}^{\infty} \|a_i u_k - v_k a_i\|_\varphi^2 &= 2 - 2 \sum_{i=1}^{\infty} \operatorname{Re} \varphi(u_k^* a_i^* v_k a_i) \\ &= 2 \operatorname{Re} \varphi \left(u_k^* \left(u_k - \sum_{i=1}^{\infty} a_i^* v_k a_i \right) \right) \\ &\leq 2 \|u_k\|_\varphi \left\| u_k - \sum_{i=1}^{\infty} a_i^* v_k a_i \right\|_\varphi < \delta. \end{aligned}$$

LEMMA 6.4. *Let (N, φ) be as in Lemma 6.3, let $x_1, \dots, x_n \in N$, and let $\varepsilon > 0$. Then there exists a finite dimensional subfactor F of N and operators $y_1, \dots, y_n \in F$, such that*

$$\varphi = \varphi|_F \otimes \varphi|_{F^c},$$

$$(F, \varphi) \cong \bigotimes_{j=1}^{m} (M_2, \varphi_\lambda) \quad \textit{for some } m \in \mathbf{N},$$

and

$$\|y_k - x_k\|_\varphi < \varepsilon, \qquad k = 1, \dots, n.$$

Proof. We can assume that N acts standardly on a Hilbert space H, so that φ is the vector state given by a cyclic and separating vector ξ_φ. Let $S = J\Delta^{1/2}$ be the modular conjugation associated with (N, ξ_φ) in Tomita-Takesaki Theory. Then $JNJ = N'$, and H becomes a normal N-bimodule, where right action on H is defined by

$$\xi a = J a^* J \xi, \quad a \in N, \quad \xi \in H.$$

Since N is spanned by its unitary operators, it is sufficient to consider $x_1, \dots, x_n$ unitary. So let $u_1, \dots, u_n \in U(N)$, and let $\varepsilon > 0$. Choose $\delta = \delta(n, \varepsilon)$, such that the conditions of Theorem 2.3 are fulfilled. By Lemma 6.3 we can choose a finite dimensional subfactor $F \subseteq N$, $v_1, \dots, v_n \in U(F)$ and $(a_i)_{i=1}^{\infty}$ in M_φ, such that

$$\varphi = \varphi|_F \otimes \varphi|_{F^c},$$

$$(F, \varphi|_F) \cong \bigotimes_{j=1}^{m} (M_2, \varphi_\lambda) \quad \text{for some } m \in \mathbf{N},$$

$$\sum_{i=1}^{\infty} a_i^* a_i = \sum_{i=1}^{\infty} a_i a_i^* = 1, \quad \sum_{i=1}^{\infty} \|v_k a_i - a_i u_k\|_\varphi^2 < \delta.$$

Put

$$\xi_k = u_m \xi_\varphi \quad \text{and} \quad \eta_k = v_k \xi_\varphi \quad \text{for } u = 1, \dots, n.$$

Note that $\|\xi_k\| = \|\eta_k\| = 1$. Since $a_i \in M_\varphi$,

$$a_i \xi_\varphi = J a_i^* J \xi_\varphi = \xi_\varphi a_i.$$

Therefore

$$\sum_{i=1}^{\infty} \|a_i \xi_k - \eta_k a_i\|^2 = \sum_{i=1}^{\infty} \|a_i u_k \xi_\varphi - v_k a_i \xi_\varphi\|^2$$
$$= \sum_{i=1}^{\infty} \|a_i u_k - v_k a_i\|_\varphi^2 < \delta.$$

Hence, if we consider H as an M_φ-bimodule, we get from Theorem 2.3 that there exists a unitary operator $w \in U(M_\varphi)$, such that

$$\|\xi_k - w^* \eta_k w\| < \varepsilon, \qquad k = 1, \dots, n.$$

Equivalently

$$\|u_k - w^* v_k w\|_\varphi < \varepsilon, \qquad k = 1, \dots, n.$$

Put $F_1 = w^* F w$. Then F_1 is a finite dimensional subfactor of N, and $w^* v_k w \in U(F_1)$ for $k = 1, \dots, n$. Moreover, since $w \in U(M_\varphi)$, φ is also a tensor product state for F_1,

$$\varphi = \varphi|_{F_1} \otimes \varphi|_{F_1^c}$$

and

$$(F_1, \varphi|_{F_1}) \cong (F, \varphi|_F) \cong \bigotimes_{j=1}^{m} (M_2, \varphi_\lambda).$$

Proof of Theorem 6.1. Theorem 6.1 follows from Lemma 6.4 by a procedure, which is standard in the type II_1-case (see, e.g., [**9**, Chapter III, §7.4]):

Let d_φ denote the metric on N given by

$$d_\varphi(x,y)=\|x-y\|_\varphi,\qquad x,y\in N.$$

Note that d_φ induces the σ-strong topology on bonded subsets of N. Let $(x_n)_{n=1}^\infty$ be a sequence that generates a σ-strongly dense *-subalgebra of N. We will construct a sequence of commuting finite dimensional subfactors $(F_m)_{m=1}^\infty$ of N, such that

(a)

$$d_\varphi\left(x_l,\bigotimes_{k=1}^m F_k\right)<\frac{1}{m},\qquad m\in\mathbf{N},\ l=1,\dots,m.$$

(b) For each $m\in\mathbf{N}$

$$\varphi=\varphi|_{F_m}\otimes\varphi|_{F_m^c}.$$

(c) For each $m\in\mathbf{N}$

$$(F_m,\varphi|_{F_m})\cong\bigotimes_{k=1}^{K_m}(M_2,\varphi_\lambda)\quad\text{for some } K_m\in\mathbf{N}.$$

