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EXTENSION OF THE THEOREMS OF CARATHÉODORY-TOEPLITZ-SCHUR AND PICK

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Let $z_1, \dots, z_k$ be distinct points in the unit disc D and let $c_{i\alpha}$ $(\alpha = 0, \dots, n_i - 1)$ be n_i complex numbers for each z_i. There exists a holomorphic function f in D with $|f| \le 1$ whose first n_i Taylor coefficients at z_i are the prescribed values $c_{i\alpha}$ if and only if the Hermitian matrix defined by z_i and $c_{i\alpha}$ is positive semidefinite.

1. Introduction. Let $D = \{z : |z| < 1\}$ be the open unit disc in $\mathbf{C}$ and let $\mathscr{B}$ denote the set of holomorphic functions f in D such that $|f| \le 1$ in D. Let $\{z_1, \dots, z_k\}$ be a k-tuple of distinct points in D. At each point z_i, let there be given a sequence of n_i complex numbers $\{c_{i0}, \dots, c_{in_i-1}\}$. We want to show that there exists an $f \in \mathscr{B}$ such that

$$f(z) = \sum_{\alpha=0}^{n_i-1} c_{i\alpha}(z - z_i)^\alpha + O((z - z_i)^{n_i}) \qquad (i = 1, \dots, k)$$

if and only if the Hermitian matrix A, defined in §3 in terms of z_i, $c_{i\alpha}$, is positive semidefinite. This result contains both Schur's theorem [6], derived from the results of Carathéodory [2] and Toeplitz [7], and Pick's theorem [5]. Garnett announced in his textbook [3] that Cantor found the matrix condition necessary and sufficient for interpolation by finitely many derivatives of a function in $\mathscr{B}$ at finitely many points in D. But, because Cantor never wrote up his result, he and Garnett encouraged us to publish this paper. Our proof is based on Marshall's method by induction [4].

2. Preliminaries. Schur's triangular matrix provides a very efficient tool. To a function

$$f(z) = \sum_{\alpha=0}^{\infty} c_\alpha (z - z_0)^\alpha$$

holomorphic at z_0 and to a positive integer n, assign a triangular $n \times n$ matrix

$$\Delta(f; z_0; n) = \begin{pmatrix} c_0 & & & \\ c_1 & c_0 & & \\ \vdots & \ddots & \ddots & \\ c_{n-1} & \cdots & c_1 & c_0 \end{pmatrix}.$$

For another $g(z)$, we have at once

$$\Delta(f + g; z_0; n) = \Delta(f; z_0; n) + \Delta(g; z_0; n);$$
$$\Delta(fg; z_0; n) = \Delta(f; z_0; n) \cdot \Delta(g; z_0; n) = \Delta(g; z_0; n) \cdot \Delta(f; z_0; n);$$
$$\Delta(1; z_0; n) = I_n \quad \text{(the unit matrix of order } n).$$

To a function

$$F(z, \bar{\zeta}) = \sum_{\alpha,\beta=0}^{\infty} a_{\alpha\beta}(z - z_0)^{\alpha}(\overline{\zeta - \zeta_0})^{\beta} \tag{1}$$

holomorphic with respect to $(z, \bar{\zeta})$ at $(z_0, \bar{\zeta}_0)$ and to a pair of positive integers (m, n), associate an $m \times n$ matrix

$$\mathbf{M}(F; z_0, \bar{\zeta}_0; m, n) = \begin{pmatrix} a_{00} & \cdots & a_{0\ n-1} \\ \vdots & & \vdots \\ a_{m-1\ 0} & \cdots & a_{m-1\ n-1} \end{pmatrix},$$

which will be our main tool. For another $G(z, \bar{\zeta})$, we have

$$\begin{aligned} \mathbf{M}(F + G; z_0, \bar{\zeta}_0; m, n) = \mathbf{M}(F; z_0, \bar{\zeta}_0; m, n) \\ + \mathbf{M}(G; z_0, \bar{\zeta}_0; m, n). \end{aligned} \tag{2}$$

