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INTEGRAL LOGARITHMIC MEANS FOR REGULAR FUNCTIONS

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For a function f, regular in the unit disc, integral logarithmic means are defined by the formulae

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\log|f(re^{i\theta})||^p \, d\theta \right\}^{1/p} \qquad (0 < r < 1)$$

for $0 < p < \infty$. These are related to

$$M_\infty(r, f) = \sup_{|z|=r} |\log|f(z)|| \qquad (0 < r < 1)$$

when the latter increases sufficiently rapidly. Thus when $\lambda_\infty(f) \geq 1$ the orders

$$\lambda_p(f) = \limsup_{r \to 1} \frac{\log M_p(r, f)}{\log 1/(1-r)}$$

are continuous at infinity in the sense that

$$\lim_{p \to \infty} \lambda_p(f) = \lambda_\infty(f),$$

a property which does not generally hold when $\lambda_\infty(f) < 1$. It transpires that in the extreme cases $\lambda_\infty(f) = \lambda_1(f) + 1$, and $\lambda_\infty(f) = \lambda_1(f) \geq 1$, $\lambda_p(f)$ is uniquely determined for $1 < p < \infty$.

1. Introduction. For a given function f, regular in the unit disc $D(0, 1) = \{z : |z| < 1\}$ let

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\log|f(re^{i\theta})||^p \, d\theta \right\}^{1/p} \qquad (0 < p < \infty),$$
$$M_\infty(r, f) = \sup_{|z|=r} \log|f(z)|$$

for $0 \leq r < 1$. We consider the asymptotic values of these quantities as $r \to 1$ in terms of the orders

$$\lambda_p(f) = \limsup_{r \to 1} \frac{\log M_p(r, f)}{\log(1/(1-r))}.$$

Note that $\lambda_1(f)$ is equal to the Nevanlinna order of f, and $\lambda_\infty(f)$ is the maximum modulus order of f, related by the classical inequalities

$$\lambda_1(f) \leq \lambda_\infty(f) \leq \lambda_1(f) + 1, \tag{1.1}$$

which are readily obtained from the Poisson-Jensen formula [**6**, p. 205].

Certain properties of $\lambda_p(f)$ follow immediately from the Hölder inequalities (see [**4**, pp. 9 and 15] for the corresponding properties of $M_p(r, f)$); for example

(A) $\lambda_p(f)$ is an increasing function of p $(0 < p < \infty)$,

(B) $p\lambda_p(f)$ is convex on the interval $(0, \infty)$.

In contrast, $\lambda_\infty(f)$ does not generally fit naturally into this context. For elementary calculations show that if $0 < \alpha < 1$ then

$$F(z) = \exp\{(1+z)^{-\alpha} - (1-z)^{-1}\} \qquad (|z| < 1)$$

satisfies $\lambda_\infty(F) = \alpha$, while $\lambda_p(F) = 1 - 1/p$ for $p > 1$. Nevertheless, we will show that

$$\lambda_\infty(f) = \lim_{p\to\infty} \lambda_p(f) \tag{1.2}$$

provided that $\lambda_\infty(f)$ is sufficiently large by proving the following result.

THEOREM 1. *If f is regular in $D(0, 1)$ and $\lambda_\infty(f) \geq 1$, then*

$$\lambda_p(f) \leq \lambda_\infty(f), \tag{1.3}$$

$$\lambda_\infty(f) \leq \lambda_p(f) + 1/p \tag{1.4}$$

for $0 < p \leq \infty$. Thus (1.2) holds when $\lambda_\infty(f) \geq 1$.

The following corollary is deduced readily from Theorem 1 in §4.

