PACIFIC JOURNAL OF MATHEMATICS
Vol. 138, No. 1, 1989

# ELEMENTS OF FINITE ORDER IN $V(ZA_4)$

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**The conjugacy classes for all elements of finite order in the unit group $V(ZA_4)$ are determined. As an application, it is shown that all normal complements to $A_4$ in $V(ZA_4)$ must be torsion free**

Let $V(ZG)$ denote the group of units of augmentation 1 in the integral group ring $ZG$. There is considerable interest in determining whether the group $G$ has a torsion free normal complement in $V(ZG)$. The authors showed in **[2]** that $S_3$ has two types of normal complements in $V(ZS_3)$, one with torsion and one without. They have also shown (see **[1]**) that $A_4$ has a torsion free normal complement in $V(ZA_4)$ and that $S_4$ has a normal complement in $V(ZS_4)$ which includes torsion elements (see **[3]**). Two questions arise naturally:

1. Can $A_4$ also have a normal complement in $V(ZA_4)$ which includes torsion?
2. Can $S_4$ also have a torsion free normal complement in $V(ZS_4)$?

This paper gives a negative answer to Question 1 by completing the task of finding all of the conjugate classes of elements of finite order in $V(ZA_4)$ and then showing that a subgroup containing any such class must also contain an element of order 2 in $A_4$. Earlier work has shown that the torsion elements of $V(ZA_4)$ are of order 2 or 3 and that all elements of order 2 are conjugate **[1]**. Sekiguchi **[4]** showed that there are four conjugate classes of subgroups of $V(ZA_4)$ which are isomorphic to $A_4$. It follows from his work that there are at least eight conjugate classes of elements of order 3; these classes include all of the elements of order 3 which lie in subgroups isomorphic to $A_4$. Our Theorem 1 shows that there are exactly four additional conjugate classes of elements of order 3 which do not lie in any subgroup isomorphic to $A_4$. Theorem 2 gives the answer to Question 1.

The results of **[1]** characterize $V(ZA_4)$ as an explicit subgroup of $\mathrm{SL}(3,Z)$ and thus permit us to utilize information about $\mathrm{SL}(3,Z)$. The characterization relies on the following definition: If $X = [x_{ij}] \in \mathrm{SL}(3,Z)$, then the *pseudotraces* $t_0$, $t_1$, and $t_2$ are given by $t_0 = x_{11} + x_{22} + x_{33}$, $t_1 = x_{12} + x_{23} + x_{31}$, and $t_2 = x_{13} + x_{21} + x_{32}$. Then we can think of $A_4$ as generated by

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and of $V(ZA_4)$ as $\{X \in \mathrm{SL}(3, Z) | X$ satisfies conditions (1) and (2)$\}$ where

*condition* (1): $X \equiv B^i \pmod 2$ for some $i$

and

*condition* (2): two of the pseudotraces $t_j$ are 0 modulo 4.

We begin by finding the centralizer of $B$ in the ring $M_3(Q)$ of all $3 \times 3$ rational matrices.

LEMMA 1. *Let $X \in M_3(Q)$. Then $XB = BX$ if and only if $X = \sum r_i B^i$ with $r_i \in Q$. Moreover, if $X \in \mathrm{SL}(3, Z)$, then*

(i) $XB \equiv BX \pmod 2$ *if and only if* $X \equiv B^i \pmod 2$ *and*

(ii) $XB = BX$ *if and only if* $X = B^i$.

*Proof*. If $X = \sum r_i B^i$, then it is clear that $XB = BX$. On the other hand, an inspection of the entries in the matrices $XB$ and $BX$ will show that $XB = BX$ implies that $X = \sum r_i B^i$ for some $r_i \in Q$. The remainder of the lemma follows from the fact that the group ring $R\langle B \rangle$ has only trivial units if $R$ is the ring of integers modulo 2 or if $R = Z$.

Let $I_2$ and $I_4$ denote, respectively, the subgroups of $\mathrm{SL}(3, Z)$ consisting of all matrices which are the identity modulo 2 and 4. It is clear that $I_4$ is contained in $V(ZA_4)$ and that $\langle B \rangle I_2$ is the subgroup consisting of all matrices which satisfy condition (1).