Moreover, if $f(z)$ and $g(\zeta)$ are holomorphic at z_0 and ζ_0, a simple calculation gives

$$\begin{aligned} &\mathbf{M}(fF\bar{g}; z_0, \bar{\zeta}_0; m, n) \\ &\quad = \Delta(f; z_0; m) \cdot \mathbf{M}(F; z_0, \bar{\zeta}_0; m, n) \cdot \Delta(g; \zeta_0; n)^*, \end{aligned} \tag{3}$$

where Δ^* denotes the transpose of the complex-conjugate of a matrix Δ. We shall use in the sequel the matrix

$$E_{mn}^{(\alpha\beta)} = \mathbf{M}((z - z_0)^{\alpha}(\overline{\zeta - \zeta_0})^{\beta}; z_0, \bar{\zeta}_0; m, n), \tag{4}$$

which is an $m \times n$ matrix whose $(\alpha + 1, \beta + 1)$-element is 1 and the other elements are all 0 $(0 \leq \alpha \leq m - 1, 0 \leq \beta \leq n - 1)$.

Now, let us establish a transformation formula. Let $F(z, \bar{\zeta})$, $z = \varphi(x)$ and $\zeta = \psi(\xi)$ be functions holomorphic at $(z_0, \bar{\zeta}_0)$, x_0 and ξ_0

respectively. Assume $z_0 = \varphi(x_0)$ and $\zeta_0 = \psi(\xi_0)$. Let (m, n) be a pair of positive integers. Define

$$G(x, \bar{\xi}) = F(\varphi(x), \overline{\psi(\xi)}).$$

For the simplicity, we write, with $E_m^{(\alpha)} = E_{mm}^{(\alpha\alpha)}$,

$$\varphi(x) = z_0 + (x - x_0)\varphi_1(x), \qquad \Phi = \Delta(\varphi_1; x_0; m),$$
$$\Phi^0 = I_m, \qquad \Phi^{\alpha+1} = \Phi^\alpha \cdot \Phi$$

and

$$\Omega(\varphi; x_0; m) = \sum_{\alpha=0}^{m-1} \Phi^\alpha \cdot E_m^{(\alpha)}. \tag{5}$$

Similarly,

$$\psi(\xi) = \zeta_0 + (\xi - \xi_0)\psi_1(\xi), \qquad \Psi = \Delta(\psi_1; \xi_0; n),$$
$$\Omega(\psi; \xi_0; n) = \sum_{\beta=0}^{n-1} \Psi^\beta \cdot E_n^{(\beta)}.$$

LEMMA. *With these notations, the transformation formula is*

$$\mathbf{M}(G; x_0, \bar{\xi}_0; m, n) = \Omega(\varphi; x_0; m) \cdot \mathbf{M}(F; z_0, \bar{\zeta}_0; m, n) \cdot \Omega(\psi; \xi_0; n)^*. \tag{6}$$

Note that if $\varphi'(x_0) = \varphi_1(x_0) \neq 0$ then $\Omega(\varphi; x_0; m)$ is a regular matrix, because $\Omega(\varphi; x_0; m)$ is triangular and its diagonal elements are $(\varphi_1(x_0))^\alpha$ $(\alpha = 0, \ldots, m-1)$.

Proof. By means of (2) and (3), we derive from (1) and

$$G(x, \bar{\xi}) = \sum_{\alpha=\beta=0}^{\infty} a_{\alpha\beta}\varphi_1(x)^\alpha (x - x_0)^\alpha \overline{(\xi - \xi_0)}^\beta \overline{\psi_1(\xi)}^\beta$$

that

$$\begin{aligned}
&\mathbf{M}(G; x_0, \bar{\xi}_0; m, n) \\
&\quad = \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} a_{\alpha\beta} \Delta(\varphi_1^\alpha; x_0; m) \cdot E_{mn}^{(\alpha\beta)} \cdot \Delta(\psi_1^\beta; \xi_0; n)^* \\
&\quad = \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} a_{\alpha\beta} (\Phi^0 \cdot E_m^{(0)} + \cdots + \Phi^{m-1} \cdot E_m^{(m-1)}) \\
&\qquad\qquad \cdot E_{mn}^{(\alpha\beta)} \cdot (\Psi^0 \cdot E_n^{(0)} + \cdots + \Psi^{n-1} \cdot E_n^{(n-1)})^*,
\end{aligned}$$

because $E_m^{(\lambda)} \cdot E_{mn}^{(\alpha\beta)} \cdot E_n^{(\mu)*} = \delta_{\lambda\alpha} \cdot \delta_{\mu\beta} \cdot E_{mn}^{(\alpha\beta)}$. Then we have (6). □

