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STATE EXTENSIONS AND A RADON-NIKODYM THEOREM FOR CONDITIONAL EXPECTATIONS ON VON NEUMANN ALGEBRAS

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Let M be a von Neumann algebra with a von Neumann subalgebra M_0. If E is a conditional expectation (i.e., projection of norm one) from M into M_0, then any faithful normal state φ_0 admits a natural extension $\varphi_0 \circ E$ with respect to E in the sense that $E = E_{\varphi_0 \cdot E}$. If E_ω is only an ω-conditional expectation, then $\varphi_0 \circ E_\omega$ is not always an extension of φ_0. This paper is devoted to the construction of an extension $\tilde{\varphi}_0$ of φ_0 generalizing the above situation for ω-conditional expectations, which leads also to a Radon-Nikodym theorem for ω-conditional expectation under suitable majorization condition.

Let M be a von Neumann algebra with a faithful normal state ω and M_0 a von Neumann subalgebra of M. A conditional expectation of M onto M_0 leaving ω invariant exists if and only if M_0 is stable under the modular group σ^ω. This is a result of Takesaki ([**15**], 10.1) and it was the reason for a generalized conditional expectation $E_\omega: M \to M_0$, which always exists and is referred to as the ω-conditional expectation, to be introduced by Accardi and Cecchini ([**1**]). If E_ω is actually a projection, then for a faithful normal state φ_0 on M_0 the composition $\tilde{\varphi}_0 = \varphi_0 \circ E_\omega$ is a natural extension of φ_0 to M and $E_\omega = E_{\tilde{\varphi}}$. In general, $\varphi_0 \circ E_\omega$ is not an extension of φ_0 and as a consequence of Theorem 4 in [**11**] (see also [**12**]) there is no extension of φ_0 possessing the same generalized conditional expectation mapping as ω. We give a construction of a $\tilde{\varphi}_0$ that can be described briefly as follows.

Assuming that $M \subset B(H)$ and ω is determined by a cyclic and separating vector $\Omega \in H$, we consider the restriction of the action of M_0 to $[M_0\Omega] = H_0$. There is a natural positive cone $P_0 \subset H_0$ with respect to M_0 such that $\omega|M_0$ and φ_0 have the vector representatives Ω and Φ_0 in P_0, respectively. We say that the vector state $\tilde{\varphi}_0(a) = \langle a\Phi_0, \Phi_0\rangle$ is the canonical extension of φ_0 with respect to ω. If the cocycle $[D\varphi_0, D(\omega|M_0)]_t$ is in the fixed point algebra of E_ω, then our $\tilde{\varphi}_0$ reduces to $\varphi_0 \circ E_\omega$, and of course, this is the case where E_ω is a projection. In fact, $\tilde{\varphi}_0$ depends rather on E_ω than ω itself; that is, if

$E_\omega = E_\psi$ then $(\varphi_0)^{\sim\omega} = (\varphi_0)^{\sim\psi}$. In general, $E_\omega(v^*av) = E_\psi(a)$ where v is an appropriate isometry in M and ψ stands for $(\varphi_0)^{\sim\omega}$.

Our references on von Neumann algebras and their modular theory are **[14]** and **[15]**. We use the standard notations of the Tomita-Takesaki theory without any explanation. H will denote always a Hilbert space and if $M \subset B(H)$ then M' is the commutant of M. For the sake of convenience, states on M' are marked with a prime, for example ω' etc.

The main results are contained in §§3 and 4.

1. Preliminaries. In this section we shall present some facts about the spatial theory of integration on von Neumann algebras, ω-conditional expectations etc., which we shall use in this paper. Those facts will be the extensions of results contained in the original papers quoted from time to time.

Let $M \subset B(H)$ be a von Neumann algebra with commutant M' and $\psi \in M_*^+$. The lineal of ψ is defined ([7], [9], [13], see also [15], 7.1) as follows:

$$D(H, \psi) = \{\xi \in H \colon \|a\xi\| \leq C_\xi \psi(a^*a) \text{ for all } a \in M\}.$$

When ψ is of the form $\psi(a) = \langle a\Psi, \Psi\rangle$ $(a \in M)$ for some $\Psi \in H$, then $D(H, \psi) = M'\Psi$.

LEMMA 1.1. *$D(H, \psi)^- = \operatorname{supp} \psi$.*

Proof. Let $p = \operatorname{supp} \psi$ and q be the projection onto closure of $D(H, \psi)$. If $\xi \in D(H, \psi)$ then $\|p^\perp \xi\| \leq C_\xi \psi(p^\perp) = 0$ and so $q \leq p$. On the other hand, $\psi(a) = \sum \langle a\eta_i, \eta_i\rangle$ with a sequence (η_i) from H. Clearly, $\eta_i \in D(H, \psi)$. Since $\psi(p - q) = \sum\langle (p - q)\eta_i, \eta_i\rangle = 0$ we obtain $q = p$.

When ω is a faithful normal state on M and $\psi \in M_*^+$ with support p then the functional $\bar{\psi}(\cdot) = \psi(\cdot) + \omega(p^\perp \cdot p^\perp)$ is faithful. This simple trick will allow us to reduce the non-faithful case to the faithful one.

LEMMA 1.2. *If $\psi \in M_*^+$, $p = \operatorname{supp} \psi$ and $\bar{\psi}$ is a faithful normal functional such that $\bar{\psi} - \psi$ is orthogonal to ψ, then $D(H, \psi) = pD(H, \bar{\psi})$.*

Proof. Let $\xi \in D(H, \bar{\psi})$. Then

$$\|ap\xi\| \leq C_\xi \bar{\psi}(pa^*ap) = C_\xi \psi(a^*a)$$

for every $a \in M$ and hence $p\xi \in D(H, \psi)$. The other inclusion is obvious.