LEMMA 2. *$V(ZA_4)$ is a normal subgroup of $\langle B \rangle I_2$. The factor group is the elementary group of order 4 with coset representatives $R_0 = I$,*

$$R_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

*and $R_3 = R_1 R_2$.*

*Proof*. If $M \in I_2$ then one of the $MR_i$ must belong to $V(ZA_4)$ since multiplying $M$ on the right by $R_1$ or $R_2$ has the effect, modulo 4, of adding 2 to $t_1$ or $t_2$. Thus precisely one of the $MR_i$ will satisfy condition (2). Since $B \in V(ZA_4)$ it follows that the $R_i$ are a full set of coset representatives of $V(ZA_4)$ in $\langle B \rangle I_2$. The square of each of these representatives is in $I_4$ and thus in $V(ZA_4)$. A direct calculation shows that $B^{R_1}$ and $B^{R_2}$ are in $V(ZA_4)$, thus the normality of $V(ZA_4)$ follows from the fact that $I_2$ is abelian modulo the subgroup $I_4$ of $V(ZA_4)$.

The next lemma is well known. In fact, Tahara [5] describes all of the conjugate classes of finite subgroups of $\mathrm{SL}(3, Z)$.

LEMMA 3. $\mathrm{SL}(3, Z)$ *contains exactly two conjugate classes of elements of order* 3. *One of these classes contains* $B$. *The other one contains*

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

In $\mathrm{SL}(3, Z)$, $B$ is conjugate to $B^2$, but this cannot happen in $V(ZA_4)$, or even in $\langle B\rangle I_2$, since conjugating $B$ by an element in $\langle B\rangle I_2$ produces an element in $BI_2$. We shall restrict our attention to the conjugate classes in $V(ZA_4)$ of elements congruent to $B$ modulo 2; the squares of the elements in each class will be a conjugate class of elements congruent to $B^2$ modulo 2.

Lemmas 1 and 2 yield a complete description of all of the conjugate classes in $V(ZA_4)$ of elements which are conjugate in $B$ in $\mathrm{SL}(3, Z)$ and are congruent to $B$ modulo 2. As we will see later, additional classes arise from conjugates of $W$.

LEMMA 4. *Conjugating* $B$ *by the four coset representatives* $R_i$ *of Lemma* 2 *produces elements of four conjugate classes in* $V(ZA_4)$. *Any conjugate of* $B$ *in* $\mathrm{SL}(3, Z)$ *which is congruent to* $B$ *modulo* 2 *and belongs to* $V(ZA_4)$ *will lie in one of these classes.*

*Proof*. Suppose that $B^{R_i} = B^{R_j M}$ for some $M \in V(ZA_4)$. Then $R_j M R_i^{-1}$ commutes with $B$ so, by Lemma 1,

$$R_j M R_i^{-1} \in \langle B\rangle.$$

It follows from the normality of $V(ZA_4)$ that $R_j R_i^{-1} \in V(ZA_4)$, thus $R_i = R_j$. Consequently, the $B^{R_i}$ lie in distinct conjugate classes of $V(ZA_4)$.

Next, suppose that $X \equiv B \pmod 2$, that $X \in V(ZA_4)$, and that $X = B^M$ for some $M \in \mathrm{SL}(3, Z)$. Then $B \equiv B^M \pmod 2$ so by Lemma 1, $M \equiv B^i \pmod 2$ for some $i$. It follows from Lemma 2 that $M = R_j N$ for some $j$ and some $N \in V(ZA_4)$. Thus $X$ is conjugate in $V(ZA_4)$ to $B^{R_j}$.

There are elements of order 3 in $V(ZA_4)$ which are congruent to $B$ modulo 2 and conjugate to $W$ in $\mathrm{SL}(3, Z)$. (Because of Lemma 3, such elements cannot be conjugate to any of the $B^{R_i}$.) In fact, if

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

then

$$W^T = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & -3 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad W^{TR_1} = \begin{bmatrix} 0 & 5 & 8 \\ 0 & -2 & -3 \\ 1 & 4 & 2 \end{bmatrix}$$

are in $V(ZA_4)$ and are congruent to $B$ modulo 2. We shall show that these two elements lie in different conjugate classes in $V(ZA_4)$ and that every element of $V(ZA_4)$ which is congruent to $B$ modulo 2, and conjugate to $W$ in $\mathrm{SL}(3, Z)$, is conjugate in $V(ZA_4)$ to one of them.

LEMMA 5. *$W^T$ and $W^{TR_1}$ are not conjugate in $V(ZA_4)$.*

*Proof*. We begin by observing that if $M \in V(ZA_4)$ then conditions (1) and (2) imply that the sum of the entries in $M$ must be 3 modulo 4. By condition (1), the entries on one pseudotrace are $1 + e_1$, $1 + e_2$, $1 + e_3$ where the $e_i$ are even, and all other entries of $M$ are even. Consequently, $1 = |M| \equiv 1 + e_1 + e_2 + e_3 \pmod 4$, so the pseudotrace with odd entries is 3 modulo 4 and it follows from condition (2) that the sum of the entries of $M$ is 3 modulo 4.