Let $f(z) = \sum_{\alpha=0}^{\infty} c_\alpha (z-z_0)^\alpha$ and $z = \varphi(x)$ be holomorphic functions at z_0 and x_0 with $z_0 = \varphi(x_0)$ and set

$$g(x) = f(\varphi(x)) = \sum_{\alpha=0}^{\infty} d_\alpha (x - x_0)^\alpha.$$

Write $\Omega = \Omega(\varphi; x_0; m)$. Calculating directly or putting $F(z, \bar{\zeta}) = f(z)$ and $n = 1$ in (6), we obtain

$$(d_0, \dots, d_{m-1}) = (c_0, \dots, c_{m-1}) \cdot {}^t\Omega. \tag{7}$$

A Blaschke product of degree n ($n \geq 0$) is a function $f \in \mathscr{B}$ of the form

$$f(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z} \qquad (\theta \in \mathbf{R},\ |z_j| < 1\ (j = 1, \dots, n)).$$

If f is a Blaschke product of degree n and if τ is a Möbius transformation, that is an analytic automorphism of the (open or closed) unit disc, then both $f \circ \tau$ and $\tau \circ f$ are Blaschke products of degree n.

3. Main theorem. Let $\{z_1, \dots, z_k\}$ be a finite set of distinct points in D ($k \geq 1$) and, for each $i = 1, \dots, k$, let $\{c_{i0}, \dots, c_{in_i-1}\}$ be a finite sequence of n_i complex numbers ($n_i \geq 1$). Set $n = n_1 + \cdots + n_k$ and write

$$C_i = \begin{pmatrix} c_{i0} & & \\ \vdots & \ddots & \\ c_{in_i-1} & \cdots & c_{i0} \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_k \end{pmatrix},$$

$$\Gamma_{ij} = \mathbf{M}(1/(1 - z\bar{\zeta}); z_i, \bar{z}_j; n_i, n_j), \quad A_{ij} = \Gamma_{ij} - C_i \cdot \Gamma_{ij} \cdot C_j^*,$$

$$\Gamma = \begin{pmatrix} \Gamma_{11} \cdots \Gamma_{1k} \\ \cdots\cdots \\ \Gamma_{k1} \cdots \Gamma_{kk} \end{pmatrix}, \qquad A = \begin{pmatrix} A_{11} \cdots A_{1k} \\ \cdots\cdots \\ A_{k1} \cdots A_{kk} \end{pmatrix}.$$

Then

$$A = \Gamma - C \cdot \Gamma \cdot C^*.$$

We see easily that both Γ and A are Hermitian $n \times n$ matrices. We shall write $A \geq 0$ if A is positive semidefinite (nonnegative).

A function f is said to be solution of (EI) in $\mathscr{B}$ if $f \in \mathscr{B}$ and if the extended interpolation conditions

$$f(z) = \sum_{\alpha=0}^{n_i-1} c_{i\alpha} (z - z_i)^\alpha + O((z - z_i)^{n_i}) \qquad (i = 1, \dots, k) \tag{EI}$$

are satisfied. With these notations and terminologies, we state our main

THEOREM. *There exists a solution of* (EI) *in* $\mathscr{B}$ *if and only if* $A \geq 0$. *If* $A \geq 0$ *and* $\det A = 0$, *then the solution of* (EI) *in* $\mathscr{B}$ *is unique and is a Blaschke product, whose degree is equal to the rank of* A. *If* $A \geq 0$ *and* $\det A > 0$, *then there are infinitely many solutions of* (EI) *in* $\mathscr{B}$ *and, among them, there is a Blaschke product of degree* n.

Note that if $A \geq 0$, $\det A \neq 0$ and if z_0 is a point in D different from $z_1, \dots, z_k$ then the set of values $\{f(z_0) \colon f$ is solution of (EI) in $\mathscr{B}\}$ as well as $\{f^{(n_i)}(z_i) \colon f$ is solution of (EI) in $\mathscr{B}\}$ is a nondegenerate closed disc in $\mathbf{C}$. This is an easy consequence of the theorem.

4. Invariance under the Möbius transformations. Before the proof of the theorem, we show that the positivity and the rank of the Hermitian matrix A corresponding to (EI) are invariant under the Möbius transformations of the variable and the function.