Set (Ψ, π_ψ, H_ψ) as the GNS-triple corresponding to ψ. It is possible to define for $\xi \in D(H, \psi)$ a bounded operator

$$R^\psi(\xi)\colon H_\psi \to H$$

such that

$$R^\psi(\xi)\pi_\psi(a) = a\xi \qquad (a \in M).$$

It was proved in **[7]** that

$$\Theta^\psi(\xi) = R^\psi(\xi)R^\psi(\xi)^* \in M'.$$

(See also **[15]**, 7.1.)

Lemma 1.3. *Let $\psi, \omega \in M_*^+$ such that $\psi \leq \lambda\omega$. Then $D(H, \psi) \subset D(H, \omega)$ and $\Theta^\omega(\xi) \leq \lambda^2\Theta^\psi(\xi)$ for $\xi \in D(H, \psi)$.*

Proof. $D(H, \psi) \subseteq D(H, \omega)$ follows immediately from the definition. Define $v\colon H_\omega \to H_\psi$ by $v\pi_\omega(a)\Omega = \pi_\psi(a)\Psi$ $(a \in M)$. Then the diagram

$$\begin{array}{ccc} H & \xleftarrow{R^\omega(\xi)} & H_\omega \\ {\scriptstyle R^\psi(\xi)}\nwarrow & & \swarrow{\scriptstyle v} \\ & H_\psi & \end{array}$$

is commutative. Since $\|v\| \leq \lambda$ we have

$$R^\omega(\xi)R^\omega(\xi)^* = R^\psi(\xi)vv^*R^\psi(\xi)^* \leq \lambda^2 R^\psi(\xi)R^\psi(\xi)^*.$$

Lemma 1.4. *Let $\psi \in M_*^+$ and M_0 be a von Neumann subalgebra of M. If ω stands for $\psi|M_0$, then $D(H, \psi) \subset D(H, \omega)$ and $\Theta^\omega(\xi) \leq \Theta^\psi(\xi)$ for $\xi \in D(H, \psi)$.*

Proof. We proceed as in the proof of Lemma 1.3, but we use the diagram

$$\begin{array}{ccc} H & \xleftarrow{R^\omega(\xi)} & H_\omega \\ {\scriptstyle R^\psi(\xi)}\nwarrow & & \nearrow{\scriptstyle i} \\ & H_\psi & \end{array}$$

where $i\colon H_\omega \to H_\psi$ is the natural embedding.

If $\psi \in M_*^+$ is faithful and $\psi' \in (M')_*^+$ then there exists a positive selfadjoint operator $(d\psi'/d\psi)$ on H such that

(i) $D(H,\psi)$ is a core for $(d\psi'/d\psi)^{1/2}$ and $\|(d\psi'/d\psi)^{1/2}\xi\|^2 = \psi'(\Theta^\psi(\xi))$ for $\xi \in D(H,\psi)$,

(ii) $\mathrm{supp}(d\psi'/d\psi) = \mathrm{supp}\,\psi'$.

(See [7] or [15], 7.3.)

PROPOSITION 1.5 ([7], p. 158). *If $\psi \in M_*^+$ is faithful and $\psi_1', \psi_2' \in (M')_*^+$ then*

$$(d(\psi_1' + \psi_2')/d\psi) = (d\psi_1'/d\psi) + (d\psi_2'/d\psi).$$

Here one should add that the sum means form sum. However, we need this result in the case of orthogonal supports when there is no difference.

LEMMA 1.6. *Let $\psi \in M_*^+$ be faithful and $\bar\psi', \psi' \in (M')_*^+$. If ψ' and $\bar\psi' - \psi'$ are orthogonal and $p' = \mathrm{supp}\,\psi'$ then*

$$(d\psi'/d\psi)^\alpha = (d\bar\psi'/d\psi)^\alpha p = p(d\bar\psi'/d\psi)^\alpha$$

for $\alpha \in \mathbf{C}$ with $\mathrm{Re}\,\alpha \geq 0$.

Proof. Due to property (ii) above the operators $(d\psi'/d\psi)$ and $(d(\bar\psi' - \psi')/d\psi)$ have orthogonal supports. Proposition 1.5 and Lemma 1.2 make the proof complete.

PROPOSITION 1.7 ([7], p. 158; [15], 7.4). *If both $\psi \in M_*^+$ and $\psi' \in (M')_*^+$ are faithful then $(d\psi'/d\psi)^{-1} = (d\psi/d\psi')$ and*

$$(d\psi/d\psi')^{it} a (d\psi/d\psi')^{-it} = \sigma_t^\psi(a) \qquad (t \in \mathbf{R},\ a \in M).$$

PROPOSITION 1.8 (([5], 2.2). *If $\psi \in M_*^+$ and $\psi' \in (M')_*^+$ are faithful then*

$$\Theta^\psi((d\psi/d\psi')^{it}\xi') = \sigma_{-t}^{\psi'}(\Theta^\psi(\xi'))$$

for every $\xi \in D(H,\psi)$ and $t \in \mathbf{R}$.

LEMMA 1.9 (cf. [8], 3.1). *Let $\psi \in M_*^+$ be faithful and $\psi' \in (M')_*^+$. Then*

$$(d\psi'/d\psi)^{1/2} D(H,\psi) \subset D(H,\psi').$$

Proof. Let $\xi \in D(H, \psi)$ and $a' \in M'$. Using Hilsum's notation and results ([9]) we have

$$\begin{aligned}\|a'(d\psi'/d\psi)^{1/2}\xi\|^2 &= \|\,|a'(d\psi'/d\psi)^{1/2}|\,\|^2 \\ &= \int (d\psi'/d\psi)^{1/2}a'^*a'(d\psi'/d\psi)^{1/2}\Theta^\psi(\xi)\,d\psi \\ &\le \|\Theta^\psi(\xi)\| \int (d\psi'/d\psi)^{1/2}a'^*a'(d\psi'/d\psi)^{1/2}\,d\psi \\ &= \|\Theta^\psi(\xi)\|\psi'(a'^*a').\end{aligned}$$

Therefore, $\xi \in D(H, \psi')$.