Now suppose that $W^{TM} = W^{TR_1}$ for some $M \in V(ZA_4)$. Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

and observe that $B^P = W^T$, so $B^{PM} = B^{PR_1}$. By Lemma 1, if $X = PMR_1^{-1}P^{-1}$, then

$$X = sI + tB + uB^2$$

for some rational numbers $s$, $t$, and $u$. Each of the column sums of $X$ is $s+t+u$; thus, if we start to evaluate $|X|$ by adding the first two rows to the third, we see that $|X| = (s + t + u)(s^2 + t^2 + u^2 - st - su - tu)$. Next, note that $X^P = MR_1^{-1}$ is an integer matrix of determinant 1 which also has column sums $s + t + u$ since $I$, $B^P$, and $(B^2)^P$ have column sums of 1. Therefore, $s + t + u$ is an integer. If we start to evaluate $|X^P|$ by adding the first two rows to the third, then factoring out $s+t+u$, we see that

$$1 = |X^P| = (s+t+u)n$$

for some integer $n$. It is now clear from the form of $|X|$ that $s+t+u$ and $s^2+t^2+u^2-st-su-tu$ are both 1 or both $-1$. If

$$s^2+t^2+u^2 = st+su+tu-1$$

then $st+su+tu \geq 1$. Hence

$$1 = (s+t+u)^2 = s^2+t^2+u^2+2(st+su+tu) \geq 2,$$

a contradiction. Therefore $s+t+u=1$.

We now know that $X^P = MR_1^{-1}$ where the column sums of $X^P$ are each 1. Multiplying $X^P$ on the right by $R_1$ adds twice the first column to the second, thus the column sums of $M$ are, respectively, 1, 3, and 1. But then the sum of the entries of $M$ is 1 modulo 4, a contradiction.

We found that the $B^{R_i}$ come from four different classes. One might expect that the $W^{TR_i}$ would come from four new classes. Lemma 5 has shown that $W^T$ and $W^{TR_1}$ do come from different classes. These turn out to be the only new classes.

LEMMA 6. *Each $W^{TR_i}$ is conjugate in $V(ZA_4)$ either to $W^T$ or to $W^{TR_1}$.*

*Proof*. It suffices to show that $W^{TR_1R_2}$ is $W^{TM}$ for some $M$ in $V(ZA_4)$. It will follow that $W^{TR_1}$ and $W^{TR_2}$ are in the same class, since $R_2^2 \in V(ZA_4)$. The matrix

$$M = \begin{pmatrix} 1 & 4 & 4 \\ -2 & -7 & -6 \\ 2 & 6 & 5 \end{pmatrix}$$

has the required properties.

The next lemma shows that the 6 classes found in Lemmas 4 and 5 account for all of the elements of order 3 in $V(ZA_4)$ which are congruent to $B$ modulo 2.

LEMMA 7. *Suppose that $X \in V(ZA_4)$, that $X \equiv B \pmod 2$, and that $X = W^M$ for some $M \in \mathrm{SL}(3, Z)$. Then $X$ is conjugate in $V(ZA_4)$ to one of $W^T$ and $W^{TR_1}$.*

*Proof*. By hypothesis, $WM \equiv MB \pmod 2$. If $M = [m_{ij}]$, then a comparison of the entries of $WM$ and $MB$ shows that

$$M \equiv \begin{bmatrix} m_{11} & m_{11} & m_{11} \\ m_{21} & m_{22} & m_{21}+m_{22} \\ m_{21}+m_{22} & m_{21} & m_{22} \end{bmatrix} \pmod 2.$$

Since $|M| = 1$, $m_{11}$ must be odd, and not both of $m_{21}$, $m_{22}$ can be even. Thus, modulo 2, $M$ is one of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We shall need the matrix $P$ such that $B^P = W^T$ (see Lemma 5), the matrix

$$U = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

which can be seen to have the property that $B^U = W$, and the matrix

$$K = sI + tW^T + u(W^2)^T = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 2 \\ 0 & -1 & -2 \end{pmatrix}$$

where $s = t = -2/3$, $u = 1/3$.

We now let

$$G = K^{-1}P^{-1}UM = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} M.$$

Then $G \in \mathrm{SL}(3, Z)$ and it follows from our information about the form of $M$ modulo 2 that $G \in \langle B \rangle I_2$. Thus, By Lemma 2, $GR_i \in V(ZA_4)$ for some $i$.