COROLLARY 1. *If f is regular in $D(0, 1)$ and $\lambda_\infty(f) \geq 1$, then*

(i) *$p(\lambda_\infty(f) - \lambda_p(f))$ is an increasing function of p on $(0, \infty)$, with range contained in $[0, 1]$,*

(ii) *$\lambda_p(f) + 1/p$ is a decreasing function of p on $(0, \infty]$.*

When $p = 1$, the inequalities (1.3) and (1.4) are equivalent to (1.1); in the case $p = 2$ they have been obtained by Sons [**5**]. As far as one extreme case of the inequalities (1.1) is concerned, it is readily observed that condition (A) shows that the equality $\lambda_\infty(f) = \lambda_1(f)$ implies that $\lambda_p(f) = \lambda_1(f)$ for $p \geq 1$. In the other extreme case represented by

$$\lambda_\infty(f) = \lambda_1(f) + 1 \tag{1.5}$$

$\lambda_p(f)$ is also completely determined when $p \geq 1$, since Corollary 1(ii) implies

$$\lambda_\infty(f) \leq \lambda_p(f) + 1/p \leq \lambda_1(f) + 1 \qquad (1 \leq p \leq \infty).$$

Thus we obtain a second corollary.

COROLLARY 2. *If f is regular in $D(0,1)$ and $\lambda_\infty(f) = \lambda_1(f) + 1$, then*

$$\lambda_p(f) = \lambda_\infty(f) - 1/p \qquad (p \geq 1).$$

2. Preliminaries for the proof of Theorem 1. In this section, we assemble some background material needed for the proof of Theorem 1. We put $\lambda_\infty(f) = \lambda$ and, when λ is finite, let μ be the integer satisfying

$$\lambda < \mu \leq \lambda + 1.$$

Then for each given positive number ε we have

$$\text{(2.1)} \qquad \log|f(re^{i\theta})| < (1-r)^{-\lambda-\varepsilon} \qquad (r_0 \leq r < 1,\ 0 \leq \theta < 2\pi)$$

for some r_0 in $(0,1)$.

We later seek lower bounds for $\log|f(\mathrm{r}e^{i\theta})|$ by considering a factorisation based on the zero sequence $\{a_m\}$ of f in $D(0,1)\backslash\{0\}$, each zero being counted according to multiplicity. Let

$$b(z, a_m, \mu) = \left(1 - \frac{1-|a_m|^2}{1-z\overline{a}_m}\right) \exp \sum_{j=1}^{\mu} \frac{1}{j}\left(\frac{1-|a_m|^2}{1-z\overline{a}_m}\right)^j.$$

This leads to the factorisation

$$\text{(2.2)} \qquad f(z) = g(z) z^s B(z, \{a_m\}, \mu),$$

where

$$\text{(2.3)} \qquad B(z) = B(z, \{a_m\}, \mu) = \prod_m b(z, a_m, \mu),$$

s is a nonnegative integer, and $g(z)$ is regular and nonzero in $D(0,1)$.

The result (1.4), is readily obtained for $g(z)$ by a simple application of a known theorem [**1**, p. 84]. We need to show that it also applies to the factor $B(z)$. We require some known results, the first being a theorem of Tsuji [**6**, p. 224].

THEOREM A. *For the canonical product $B(z)$ defined by* (2.3), *and positive ε we have*

$$\text{(2.4)} \qquad \log|B(z)| \leq K \sum_m \left|\frac{1-|a_m|^2}{1-z\overline{a}_m}\right|^{\mu+1+\varepsilon} \qquad (\tfrac{1}{2} \leq |z| < 1),$$

and, if C_m denotes the disc $D(a_m, (1-|a_m|^2)^{\mu+4})$ then

$$\text{(2.5)} \qquad \log|B(z)| \geq K \log(1-|z|) \sum_m \left|\frac{1-|a_m|^2}{1-z\overline{a}_m}\right|^{\mu+1+\varepsilon}$$

when $\frac{1}{2} \leq |z| < 1$, $z \notin \bigcup_m C_m$.

The constant K in (2.4) and (2.5) depends on ε, μ and $\{a_m\}$, or on ε and f if we regard $B(z)$ as defined by (2.2). As here, we will subsequently use K to denote a positive constant, not necessarily the same at each occurrence, but depending on parameters which will normally be stipulated as appropriate. The symbol r_0 will be used similarly, but always restricted to the interval $(0, 1)$.