LEMMA 1.10 (cf. [3] AND [16]). *Let $\psi \in M_*^+$ and $\psi' \in (M')_*^+$ be faithful. Then the mapping*

$$i_\psi : a \to \int (d\psi/d\psi')^{1/2}a(d\psi/d\psi')^{1/2}(\cdot)\,d\psi' \qquad (a \in M)$$

is a positive linear mapping of M into M_*. It does not depend on ψ'. For $a \in M_+$ the majorization $i_\psi(a) \le \|a\|\psi$ holds. If ψ is faithful then i_ψ is injective and $i_\psi(M_+)$ consists of all $\omega \in M_*^+$ such that $\omega \le \lambda\psi$ with some $\lambda > 0$.

Proof. Since $(d\psi/d\psi')^{1/2}a(d\psi/d\psi')^{1/2} \in L^1(M, \psi)$, the mapping i_ψ is well-defined, positive and linear. For faithful ψ the statement is completely covered by 3.7 and 3.9 of [3]. Let $p = \operatorname{supp}\psi$ and take an auxiliary $\bar\psi \in M_*^+$ such that it is faithful and $\bar\psi - \psi$ is positive and orthogonal to ψ. Then

$$\begin{aligned}i_\psi(a)(b) &= \int (d\psi/d\psi')^{1/2}a(d\psi/d\psi')^{1/2}b\,d\psi' \\ &= \int p(d\psi/d\psi')^{1/2}a(d\psi/d\psi')^{1/2}pb\,d\psi' \\ &= \int (d\psi/d\psi')^{1/2}a(d\psi/d\psi')^{1/2}pbp\,d\psi' = i_{\bar\psi}(a)(pbp)\end{aligned}$$

for $a, b \in M$. Now clearly $i_\psi(a)$ does not depend on ψ' and

$$i_\psi(a)(b) = i_{\bar\psi}(a)(pbp) \le \|a\|\bar\psi(pbp) = \|a\|\psi(b)$$

if $a, b \ge 0$.

Let M_0 be a von Neumann subalgebra of M, ψ a normal state of M such that $\psi|M_0 = \psi_0$ is faithful. In the light of the previous lemma for $a \in M$ there is an element $i_{\psi_0}^{-1}(i_\psi(a)|M_0) \in M_0$. We define the ψ-conditional expectation $E_\psi : M \to M_0$ by setting

$$E_\psi(a) = i_{\psi_0}^{-1}(i_\psi(a)|M_0).$$

So E_ψ is a positive, unital linear contraction. It generalizes the notion of ψ-conditional expectation introduced for a faithful state ψ first in **[1]** as it follows from **[2]**, **[3]** and **[10]**.

PROPOSITION 1.11 (cf. **[4]**). *Let $\psi \in M_*^+$ and M_0 a von Neumann subalgebra of M. Assume that $\psi_0 = \psi | M_0$ is faithful. Then for $\xi' \in D(H, \psi')$ we have*

$$E_\psi(\Theta^{\psi'}(\xi')) = \Theta^{\psi_0'}((d\psi_0/d\psi_0')^{-1/2}(d\psi/d\psi')^{1/2}\xi')$$

if $\psi_0' \in (M_0')_*^+$ is faithful and $\psi' = \psi_0' | M'$.

Proof. Due to Lemma 1.9 $(d\psi/d\psi')^{1/2}\xi' \in D(H, \psi) \subset D(H, \psi_0)$, and the right hand side makes sense. By simple calculation we have for $a_0 \in (M_0)_+$

$$\begin{aligned} & i_{\psi_0}(\Theta^{\psi_0'}((d\psi_0/d\psi_0')^{-1/2}(d\psi/d\psi')^{1/2}\xi'))(a_0) \\ &= \int (d\psi_0/d\psi_0')^{1/2} a \Theta^{\psi_0'}((d\psi_0/d\psi_0')^{-1/2}(d\psi/d\psi')^{1/2}\xi') \\ &\quad \times (d\psi_0/d\psi_0')^{1/2} a_0 \, d\psi' \\ &= \| \, |a_0^{1/2}(d\psi_0/d\psi_0')^{1/2}|(d\psi_0/d\psi_0')^{-1/2}(d\psi/d\psi')^{1/2}\xi')\|^2 \\ &= \|a_0^{1/2}(d\psi/d\psi')^{1/2}\xi')\|^2 = i_\psi(\Theta^{\psi'}(\xi'))(a_0). \end{aligned}$$

We note since the linear span of $\{\Theta^{\psi'}(\xi') \colon \xi' \in D(H, \psi')\}$ is dense in M, the above formula characterizes E_ψ.

2. Analytic continuation. Let $S = \{z \in \mathbf{C} \colon 0 \le \operatorname{Re} z \le 1/2\}$. For the sake of brevity we say that a function is analytic on S if it is holomorphic on $\operatorname{Int} S$ and continuous and bounded on S. In this section we consider vector-valued functions defined primarily on the imaginary line and prove that they admit analytic extension to S. Most of the results are of auxiliary nature and will be used in the rest of the paper, but some of them are interesting in their own right.

M will be always a von Neumann algebra with commutant M' and ω' a faithful normal state on M'.