In order to avoid complicated calculations with the coefficients, it is very convenient to introduce the notion of local solution and to make use of the formulas given in §2. A function f holomorphic in some neighborhood of the finite set $\{z_1, \dots, z_k\}$ is said to be local solution of (EI) if the conditions (EI) are satisfied. Of course, a local solution exists in any case and a global solution is a local solution. Note that in place of this notion we may use the notion of germ or the notion of formal power series.

If f is a local solution of (EI) then, writing

$$F(z, \bar{\zeta}) = \frac{1 - f(z)\overline{f(\zeta)}}{1 - z\bar{\zeta}} = \frac{1}{1 - z\bar{\zeta}} - f(z) \cdot \frac{1}{1 - z\bar{\zeta}} \cdot \overline{f(\zeta)},$$

we have by (2) and (3) the fundamental relation

$$A_{ij} = \Gamma_{ij} - C_i \cdot \Gamma_{ij} \cdot C_j^* = \mathbf{M}(F; z_i, \bar{z}_j; n_i, n_j).$$

1°. Consider a Möbius transformation

$$z = \varphi(x) = e^{i\theta}(x - c)/(1 - \bar{c}x)$$

with $\theta \in \mathbf{R}$ and $|c| < 1$. Take a local solution f of (EI) and put $x_i = \varphi^{-1}(z_i)$. Then we see by (7) that $g(x) = f(\varphi(x))$ is a local solution of another extended interpolation problem

$$\text{(EI)}^* \qquad g(x) = \sum_{\alpha=0}^{n_i - 1} d_{i\alpha}(x - x_i)^\alpha + O((x - x_i)^{n_i}) \qquad (i = 1, \dots, k),$$

with the coefficients given by

$$(d_{i0}, \dots, d_{in_i-1}) = (c_{i0}, \dots, c_{in_i-1}) \cdot {}^t\Omega_i,$$

where $\Omega_i = \Omega(\varphi; x_i; n_i)$ is defined in (5) of §2. Set

$$\begin{aligned} G(x, \bar{\xi}) &= \frac{1 - g(x)\overline{g(\xi)}}{1 - x\bar{\xi}} \\ &= (1 - |c|^2) \cdot \frac{1}{1 - \bar{c}x} \cdot \frac{1 - f(\varphi(x))\overline{f(\varphi(\xi))}}{1 - \varphi(x)\overline{\varphi(\xi)}} \cdot \frac{1}{1 - c\bar{\xi}}. \end{aligned}$$

The formula (3) and the lemma in §2 give

$$\mathbf{M}(G; x_i, \bar{x}_j; n_i, n_j) = (1 - |c|^2) M_i \cdot \Omega_i \cdot A_{ij} \cdot \Omega_j^* \cdot M_j^*,$$

where $M_i = \Delta(1/(1 - \bar{c}x); x_i; n_i)$ and Ω_i are all regular matrices.

Hence the Hermitian matrix corresponding to (EI)* for g is

$$(1 - |c|^2) S \cdot A \cdot S^*, \quad \text{where } S = \begin{pmatrix} M_1\Omega_1 & & \\ & \ddots & \\ & & M_k\Omega_k \end{pmatrix}, \quad |c| < 1,$$

which proves the invariance under the Möbius transformation φ.

2°. Suppose $|c_{i0}| \leq 1$ $(i = 1, \dots, k)$. Take a local solution f of (EI) and put $g(z) = e^{i\theta}(f(z) - c)/(1 - \bar{c}f(z))$ with $\theta \in \mathbf{R}$ and $|c| < 1$. $g(z)$ is holomorphic in a sufficiently small neighborhood of $\{z_1, \dots, z_k\}$ and is a local solution of another extended interpolation problem determined by $\Delta(g; z_i; n_i) = e^{i\theta}(C_i - cI_{n_i})(I_{n_i} - \bar{c}C_i)^{-1}$. Setting

$$\begin{aligned} G(z, \bar{\zeta}) &= \frac{1 - g(z)\overline{g(\zeta)}}{1 - z\bar{\zeta}} \\ &= (1 - |c|^2) \frac{1}{1 - \bar{c}f(z)} \cdot \frac{1 - f(z)\overline{f(\zeta)}}{1 - z\bar{\zeta}} \cdot \frac{1}{1 - c\overline{f(\zeta)}}, \end{aligned}$$

we have by (3)

$$\mathbf{M}(G; z_i, \bar{z}_j; n_i, n_j) = (1 - |c|^2) N_i \cdot A_{ij} \cdot N_j^*,$$

where $N_i = \Delta(1/(1 - \bar{c}f(z)); z_i; n_i)$ is a regular matrix. As in 1°, the Hermitian matrix corresponding to this problem is

$$(1 - |c|^2) N \cdot A \cdot N^*, \quad \text{where } N = \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_k \end{pmatrix}, \quad |c| < 1.$$