We require some information regarding the zero distribution of f when $\lambda_\infty(f) = \lambda \geq 1$. Let the disc $D(0, 1)$ be covered by sets of the form

$$S(q, k) = \{z\colon 1 - 2^{-q} \leq |z| < 1 - 2^{-q-1}, \pi k 2^{-q} \leq \arg z < \pi(k+1)2^{-q}\}$$

for integers q and k satisfying

$$q = 0, 1, 2, \ldots, \quad -2^q \leq k < 2^q - 1. \tag{2.6}$$

For the given function f let $N(q, k, f)$ denote the number of zeros of f in $S(q, k)$. Then for any positive ε there is a number q_0, such that

$$N(q, k, f) < 2^{(\lambda+\varepsilon)q} \qquad (q \geq q_0), \tag{2.7}$$

for all relevant k in (2.6) [**3**, p. 21]. This inequality gives rise to a bound to the sums occurring in (2.4) and (2.5), as estimated in [**3**, pp. 23–25].

THEOREM B. *Let f be regular in $D(0, 1)$ with factorisation* (2.2). *Then for each positive ε, and $\alpha > \lambda = \lambda_\infty(f) \geq 1$, we have*

$$\sum_m \left| \frac{1 - |a_m|^2}{1 - z\bar{a}_m} \right|^{\alpha+1} < K(1 - |z|)^{-\lambda-\varepsilon} \qquad (r_0 \leq |z| < 1)$$

for some r_0 in $(0, 1)$.

As a final preliminary to the proof of (1.3) of Theorem 1, we estimate $M_p(r, f)$ according to the following lemma.

LEMMA 1. *Let f be regular in $D(0, 1)$ and $\lambda_\infty(f) = \lambda \geq 1$. Then, if $\varepsilon > 0$ and $1 \leq p < \infty$ we have*

$$\int_0^{2\pi} |\log|B(re^{i\theta}, \{a_m\}, \mu)||^p \, d\theta < K(1 - r)^{-p(\lambda+\varepsilon)} \tag{2.8}$$

for some constant K and $0 \leq r < 1$.

We deal with the integral in (2.8) by covering the range of integration by $[\pi/(1-r)]+1$ intervals of the form $[\tau+r-1, \tau+1-r]$ for $\tau = 2k(1-r)$ and $k = 0, 1, 2, \dots, [\pi/(1-r)]$, showing that

$$\int_{\tau+r-1}^{\tau+1-r} |\log|B(re^{i\theta})||^p \, d\theta < K(1-r)^{1-p(\lambda+\varepsilon)} \tag{2.9}$$

for each τ. The method of proof indicates that the constant K need not depend on τ. However, for convenience and without loss of generality, we suppose that $\tau = 0$ in the following proof. Thence we obtain (2.8), as stated.

Without loss of generality, we assume

$$\tfrac{1}{2} \le r < 1, \qquad \tfrac{3}{4} \le |a_m| < 1, \tag{2.10}$$

since the contribution to the integral (2.8), due to any zeros not satisfying this latter inequality is clearly bounded. For given r, let E denote the set of integers m for which the exceptional discs C_m of Theorem A intersect $\gamma_r = \{z \colon z = re^{i\theta},\ r-1 \le \theta \le 1-r\}$. By application of (2.7), we have

$$\#(E) < K(1-r)^{-\lambda-\varepsilon}, \tag{2.11}$$

where $\#(E)$ denotes the number of elements in the set E. We consider the factorisation $B = B_1 B_2 B_3$, where

$$B_1(z) = \prod_{m \notin E} b(z, a_m, \mu),$$

$$B_2(z) = \prod_{m \in E} \exp \sum_{j=1}^{\mu} \frac{1}{j} \left(\frac{1-|a_m|^2}{1-z\bar{a}_m} \right)^j,$$

$$B_3(z) = \prod_{m \in E} 1 - \frac{1-|a_m|^2}{1-z\bar{a}_m} = \prod_{m \in E} \frac{\bar{a}_m(a_m - z)}{1-z\bar{a}_m}.$$

First we note that for any positive number ε, Theorems A and B give

$$\begin{aligned} \int_{r-1}^{1-r} |\log|B_1(re^{i\theta})||^p \, d\theta &< K(1-r)^{1-p(\lambda+\varepsilon/2)} \log\left(\frac{1}{1-r}\right)^p \\ &< K(1-r)^{1-p(\lambda+\varepsilon)}, \end{aligned} \tag{2.12}$$

where the constants K in (2.12) can be chosen to depend only on ε, μ, p, and the whole sequence $\{a_m\}$.