LEMMA 2.1. *Let $z \to f(z) \in H$ be an analytic function on S. If $A \ge 0$ is a selfadjoint operator on a Hilbert space H and $\|A^{1/2} f(1/2 + it)\|$ is bounded on $\mathbf{R}$ then $z \to A^z f(z)$ is analytic on S.*

Proof. Let $\eta \in D(A^{1/2})$. So $z \to \langle f(z), A^{\bar z}\eta\rangle$ is analtyic on S. If $\|f(it)\| \le K$ and $\|f(1/2 + it)\| \le L$ for all $t \in \mathbf{R}$, then

$$|\langle f(it), A^{-it}\eta\rangle| \le K\|\eta\| \quad \text{and} \quad |\langle f(1/2 + it), A^{1/2 - it}\eta\rangle| \le L\|\eta\|$$

for all $t \in \mathbf{R}$. Applying the three lines theorem ([**8**], VI.10.3) we have

$$|\langle f(z), A^{\bar{z}}\eta\rangle| \leq C\|\eta\| \qquad (\eta \in D(A^{1/2}))$$

with some constant C. Since $D(A^{1/2})$ is a core for $A^{\bar{z}}$ we conclude that $f(z) \in D(A^z)$. Moreover, $\|A^z f(z)\| \leq C$. The analyticity of the function

$$z \to \langle A^z f(z), \eta\rangle \qquad (\eta \in D(A^{1/2}))$$

implies that $A^z f(z)$ is analytic, indeed.

PROPOSITION 2.2 (cf. [**5**], 2.3). *Let $\varphi, \omega \in M_*^+$ and assume that ω is faithful. Then the function*

$$z \to (d\varphi/d\omega')^z (d\omega/d\omega')^{-z}\xi$$

is analytic on S for $\xi \in D(H, \omega)$.

Proof. First assume that φ is faithful. By an application of Proposition 1.8 we have

$$\begin{aligned}\|(d\varphi/d\omega')^{1/2}(d\omega/d\omega')^{-1/2-it}\xi\|^2 &= \varphi(\Theta^{\omega'}((d\omega/d\omega')^{-1/2-it}\xi)) \\ &= \varphi(\sigma_{-t}^{\omega}(\Theta^{\omega'}((d\omega/d\omega')^{-1/2}\xi))) \leq \|\varphi\|\,\|\Theta^{\omega'}((d\omega/d\omega')^{-1/2}\xi)\|.\end{aligned}$$

Since $(d\omega/d\omega')^{-1/2}\xi \in D(H, \omega')$, this upper estimate is finite and reference to Lemma 2.1 completes the proof in the faithful case.

In the general case, we consider $\bar{\varphi} = \varphi + \omega(p^{\perp} \cdot p^{\perp})$, where $p = \operatorname{supp}\varphi$. Due to Lemma 1.6 $(d\varphi/d\omega')^z = p(d\bar{\varphi}/d\omega')^z$ and this formula reduces the case to the faithful one.

COROLLARY 2.3. *Let φ, ω and ξ be as above. Then*

$$a'(d\varphi/d\omega')^z(d\omega/d\omega')^{-z}\xi = (d\varphi/d\omega')^z(d\omega/d\omega')^{-z}a'\xi$$

for every $a' \in M'$ and $z \in S$.

Proof. Since $a'\xi \in D(H, \omega)$ both sides are analytic on S. Therefore, it is sufficient to prove the equality on the imaginary line. Let $\bar{\varphi}$ be as in the proof of the previous proposition. Then we have

$$\begin{aligned}a'(d\varphi/d\omega')^{it}(d\omega/d\omega')^{-it}\xi &= a'p(d\bar{\varphi}/d\omega')^z(d\omega/d\omega')^{-z}\xi \\ &= a'p[D\bar{\varphi}, D\omega]_t\xi = p[D\bar{\varphi}, D\omega]_t a'\xi = (d\varphi/d\omega')^z(d\omega/d\omega')^{-z}a'\xi\end{aligned}$$

since $(d\varphi/d\omega')^{it}(d\omega/d\omega')^{-it}$ is the Radon-Nikodym cocycle belonging to M ([**7**] or [**15**], 7.4).

LEMMA 2.4. *Let $\varphi \in M_*^+$ and $\xi' \in D(H, \omega)$. Then the function*

$$t \to \|\Theta^{\varphi}((d\varphi/d\omega')^{1/2+it}\xi')\| \qquad (t \in \mathbf{R})$$

is bounded.

Proof. Let ω be a faithful normal state on M and set $\bar{\varphi} = \varphi + \omega(p^{\perp} \cdot p^{\perp})$, where $p = \operatorname{supp}\varphi$. By Lemma 1.3 we have

$$\|\Theta^{\varphi}((d\varphi/d\omega')^{1/2+it}\xi')\| \le \|\Theta^{\bar{\varphi}}((d\bar{\varphi}/d\omega')^{1/2+it}p\xi')\|$$

and the latest term is bounded due to Proposition 1.8 since $p\xi' \in D(H, \omega)$.

PROPOSITION 2.5. *Let M be a von Neumann algebra with commutant M' and a subalgebra M_0. Let φ (ω_0, ω_0') be a normal state on M (M_0, M_0') and set $\omega' = \omega_0'|M'$. Assume that φ_0, ω_0 and ω_0' are faithful. Then for $\xi' \in D(H, \omega')$ the function*

$$z \to (d\omega_0/d\omega_0')^z(d\varphi_0/d\omega_0')^{-z}(d\varphi/d\omega')^z\xi'$$

is analytic on S.

Proof. For an iterated application of Lemma 1.12 we show that the functions

$$f: t \to \|(d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2+it}\xi'\|^2,$$
$$g: t \to \|(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2-it}(d\varphi/d\omega')^{1/2+it}\xi'\|^2$$

are bounded on $\mathbf{R}$. First by Lemma 1.4

$$f(t) = \omega_0'(\Theta^{\varphi_0}((d\varphi/d\omega')^{1/2+it}\xi')) \le \|\Theta^{\varphi}((d\varphi/d\omega')^{1/2+it}\xi')\|$$

and we can refer to Lemma 2.4 above.