This shows that the positivity and the rank of A are invariant under the Möbius transformations of the function.

5. Proof of the theorem. First of all, note $|c_{i0}| \leq 1$. In fact, if there exists a solution of (EI) in $\mathscr{B}$, then $|c_{i0}| \leq 1$ trivially; conversely, if $A \geq 0$, then the $(1,1)$-element $(1-|c_{i0}|^2)/(1-|z_i|^2)$ of A_{ii} is nonnegative.

1°. Now, we treat the case where one of the $|c_{i0}|$ is 1. If $f \in \mathscr{B}$ is a solution of (EI) with $|c_{10}| = 1$, then f is constant by the maximum principle, so that $c_{10} = \cdots = c_{k0}$ and $c_{i\alpha} = 0$ for $\alpha > 0$. Therefore, $A_{ij} = \Gamma_{ij} - c_{i0}\overline{c_{j0}}\Gamma_{ij} = \mathbf{0}$ and $A = \mathbf{0}$. Conversely, when $A \geq 0$ and $|c_{10}| = 1$, we claim that $c_{10} = \cdots = c_{k0}$ and $c_{i\alpha} = 0$ for $\alpha > 0$. Let $a^{(ij)}_{\alpha\beta}$ be the $(\alpha+1, \beta+1)$-element of A_{ij}. Then $a^{(11)}_{00} = 0$. As $A \geq 0$, we see

$$a^{(11)}_{00}a^{(ii)}_{00} - a^{(i1)}_{00}a^{(1i)}_{00} = -|a^{(i1)}_{00}|^2 \geq 0.$$

Hence $a^{(i1)}_{00} = (1 - c_{i0}\overline{c_{10}})/(1 - z_i\bar{z}_1) = 0$, which shows $c_{i0} = c_{10}$ and $a^{(ii)}_{00} = 0$. As above,

$$a^{(ii)}_{00}a^{(ii)}_{\alpha\alpha} - a^{(ii)}_{\alpha 0}a^{(ii)}_{\alpha 0} = -|a^{(ii)}_{\alpha 0}|^2 \geq 0,$$

so that $a^{(ii)}_{\alpha 0} = 0$. Suppose now $1 \leq \alpha \leq n_i$ and, if $\alpha > 1$, that we have seen $c_{i1} = \cdots = c_{i\alpha-1}$. Calculating directly, from $A_{ii} = \Gamma_{ii} - C_i \cdot \Gamma_{ii} \cdot C_i^*$, we derive

$$a^{(ii)}_{\alpha 0} = -c_{i\alpha}\overline{c_{i0}}\gamma^{(ii)}_{00} = 0,$$

where the $(1,1)$-element $\gamma^{(ii)}_{00}$ of Γ_{ii} is $(1-|z_i|^2)^{-1} \neq 0$. Thus $c_{i\alpha} = 0$. By induction, this proves the assertion. In this case, $A = \mathbf{0}$ and the constant c_{10} is the unique solution of (EI) in $\mathscr{B}$, which is a Blaschke product of degree 0.

2°. Let us prove the theorem by induction on $n = \sum_{i=1}^{k} n_i$. The case $n = 1$ holds because A is reduced to

$$\frac{1-|c_{10}|^2}{1-|z_1|^2}.$$

Assume $n > 1$. By 1°, we assume $|c_{i0}| < 1$ $(i = 1, \ldots, k)$. By virtue of the invariance under the Möbius transformations established in §4, we can assume $z_1 = 0$ and $c_{10} = 0$.