Note that

$$UM = PKG = (PKP^{-1})PG$$

where $PKP^{-1}$ commutes with powers of $B$ since it is a sum of powers of $B$. Therefore,

$$X = W^P = B^{UM} = B^{PG} = W^{TG}.$$

Thus, $X^{R_i} = W^{T(GR_i)}$ is a conjugate of $W^T$ in $V(ZA_4)$. It follows from Lemmas 2 and 6 that $X$ is conjugate in $V(ZA_4)$ either to $W^T$ or to $W^{TR_1}$.

THEOREM 1. *$V(ZA_4)$ contains precisely 12 conjugacy classes of elements of order 3. The elements $B^{R_i}$, $i = 0, 1, 2, 3$, together with $W^T$*

*and* $W^{TR_1}$ *are representatives of the* 6 *conjugacy classes that are congruent to* $B$ *modulo* 2; *their squares are representatives of the other* 6 *classes.*

*Proof*. The theorem is immediate in view of Lemmas 3–7. As we noted after stating Lemma 3, it suffices to find the classes for elements congruent to $B$ modulo 2; there are then corresponding classes for elements congruent to $B^2$ modulo 2. Lemma 3 narrowed the search to conjugates of $B$ and $W$ in $\mathrm{SL}(3, Z)$. Lemma 4 described the classes arising from conjugates of $B$. Lemma 5 exhibited two distinct classes arising from conjugates of $W$; Lemma 6 showed that these were the only new classes generated from $W^T$ by the $R_i$; Lemma 7 showed that any class arising from a conjugate of $W$ has to be one produced from $W^T$ by an $R_i$.

The authors have shown (see **[1]**)

LEMMA 8. *All elements of order* 2 *in* $V(ZA_4)$ *are conjugate in* $V(ZA_4)$.

Theorem 1 and Lemma 8 account for all the conjugacy classes of elements of finite order in the unit group $V(ZA_4)$. If $N$ is any normal subgroup of $V(ZA_4)$ containing an element of order 2, then it follows from Lemma 8 that $A \in N$. Thus, a normal complement to $A_4$ in $V(ZA_4)$ cannot contain an element of order 2. We shall now show that any normal subgroup containing an element of order 3 must also contain an element of order 2 and thus establish

THEOREM 2. *All normal complements to* $A_4$ *in* $V(ZA_4)$ *are torsion free.*

*Proof*. Let $N$ be a normal subgroup of $V(ZA_4)$ containing an element of order 3. In view of Theorem 1, it follows that $N$ contains one of the $B^{R_i}$ or $W^T$ or $W^{TR_1}$.

*Case* 1. Suppose $B^{R_i} \in N$. A routine calculation shows that $A^{R_i} \in V(ZA_4)$ for each $i$. In $A_4$, the commutator $(A, B)$ is an element of order 2; therefore $(A, B)^{R_i}$ is an element of order 2 which lies in $N$.

*Case* 2. Suppose that $N$ contains $W^T$ or $W^{TR_i}$.

Let

$$M_1 = \begin{pmatrix} -1 & 0 & -2 \\ 2 & -1 & 2 \\ -2 & 0 & -3 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -3 & 0 \\ 2 & 2 & 1 \end{pmatrix},$$

and note that the $M_i \in V(ZA_4)$.

Let

$$X = W^T W^{TM_1} W^{TM_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 8 & 8 & 1 \end{pmatrix}.$$

Then $X$ is an element of order 2 in $V(ZA_4)$ which will lie in $N$ if $N$ contains $W^T$. Also, since $R_1$ normalizes $V(ZA_4)$, any normal subgroup containing $W^{TR_1}$ must contain $W^{TR_1 H_i}$ where $H_i = M_i^{R_1}$ and thus will contain $X^{R_1}$.

REMARK. The proof for Case 1 amounted to showing that any normal subgroup containing a $B^{R_1}$ must contain a conjugate of $A_4$. As Sekiguchi showed in **[4]**, $V(ZA_4)$ contains just 4 conjugate classes of groups isomorphic to $A_4$. Our elements $W^T$ and $W^{TR_1}$ are not contained in subgroups of $V(ZA_4)$ which are isomorphic to $A_4$. For example, $W^T = B^P$ but $A^P \notin V(ZA_4)$ so $\langle B^P, A^P \rangle$ is isomorphic to $A_4$ but it is not contained in $V(ZA_4)$.

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Received February 28, 1987.

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