We proceed similarly for g.

$$\begin{aligned} g(t) &= \omega_0(\Theta^{\omega_0'}((d\varphi_0/d\omega_0')^{-1/2-it}(d\varphi/d\omega')^{1/2+it}\xi')) \\ &\le \|\Theta^{\omega_0'}((d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2+it}\xi')\|. \end{aligned}$$

Here we need 3.5 of **[8]**, by which this equals

$$\|\Theta^{\varphi_0}((d\varphi/d\omega')^{1/2+it}\xi')\|.$$

So the above argument completes the proof.

THEOREM 2.6. *Let φ, φ_0, ω, ω_0, ω' and ω_0' be as in Proposition 2.5. If the operator*

$$T = (d\omega/d\omega')^{1/2}(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2}$$

is defined on $D(H, \omega')$ *and has a bounded bilinear form (i.e.,* $\langle T\xi', \eta'\rangle \le C\,\|\xi'\|\,\|\eta\|$ *for all* $\xi, \eta' \in D(H, \omega')$*), then the closure of* T *belongs to* M *and does not depend on* ω_0'.

Proof. $\langle T\xi', \eta'\rangle$ is the value of the analytic function

$$F(z) = \langle (d\omega_0/d\omega_0')^z (d\varphi_0/d\omega_0')^{-z} (d\varphi/d\omega')^z \xi',\ (d\omega/d\omega')^{-\bar z}\eta'\rangle$$

defined on S as it follows from Proposition 2.5. If $\bar\varphi = \varphi + \omega(p\cdot p)$ (p stands for the support of φ), then

$$F(it) = \langle \sigma_{-t}^{\omega}([D\omega_0, D\varphi_0]_t)[D\omega, D\bar\varphi]_{-t}p\xi', \eta'\rangle$$

does not depend on on ω_0' and neither does $\langle T\xi', \eta'\rangle$.

From now on we assume that ω, ω_0, ω' and ω_0' are vector states given by the same vector $\Omega \in H$. Then simply $D(H, \omega) = M'\Omega$ and $D(H, \omega') = M\Omega$. Take $\xi' \in D(H, \omega) \cap D(H, \omega')$ and $a' \in M'$ such that $a'\xi' \in D(H, \omega')$. Considering the functions

$$z \to \langle (d\omega_0/d\omega_0')^z (d\varphi_0/d\omega_0')^{-z} (d\varphi/d\omega')^z a'\xi',\ (d\omega/d\omega')^{-\bar z}\eta'\rangle,$$

$$z \to \langle (d\omega_0/d\omega_0')^z (d\varphi_0/d\omega_0')^{-z} (d\varphi/d\omega')^z \xi',\ (d\omega/d\omega')^{-\bar z} a' * \eta'\rangle$$

we establish that they are analytic on S and coincide on the imaginary line. Hence

$$\langle Ta'\xi', \xi'\rangle = \langle a'T\xi', \xi'\rangle.$$

Due to the properties of the Tomita algebra ([**14**], 10.20–21) $D(H, \omega) \cap D(H, \omega')$ is dense in H and $\{a' \in M' : a'D(H, \omega) \cap D(H, \omega') \subset D(H, \omega')\}$ is wo-dense in M'. Therefore we can conclude that the bounded closure of T is in M.

3. State extension. Let M_0 and M be von Neumann algebras with $M_0 \subset M$. We consider a faithful normal state φ_0 (ω) on M_0 (M) and intend to construct a canonical extension $\tilde\varphi_0$ of φ_0 with respect to ω. We assume that M acts on a Hilbert space H and the cyclic and separating vector Ω determines ω. As above ω_0' will be an auxiliary faithful normal state on M_0' and we use the notation $\omega|M = \omega_0$ and $\omega_0'|M' = \omega'$.

We set $\Phi_0 = (d\varphi_0/d\omega_0')^{1/2}(d\omega_0/d\omega_0')^{-1/2}\Omega$.

LEMMA 3.1. Φ_0 *is cyclic for* M.

Proof. We show that Φ_0 is separating for $M' \subset M_0'$. Let $a' \in M'$ and assume that $a'\Phi_0 = 0$. According to Corollary 2.3 we have

$$\begin{aligned} a'\Phi_0 &= a'(d\varphi_0/d\omega_0')^{1/2}(d\omega_0/d\omega_0')^{-1/2}\Omega \\ &= (d\varphi_0/d\omega_0')^{1/2}(d\omega_0/d\omega_0')^{-1/2}a'\Omega. \end{aligned}$$

Since the spatial derivatives involved are injective $a'\Phi_0 = 0$ implies $a'\Omega = 0$ and $a' = 0$.

We define now the canonical extension of φ_0 as the vector state corresponding to Φ_0: $\tilde{\varphi}_0(a) = \langle a\Phi_0, \Phi_0\rangle$ $(a \in M)$.

PROPOSITION 3.2. *Let φ_0, ω_0 and ω be as above. Then the function*

$$F(it) = \omega([D\varphi_0, D\omega_0]_t^* a [D\varphi_0, D\omega_0]_t) \qquad (t \in \mathbf{R}, a \in M)$$

admits an analytic continuation $\tilde{F}$ to S and $\tilde{\varphi}_0(a) = \tilde{F}(1/2)$.

Proof. The function

$$z \to \langle a(d\varphi_0/d\omega_0')^z (d\omega_0/d\omega_0')^{-z}\Omega,\ (d\varphi_0/d\omega_0')^{-\bar{z}}(d\omega_0/d\omega_0')^{-\bar{z}}\Omega\rangle$$

is an extension of F. Since

$$z \to a(d\varphi_0/d\omega_0')^z (d\omega_0/d\omega_0')^{-z}\Omega$$

is analytic (Proposition 2.2), it is also analytic.