Next, the inequality

$$|1 - z\bar{a}_m| > \tfrac{1}{2}(1 - |a_m|^2)$$

yields

$$|\log|B_2(z)|| < \sum_{m\in E}\frac{1}{j}\left|\frac{1-|a_m|^2}{1-z\bar{a}_m}\right|^j \le K\#(E).$$

Hence (2.11) implies

$$\int_{r-1}^{1-r}|\log|B_2(re^{i\theta})||^p\,d\theta < K(1-r)^{1-p(\lambda+\varepsilon)}. \tag{2.13}$$

It remains to consider B_3.

Given $z = \mathrm{re}^{i\theta}$ in $D(0,1)$ we have

$$1 \ge |B_3(z)|^2 = \prod_{m\in E} r_m^2\left\{1+\frac{(1-r^2)(1-r_m^2)}{|z-a_m|^2}\right\}^{-1} \tag{2.14}$$

where $a_m = r_m e^{i\theta_m}$. For each m in E we can find w with $|w| = r$ such that

$$|w-a_m| \le (1-|a_m|^2)^{\mu+4} \le \left(\frac{7}{16}\right)^3(1-r_m^2) < \frac{1}{8}(1-r_m^2).$$

Thus

$$1-r_m^2 < 2(1-r_m) \le 2(1-r+|w-a_m|),$$

from which we obtain

$$1-r_m^2 < \frac{8}{3}(1-r) < \frac{8}{3}(1-r^2).$$

Since

$$|z-a_m|^2 \ge 4rr_m\sin^2\frac{1}{2}(\theta-\theta_m) \ge \frac{3}{2}\sin^2\frac{1}{2}(\theta-\theta_m),$$

and in (2.14),

$$\sum_{m\in E}\log\left(\frac{1}{r_m}\right) < \#(E)\log\left(\frac{4}{3}\right),$$

Minkowski's inequality yields

$$\begin{aligned}
&\int_{r-1}^{1-r}|\log|B_3(re^{i\theta})||^p\,d\theta\\
&\quad< K\#(E)^p(1-r)+K\int_{r-1}^{1-r}\left(\sum_{m\in E}\log 1+\frac{16(1-r^2)^2}{9\sin^2\frac{1}{2}(\theta-\theta_m)}\right)^p d\theta\\
&\quad< K\#(E)^p(1-r)+K\#(E)^p\int_{r-1}^{1-r}\left(\log\left(1+\frac{16(1-r^2)^2}{9\sin^2\frac{1}{2}t}\right)\right)^p dt\\
&\quad< K\#(E)^p(1-r).
\end{aligned}$$

The inequality (2.9) with $\tau = 0$ now follows from (2.11), (2.12), (2.13) and this last inequality, so that Lemma 1 is proved.

3. The Proof of Theorem 1. We begin the proof of Theorem 1 by using the results of the last section to verify (1.3). The property (A) shows that, without any loss of generality, we may assume $p > 1$.

Let ε be a given number in the interval $(0, \mu - \lambda)$. Then in applying Tsuji's Theorem A, we note

$$\sum_{r<|a_m|<1} (1 - |a_m|^2)^{\mu+4} < (1 - r^2)^2 \sum_{r<|a_m|<1} (1 - |a_m|^2)^{\mu+2},$$

where this latter sum converges. Therefore, there is an integer q_0 such that each interval $[1 - 2^{-q}, 1 - 2^{-q-1})$ contains a number R_q for which the circle $\{z : |z| = R_q\}$ does not intersect any of the exceptional discs of Theorem A when $q \geq q_0$. An application of Theorem A implies

$$(3.1)\quad |\log|B(z, \{a_m\}, \mu)|| < K(1 - |z|)^{-\lambda-\varepsilon}\log(1/(1 - |z|)) \qquad (|z| = R_q, q \geq q_0).$$

By using the factorisation (2.2), we now have

$$\begin{aligned}\log|g(z)| &\leq \log|f(z)| + |\log|B(z, \{a_m\}, \mu)|| - s\log|z| \\ &\leq K(1 - |z|)^{-\lambda-\varepsilon}\log(1/(1 - |z|)) \qquad (|z| = R_q, q \geq q_0).\end{aligned}$$