Let f be a local solution of (EI) in $\mathscr{B}$ and put $g(z) = f(z)/z$. For the simplicity, put $m_1 = n_1 - 1$, $m_i = n_i$ $(i = 2, \ldots, k)$ and $Z_i = \Delta(z; z_i; m_i)$. Because $\Delta(f; z_i; m_i) = Z_i \cdot \Delta(g; z_i; m_i)$, $g(z)$ is a local solution of another extended interpolation problem

$$\text{(EI)}^\circ \qquad g(z) = \sum_{\alpha=0}^{m_i-1} d_{i\alpha}(z-z_i)^\alpha + O((z-z_i)^{m_i}) \qquad (i = 1, \ldots, k)$$

with the coefficients $d_{i\alpha}$ given by

$$\begin{cases} d_{1\alpha} = c_{1\alpha+1} \quad (\alpha = 0, \dots, m_1 - 1), \\ (d_{i0}, \dots, d_{im_i-1}) = (c_{i0}, \dots, c_{im_i-1})^t Z_i^{-1} \quad (i = 2, \dots, k), \end{cases}$$

where Z_i is a regular matrix for $i > 1$. Conversely, if $g(z)$ is a local solution of (EI)° then $f(z) = z \cdot g(z)$ is a local solution of (EI).

Let $B_{ij} = \mathbf{M}((1 - g(z)\overline{g(\zeta)})/(1 - z\bar{\zeta}); z_i, \bar{z}_j; m_i, m_j)$ and set

$$B = \begin{pmatrix} B_{11} \cdots B_{1k} \\ \cdots\cdots \\ B_{k1} \cdots B_{kk} \end{pmatrix}.$$

Then B is the Hermitian matrix of order $n-1$ corresponding to (EI)°. From

$$\frac{1 - f(z)\overline{f(\zeta)}}{1 - z\bar{\zeta}} = 1 + z\frac{1 - g(z)\overline{g(\zeta)}}{1 - z\bar{\zeta}}\bar{\zeta},$$

we derive, by (2) and (3), with $E_{ij} = E^{(00)}_{n_i n_j}$ given in (4),

$$\begin{aligned} A_{11} &= E_{11} + \begin{pmatrix} 0 & \cdots & 0 \\ & I_{m_1} & \vdots \\ & & 0 \end{pmatrix} \cdot \begin{pmatrix} & & * \\ & B_{11} & \vdots \\ * & \cdots & * \end{pmatrix} \cdot \begin{pmatrix} 0 & & \\ \vdots & I_{m_1} & \\ 0 & \cdots & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B_{11} & \\ 0 & & & \end{pmatrix}, \\ A_{i1} &= E_{i1} + Z_i \cdot \begin{pmatrix} & * \\ B_{i1} & \vdots \\ & * \end{pmatrix} \cdot \begin{pmatrix} 0 & & \\ \vdots & I_{m_1} & \\ 0 & \cdots & 0 \end{pmatrix} \\ &= E_{i1} + \begin{pmatrix} 0 & \\ \vdots & Z_i \cdot B_{i1} \\ 0 & \end{pmatrix}, \\ A_{ij} &= E_{ij} + Z_i \cdot B_{ij} \cdot Z_j^* \qquad (i,j = 2, \dots, k). \end{aligned}$$

Hence

$$A = P \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix} \cdot P^*, \quad \text{where } P = \begin{pmatrix} I_{n_1} & & & \\ E_{21} & Z_2 & & \\ \vdots & & \ddots & \\ E_{k1} & & & Z_k \end{pmatrix}.$$

P is an $n \times n$ regular matrix because Z_i is regular for $i = 2, \dots, k$. This shows that $A \geq 0 \Leftrightarrow B \geq 0$, that $\det A = 0 \Leftrightarrow \det B = 0$, and that $\operatorname{rank} A = \operatorname{rank} B + 1$.

Clearly, a function $f(z)$ is a solution of (EI) in $\mathscr{B}$ if and only if $g(z) = f(z)/z$ is a solution of $(\mathrm{EI})^{\circ}$ in $\mathscr{B}$. By induction, this is the case if and only if $B \geq 0$, which is equivalent to $A \geq 0$. This proves the first statement of the theorem.

A function $f(z)$ is a Blaschke product of degree r with $f(0) = 0$ if and only if $g(z) = f(z)/z$ is a Blaschke product of degree $r - 1$. The rest of the theorem is an immediate consequence of what we have mentioned. □

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