For an arbitrary $\psi \in M_*^+$ we set $\Omega(\psi) = (d\psi/d\omega')^{1/2}(d\omega/d\omega')^{-1/2}\Omega$. We know from Proposition 2.2 that $z \to (d\psi/d\omega')^z (d\omega/d\omega')^{-z}\Omega$ is analytic on S. On the imaginary axis this is independent of ω'. Consequently, $\Omega(\psi)$ is independent of ω'. Considering $\omega'(\cdot) = \langle \cdot\Omega, \Omega\rangle$ we conclude that $\Omega(\psi)$ is the vector representative of ψ in the natural positive cone associated with Ω.

LEMMA 3.3. *Let $\psi \in M_*^+$ and Φ_0, φ_0, $\tilde{\varphi}_0$ be as above. Then the operator*

$$v_\psi' \colon a\Phi_0 \to a\Omega(\psi) \qquad (a \in M)$$

is bounded if and only if $\psi \leq \lambda\tilde{\varphi}_0$. When it is bounded, its closure belongs to M'.

Proof. $\|v_\psi'(a\Phi_0)\|^2 = \|a\Omega(\psi)\|^2 = \psi(a^*a)$. That is majorized by $\lambda\|a\Phi_0\|^2 = \lambda\tilde{\varphi}_0(a^*a)$ if and only if $\psi \leq \lambda\tilde{\varphi}_0$. If this holds then $v_\psi' b = b v_\psi'$ for all $b \in M$.

THEOREM 3.4. *Let φ_0, ω_0, ω (ω_0', ω') be as above. If φ is a positive normal extension of φ_0 to M such that $\varphi \leq \lambda\tilde{\varphi}_0$, then the operator*

$$S = (d\omega/d\omega')^{-1/2}(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2}$$

is defined on $D(H, \omega')$ and its bounded closure lies in M.

Proof. We know from Proposition 2.5 that

$$(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2}(d\omega/d\omega')^{-1/2}(d\omega/d\omega')^{1/2}\xi'$$

makes sense for $\xi \in D(H, \omega')$. As $(d\omega/d\omega')^{1/2}\xi' \in D(H, \omega)$, so it can be expressed as $a'\Omega$ for some $a' \in M'$. By repeated application of Corollary 2.3 and using Lemma 3.3 we have

$$\begin{aligned}(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2}(d\omega/d\omega')^{-1/2}a'\Omega \\ = a'(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2}v_\varphi'\Phi_0 = a'v_\varphi'\Omega\end{aligned}$$

that is in $d(H, \omega)$. Hence

$$\begin{aligned}\|S\xi'\|^2 &= \|(d\omega/d\omega')^{-1/2}a'v_\varphi'\Omega\|^2 \\ &= \omega'(\Theta^\omega(a'v_\varphi'\Omega)) = \omega'(a'v_\varphi'\Theta^\omega(\Omega)v_\varphi'a'^*) \le \|v_\varphi'\|^2\omega'(a'a'^*).\end{aligned}$$

(Note that $\Theta^\omega(\Omega) = I$.) On the other hand,

$$\|\xi'\|^2 = \|(d\omega/d\omega')^{-1/2}a'\Omega\|^2 = \omega'(\Theta^\omega(a'\Omega)) = \omega'(a'a'^*).$$

We have proved that S is bounded and now Theorem 2.6 gives that its closure is in M.

THEOREM 3.5. *Let φ_0, ω_0, ω (ω_0', ω') be as above and stand $\tilde{\varphi}_0$ for the extension of φ_0 to M with respect to ω. Then the closure of the operator*

$$S = (d\omega/d\omega')^{-1/2}(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2}(d\tilde{\varphi}_0/d\omega')^{1/2}$$

(defined on $D(H, \omega')$) is a partial isometry with initial projection $p =$ supp $\tilde{\varphi}_0$, *and with range H.*

Proof. Taking the auxiliary faithful functional $\bar{\varphi}(\cdot) = \tilde{\varphi}_0(\cdot) + \omega(p^\perp \cdot p^\perp)$ we consider the operator

$$T = (d\bar{\varphi}/d\omega')^{-1/2}p(d\varphi_0/d\omega_0')^{1/2}(d\omega_0/d\omega_0')^{-1/2}(d\omega/d\omega')^{1/2}$$

and show that it is a contraction on $D(H, \omega')$. Let $\xi' \in D(H, \omega')$. Then $(d\omega/d\omega')^{1/2}\xi' = a'\Omega$ for some $a' \in M'$.

$$\begin{aligned}\|T\xi'\|^2 &= \|(\bar{\varphi}/d\omega')^{-1/2}pa'(d\varphi_0/d\omega')^{1/2}(d\omega_0/d\omega_0')^{-1/2}\Omega\|^2 \\ &= \|(d\bar{\varphi}/d\omega')^{-1/2}a'p\Phi_0\|^2 = \|(d\bar{\varphi}/d\omega')^{-1/2}a'\Phi_0\|^2 \\ &= \omega'(\Theta^{\bar{\varphi}}(a'\Phi_0)) = \omega'(a'\Theta^{\bar{\varphi}}(\Phi_0)a'^*).\end{aligned}$$

Since $R^{\bar{\varphi}}(\Phi_0)$ is a partial isometry with range H, we have

$$\|T\xi'\|^2 = \omega'(a'a'^*) = \|\xi'\|^2.$$

We establish $\overline{TS} = p$. Since $\|S\| \le 1$ the restriction of $\overline{S}$ to pH must be an isometry. On the other hand, $Sp^\perp = 0$, so $\overline{S}$ is a partial isometry with initial projection p. Since $\operatorname{Rng}\overline{T} \subset pH$, we have $\operatorname{Rng}\overline{S} = H$. (Of course, $S^* = \overline{T}$.)