Hence, for any r in $[1 - 2^{-q}, 1 - 2^{-q-1})$, the maximum modulus principle implies

$$\begin{aligned}M_\infty(r, g) \leq M(R_{q+1}, g) &\leq K(q + 1)2^{(q+1)(\lambda+\varepsilon)} \\ &< K(1 - r)^{-\lambda-\varepsilon}\log(1/(1 - r))\end{aligned}$$

when $q \geq q_0$. Since $\lambda \geq 1$, and g has no zeros in $D(0, 1)$, it follows [2] that

$$(3.2)\quad |\log|g(z)|| \leq K(1 - |z|)^{-\lambda-\varepsilon}\log(1/(1 - |z|)) \qquad (1 - 2^{-q_0} \leq |z| < 1).$$

The inequality (3.2) leads to

$$M_p(r, g) \leq K(1 - r)^{-\lambda-\varepsilon}\log(1/(1 - r)),$$

from which the Minkowski inequalities and Lemma 1 yield

$$\begin{aligned}M_p(r, f) &\leq M_p(r, g) + M_p(r, B) \\ &\leq K(1 - r)^{-\lambda-\varepsilon}\log(1/(1 - r))\end{aligned}$$

when r is sufficiently close to 1. We now have $\lambda_p(f) \leq \lambda + \varepsilon$ for all positive ε, so that

$$\lambda_p(f) \leq \lambda = \lambda_\infty(f). \tag{3.3}$$

The inequality (1.3) has been proved.

The proof of (1.4) when $p > 1$ is obtained by applying the method of proof of Theorem 5.9 [**1**, p. 84]. The Poisson-Jensen formula, together with Hölder's inequality, yields

$$\begin{aligned}\log|f(re^{i\theta})| &\leq \frac{1}{2\pi}\int_0^{2\pi} |\log|f(Re^{i\phi})||P(R,r,\theta-\phi)\,d\phi \\ &\leq M_p(r,f)\left(\frac{1}{2\pi}\int_0^{2\pi} P(R,r,\theta-\phi)^{p/p-1}\,d\phi\right)^{(p-1)/p},\end{aligned}$$

for $0 < r < R < 1$, $0 \leq \theta < 2\pi$. We put $R = \frac{1}{2}(1+r)$, and use a standard estimate [**1**, p. 84] for the Poisson kernel to obtain

$$M_\infty(r,f) \leq KM_p(r,f)(1-r)^{-1/p}.$$

The inequality (1.4) follows for $1 < p < \infty$, and so does (1.2).

We have already noted that (1.1) implies (1.4) when $p = 1$, so it remains to consider $0 < p < 1$. The property (B) shows that

$$p(\lambda_s(f) - \lambda_p(f)) \leq q\left(\frac{s-p}{s-q}\right)(\lambda_s(f) - \lambda_q(f)) \qquad (0 < p < q < s),$$

with limiting form

$$p(\lambda_\infty(f) - \lambda_p(f)) \leq q(\lambda_\infty(f) - \lambda_q(f)) \qquad (0 < p < q), \tag{3.5}$$

obtained from (1.2). But we have already seen that the right-hand side of this latter inequality has upper bound 1 when $q > 1$. Hence $0 < p < 1 < q$ implies

$$p(\lambda_\infty(f) - \lambda_p(f)) \leq 1 \tag{3.6}$$

for $0 < p < 1$, and (1.4) follows for all positive p.

4. The proof of Corollary 1. Corollary 1 follows readily from the proof of Theorem 1. The inequality (3.5) shows that $p(\lambda_\infty(f) - \lambda_p(f))$ is increasing on $(0,\infty)$, and (1.3) and (1.4) show that the range of this function is included in $[0,1]$. The inequalities (3.5) and (1.4) also imply

$$p\lambda_p(f) \geq q\lambda_q(f) - (q-p)\lambda_\infty(f)$$
$$\geq q\lambda_q(f) - (q-p)(\lambda_q(f) + 1/q) = p\lambda_q(f) - 1 + p/q.$$

Corollary 1(ii) follows immediately for $0 < p < \infty$, and for $p = \infty$ by taking limits.

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