It is clear from Lemma 6.4 that we can choose $F_1\subseteq N$, such that (a), (b) and (c) are fulfilled for $m=1$. Assume next that we have found commuting finite dimensional subfactors $F_1,\dots,F_n$, such that (a), (b) and (c) are fulfilled for $m=1,\dots,n$, and let us construct F_{n+1}. Put

$$F=(F_1\cup\cdots\cup F_n)''=\bigotimes_{m=1}^n F_m.$$

By (b), each F_m is σ^φ-invariant. Hence F is σ^φ-invariant, which implies that

$$\varphi=\varphi|_F\otimes\varphi|_{F^c}$$

(cf. §4). Let $(e_{ij})_{i,j=1}^d$ be a set of matrix units for F. Then each x_m can be written in the form

$$x_m=\sum_{i,j=1}^d e_{ij}x_{ij}^{(m)}$$

where $x_{ij}^{(m)}\in F^c$. Put

$$K=\sum_{i,j=1}^m\|e_{ij}\|_\varphi.$$

By applying Lemma 6.4 to $(F^c, \varphi|_{F^c})$, we get that there exists a finite dimensional subfactor F_{n+1} of F^c and operators $y_{ij}^{(m)} \in F_{n+1}$ $(m \leq n+1)$, such that

$$\|x_{ij}^{(m)} - y_{ij}^{(m)}\|_\varphi < \frac{1}{(n+1)K}$$

for $i, j = 1, \dots, d$ and $m = 1, \dots, n+1$, and such that

$$\varphi|_{F^c} = \varphi|_{F_{n+1}} \otimes \varphi|_{(F^c \cap F_{n+1})}$$

and

$$(F_{n+1}, \varphi) \cong \bigotimes_{k=1}^{p} (M_2, \varphi_\lambda)$$

for some $p \in \mathbf{N}$. Since σ^φ and $\sigma^{\varphi|F^c}$ coincide on F^c, F^{n+1} is σ^φ-invariant and, therefore,

$$\varphi = \varphi|_{F_{n+1}} \otimes \varphi|_{(F_{n+1})^c}.$$

Put

$$y_m = \sum_{i,j=1}^{m} e_{ij} y_{ij}^{(m)}, \qquad m = 1, \dots, n+1.$$

Then

$$y_m \in (F \cup F_{n+1})'' = \bigotimes_{k=1}^{n+1} F_k$$

and

$$\|x_m - y\|_\varphi \leq \sum_{i,j=1}^{m} \|e_{ij}\|_\varphi \|x_{ij}^{(m)} - y_{ij}^{(m)}\|_\varphi$$

for $m = 1, \dots, n+1$. (Here we have used that $\varphi = \varphi|_F \otimes \varphi|_{F^c}$.) This shows that (a), (b) and (c) are fulfilled for $m = n+1$, so by induction we get a sequence $(F_m)_{m=1}^\infty$ of commuting subfactors satisfying (a), (b) and (c).

Let ε_m denote the φ-invariant conditional expectation of N onto $\bigotimes_{l=1}^m F_l$. Since ε_m is an orthogonal projection with respect to the inner product

$$(x, y)_\varphi = \varphi(y^* x),$$

it follows from (b) that for $m \geq l$

$$\|\varepsilon_m(x_l) - x_l\|_\varphi \leq d_\varphi \left(x_l, \bigotimes_{k=1}^{m} F_m \right).$$

Since d_φ induces the σ-strong operator topology on bonded sets, we get that

$$\left(\bigcup_{l=1}^{\infty} F_l\right)''$$

is strongly dense in N. By (b) the restriction of φ to $\bigotimes_{l=1}^{m} F_l$ coincides with

$$\bigotimes_{l=1}^{m} \varphi|_{F_l}$$

for every $m \in \mathbf{N}$. This implies that

$$(N, \varphi) \cong \bigotimes_{l=1}^{\infty} (F_l, \varphi|_{F_l})$$

so by (c) we get

$$(N, \varphi) \cong \bigotimes_{m=1}^{\infty} (M_2, \varphi_\lambda) = (R_\lambda, \omega_\lambda).$$

This proves Theorem 6.4.

COROLLARY 6.5. *Let φ_1, φ_2 be two normal faithful states on the Powers factor R_λ for which $\sigma_{t_0}^{\varphi_1} = \sigma_{t_0}^{\varphi_2} = \mathrm{id}$ ($t_0 = -2\pi \log \lambda$). Then there exists an automorphism α of R_λ, such that $\varphi_2 = \varphi_1 \circ \alpha$.*

Proof. By Theorem 6.1 there exists $\alpha_1, \alpha_2 \in \mathrm{Aut}(R_\lambda)$, such that

$$\varphi_i = \omega_\lambda \circ \alpha_i, \qquad i = 1, 2.$$

Hence $\alpha = \alpha_2 \alpha_1^{-1}$ can be used.

REMARK 6.6. Corollary 6.5 is probably well known, and it can be proved by other means. In fact it is not hard to see that Corollary 6.5 holds for any σ-finite factor N of type III_λ, $0 < \lambda < 1$, for which the fundamental homomorphism

$$\mathrm{mod}(\alpha)\colon\ \mathrm{Aut}(N) \to \mathrm{Aut}(F_M)$$

defined in [**8**, p. 549] is surjective.

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