THEOREM 3.6. *Let M, M_0, φ, $\tilde{\varphi}_0$, ω_0, ω, ω'_0, ω' be as above. Assume that ω_0, ω, ω'_0, ω' are given by a vector $\Omega \in H$. If φ is a normal extension of φ_0 to M such that*

(i) $D((d\omega/d\omega')^{-1/2}(d\omega_0/d\omega'_0)^{1/2}(d\varphi_0/d\omega'_0)^{-1/2}(d\varphi/d\omega')^{1/2}) \supset D(H,\omega')$,

(ii) $(d\omega/d\omega')^{-1/2}(d\omega_0/d\omega'_0)^{1/2}(d\varphi_0/d\omega'_0)^{-1/2}(d\tilde{\varphi}_0/d\omega')^{1/2}$ *has a bounded closure,*

then $\varphi \leq \lambda\tilde{\varphi}_0$. In particular, if the closure is a partial isometry with range projection I, then $\varphi = \tilde{\varphi}_0$.

Proof. Stand S for the bounded operator mentioned in (ii). Theorem 2.6 tells us that $S \in M$. Let J_ω and Δ_ω $(= (d\omega/d\omega'))$ be the standard operators of the Tomita-Takesaki theory for Ω. Set $w' = J_\omega S^* J_\omega$. From Tomita's theorem $w' \in M'$. So for $a \in M$ we have

$$\begin{aligned} w'a\Phi_0 &= aw'(d\varphi_0/d\omega'_0)^{1/2}(d\omega_0/d\omega'_0)^{-1/2}\Omega \\ &= a(d\varphi_0/d\omega'_0)^{1/2}(d\omega_0/d\omega'_0)^{-1/2}J_\omega S^* J_\omega\Omega \\ &= a(d\varphi_0/d\omega'_0)^{1/2}(d\omega_0/d\omega'_0)^{-1/2}\Delta_\omega^{1/2}S\Omega \\ &= a(d\varphi_0/d\omega'_0)^{1/2}(d\omega_0/d\omega'_0)^{-1/2}\Delta_\omega^{1/2} \\ &= \Delta_\omega^{-1/2}(d\omega_0/d\omega'_0)^{1/2}(d\varphi_0/d\omega'_0)^{-1/2}(d\varphi/d\omega')^{1/2}\Omega \\ &= a(d\varphi/d\omega')^{1/2}\Omega = a\Omega(\varphi). \end{aligned}$$

This shows that in fact $w' = v'_\varphi$ and according to Lemma 3.3, $\varphi \leq \lambda\tilde{\varphi}_0$.

If $a \in M$, then

$$\begin{aligned} \varphi(a) &= \langle a\Omega(\varphi), \Omega(\varphi)\rangle = \langle w'a\Phi_0, w'\Phi_0\rangle = \langle w'^* w'a\Phi_0, \Phi_0\rangle \\ &= \langle J_\omega SS^* J_\omega a\Phi_0, \Phi_0\rangle \end{aligned}$$

and it equals $\langle a\Phi_0, \Phi_0\rangle = \tilde{\varphi}_0(a)$ provided $SS^* = I$.

We note that the proof gives a simple relation between S and v'_φ. Namely, $J_\omega S J_\omega = (v'_\varphi)^*$.

THEOREM 3.7. *Let M, M_0, φ_0, ω'_0, and ω' be as above. If ω_1 and ω_2 are faithful normal states on M so that the generalized conditional expectations $E_\omega\colon M \to M_0$ and $E_\omega\colon M \to M_0$ coincide, then $(\varphi_0)^{\sim\omega_1} = (\varphi_0)^{\sim\omega_2}$.*

Proof. We verify that

$$(d\varphi'_0/d\omega'_0)^{1/2}(d(\omega_1|M_0)/d\omega'_0)^{-1/2}\Omega_1 \\ = (d\varphi_0/d\omega'_0)^{1/2}(d(\omega_2|M_0)/d\omega'_0)^{-1/2}\Omega_2$$

if $\Omega_1, \Omega_2 \in H$ are vector representatives of ω_1 and ω_2 in the natural positive cone.

Due to Corollary 4 in **[12]** we have

$$[D(\omega_1|M_0), D(\omega_2|M_0)]_t = [D\omega_1, D\omega_2]_t \qquad (t \in \mathbf{R})$$

at our disposal. By analytical extension (cf. Proposition 2.2) we obtain

$$(d(\omega_1|M_0)/d\omega'_0)^{1/2}(d(\omega_2|M_0)/d\omega'_0)^{-1/2}\Omega_2 \\ = (d\omega_1/d\omega'_0)^{1/2}(d\omega_2/d\omega'_0)^{-1/2}\Omega_2.$$

Hence

$$(d\varphi_0/d\omega'_0)^{1/2}(d(\omega_1|M_0)/d\omega'_0)^{-1/2}\Omega_1 \\ = (d\varphi_0/d\omega'_0)^{1/2}(d(\omega_1|M_0)/d\omega'_0)^{-1/2} \\ \times (d\omega_1/d\omega'_0)^{1/2}(d\omega_2/d\omega'_0)^{-1/2}\Omega_2 \\ = (d\varphi_0/d\omega'_0)^{1/2}(d(\omega_2|M_0)/d\omega'_0)^{-1/2}\Omega_2.$$

It has turned out that the canonical extension of φ_0 with respect to ω depends rather on E_ω than on ω itself.

THEOREM 3.8. *Let M, M_0, φ_0, ω'_0, ω', ω, ω_0 and $E_\omega\colon M \to M_0$ be as above. If $E_\omega([D\varphi_0, D\omega_0]_t) = [D\varphi_0, D\omega_0]_t$ for all $t \in \mathbf{R}$, then $(\varphi_0)^{\sim\omega} = \varphi_0 \cdot E_\omega$.*

Proof. Let M_1 be the fixed point algebra of E_ω and we denote by φ_1 and ω_1 the restrictions of φ and ω to M_1, respectively. Due to **[12]** $[D\varphi_0, D\omega_0]_t \in M_1$ implies $[D\varphi_0, D\omega_0]_t = [D\varphi_1, D\omega_1]_t$ $(t \in \mathbf{R})$. Through analytic continuation we have

$$(d\varphi_0/d\omega'_0)^{1/2}(d\omega_0/d\omega'_0)^{-1/2}\Omega = (d\varphi_1/d\omega'_1)^{1/2}d(\omega_1/d\omega'_1)^{-1/2}\Omega$$

and we obtain that the canonical extensions of φ_0 and φ_1 with respect to ω are the same.

Let F_ω be the ω-conditional expectation of M into M_1. Actually, it is a projection of norm one. Set $\varphi = \varphi_1 \cdot F_\omega$. Since

$$\langle (d\omega_1/d\omega'_1)^z(d\varphi_1/d\omega'_1)^{-z}(d\varphi/d\omega')^z\xi', (d(\omega/d\omega')^{-\bar z}\eta\rangle$$

is analytic on S for $\xi' \in D(H, \omega')$ and $\eta \in D(H, \omega)$, furthermore

$$[D\varphi, D\omega]_t = [D\varphi_1, D\omega_1]_t \qquad (t \in \mathbf{R})$$

we conclude that

$$\langle (d\omega_1/d\omega_1')^{1/2}(d\varphi_1/d\omega_1')^{-1/2}(d\varphi/d\omega')^{1/2}\xi', (d\omega/d\omega')^{-1/2}\eta\rangle = \langle \xi', \eta'\rangle.$$

Consequently, $(d\omega/d\omega')^{-1/2}(d\omega_1/d\omega_1')^{1/2}(d\varphi_1/d\omega_1')^{-1/2}(d\varphi/d\omega')^{1/2}$ is defined on $D(H, \omega')$ and admits a bounded closure, the identity. Theorem 3.6 is applicable and tells us that $\varphi = (\varphi_1)^{\sim\omega}$. So $\varphi \cdot E_\omega = \varphi_1 \cdot F_\omega \cdot E_\omega = \varphi_1 \cdot F_\omega = \varphi$ and φ is faithful. Reference to **[12]** gives that

$$[D\varphi, D\omega]_t = [D\varphi_1, D\omega_1]_t = [D(\varphi|M_0), D\omega_0]_t \qquad (t \in \mathbf{R}).$$

Therefore, $\varphi|M_0 = \varphi_0$ and we obtain $(\tilde{\varphi}_0)^\omega = \varphi = \varphi_0 \circ E_\omega$.

It follows in particular from Theorem 3.8 that if E_ω is a projection then $(\tilde{\varphi}_0)^\omega$ is always $\varphi_0 \circ E_\omega$. The following example shows that in general $\varphi_0 \circ E_\omega$ is not an extension of φ_0.

EXAMPLE 3.9. Let $M_0 \subsetneq M \subset B(H)$ and Ω be a cyclic and separating vector both for M_0 and M. If ω is the vector state on M given by Ω, then the ω-conditional expectation $E_\omega \colon M \to M_0$ is an algebra isomorphism and its range M_1 is a proper von Neumann subalgebra of M_0 (cf. **[1]**, p. 259). If φ_0 is a state on M_0 such that $\varphi_0 \neq \omega|M_0$, however $\varphi_0|M_1 = \omega|M_1$, then $\varphi_0 \circ E_\omega = \omega$, but $\varphi_0 \neq \omega|M_0$.

4. A Radon-Nikodym theorem. Connes proved (**[6]**) that if φ and ω are faithful normal states on the von Neumann algebra M and $\varphi \leq \lambda\omega$, then $\varphi(x) = \omega(axa^*)$ with an appropriate $a \in M$. Since states can be considered as conditional expectations onto the trivial subalgebra the following theorem generalizes his result.

THEOREM 4.1. *Let M and M_0 be von Neumann algebras with $M_0 \subset M$ and $\varphi, \omega \in M_*^+$. Assume that ω and $\varphi_0 = \varphi|M_0$ are faithful and $\varphi \leq \lambda(\varphi_0)^{\sim\omega}$. ($(\varphi_0)^{\sim\omega}$ stands for the ω-extension of φ_0 with respect to ω). Then there exists $a \in M$ such that $E_\omega(axa^*) = E_\varphi(x)$ for every $x \in M$.*

Proof. By two applications of Proposition 1.11, we have

$$\begin{aligned} E_\varphi(\Theta^{\omega'}(\xi')) &= \Theta^{\omega_0'}((d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2}\xi') \\ &= E_\omega(\Theta^{\omega'}((d\omega/d\omega')^{-1/2}(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2}\xi')) \end{aligned}$$

for $\xi' \in D(H, \omega')$. (ω_0' is a faithful normal state on M_0' and $\omega' = \omega_0'|M'$.) According to Theorem 3.4 the closure of

$$(d\omega/d\omega')^{-1/2}(d\omega_0/d\omega_0')^{1/2}(d\varphi_0/d\omega_0')^{-1/2}(d\varphi/d\omega')^{1/2}$$

is in M. Hence

$$E_\omega(a\Theta^{\omega'}(\xi')a^*) = E_\varphi(\Theta^{\omega'}(\xi')) \quad \text{for } \xi' \in D(H, \omega').$$

As the linear hull of $\{\Theta^{\omega'}(\xi') : \xi' \in D(H, \omega')\}$ is dense in M we proved the theorem.

COROLLARY 4.2. *There exists an isometry $v \in M$ with range projection* $\mathrm{supp}(\varphi_0)^{\sim\omega}$ *such that*

$$E_\omega(v^* x v) = E_\psi(x) \qquad (x \in M)$$

if ψ stands for $(\varphi_0)^{\sim\omega}$.

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