PACIFIC JOURNAL OF MATHEMATICS
Vol. 139, No. 1, 1989

# SPACES OF WHITNEY MAPS

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**Let $X$ be a continuum. Let $2^X$ (respectively, $C(X)$) be the hyperspace of nonempty closed subsets (respectively, subcontinua) of $X$, endowed with the Hausdorff metric. For $\mathscr{H} = C(X)$ or $2^X$, let $W(\mathscr{H})$ denote the space of Whitney maps for $\mathscr{H}$ with the "sup metric" and pointwise product. In this paper we prove that if there exists a homeomorphism $\phi$: $W(C(X)) \to W(C(Y))$ (or $\phi$: $W(2^X) \to W(2^Y)$) which preserved products and "strict order", then $X$ is homeomorphic to $Y$. We also prove that there exists an embedding $\psi$: $W(C(X)) \to W(2^X)$ such that $\psi(u)$ is an extension of $u$ for each $u \in W(C(X))$.**

**Introduction.** A *continuum* is a nondegenerate compact, connected metric space. All the spaces considered here are continua. If $X$ is a continuum, $2^X$ (respectively, $C(X)$) is the hyperspace of nonempty closed subsets (respectively, subcontinua) of $X$, endowed with the Hausdorff metric $H$. Let $\mathscr{H}$ be a nonempty closed subset of $2^X$. A *Whitney map* for $\mathscr{H}$ is a continuous function $u$: $\mathscr{H} \to [0, 1]$ such that (a) $u(A) = 0$ if and only if $A$ is a single point set; (b) $u(A) = 1$ if and only if $A = X$; and (c) if $A, B \in \mathscr{H}$ and $A \subset B \neq A$, then $u(A) < u(B)$. Let $W(\mathscr{H})$ denote the space of Whitney maps for $\mathscr{H}$. We identify $X$ with $\{\{x\}: x \in X\} \subset C(X), 2^X$. Given $u, w \in W(\mathscr{H})$, we say that $u$ is *strictly smaller* than $w$ ($u \lhd w$) if $u(A) < w(A)$ for each $A \in \mathscr{H} - (X \cup \{X\})$ and $u$ is *smaller or equal* than $w$ ($u \leq w$) if $u(A) \leq w(A)$ for each $A \in \mathscr{H}$. We consider $W(\mathscr{H})$ with the "sup metric", the pointwise product and the orders defined above.

In this paper we prove that: (a) $W(\mathscr{H})$ is a topologically complete space. This answers a question asked by S. B. Nadler, Jr. [**2**, question 14.71.4]; (b) There is a natural way to embed $2^X$ in $W(C(X))$ and in $W(2^X)$; (c) If $\mathscr{H} = C(X)$ and $\mathscr{G} = C(Y)$ or $\mathscr{H} = 2^X$ and $\mathscr{G} = 2^Y$ and there exists a homeomorphism $\phi$: $W(\mathscr{H}) \to W(\mathscr{G})$ which is a semigroup isomorphism and preserves strict order (in the sense that $u \lhd w$ if and only if $\phi(u) \lhd \phi(w)$), then $X$ is homeomorphic to $Y$ (this answers, partially, question 14.71.1 formulated by S. B. Nadler, Jr. in [**2**]). In [**3**], L. E. Ward, Jr., showed that Whitney maps for $\mathscr{H}$ can be extended to Whitney maps for $2^X$. Using his constructions, we prove: (d) There exists an embedding $\phi: W(\mathscr{H}) \to W(2^X)$ such that $\phi(u)$ is an extension of $u$ for every $u \in W(\mathscr{H})$.

**1.** $W(\mathscr{H})$ **is topologically complete.** Let $\mathscr{H}$ be a nonempty closed subset of $2^X$. Let $\mathscr{C}(\mathscr{H}) = \{f: \mathscr{H} \to [0,1] | f \text{ is continuous}\}$. We consider $\mathscr{C}(\mathscr{H})$ with the sup metric $S$ defined by $S(f,g) = \max\{|f(A) - g(A)|: A \in \mathscr{H}\}$. Then $(\mathscr{C}(\mathscr{H}), S)$ is a complete metric space. Let $W_0(\mathscr{H}) = \{f \in \mathscr{C}(\mathscr{H}): A, B \in \mathscr{H}$ and $A \subset B$ implies that $f(A) \leq f(B)$, $f(A) = 0$ for every $A \in \mathscr{H} \cap X$ and $f(X) = 1$ if $X \in \mathscr{H}\}$. It is easy to see that $W_0(\mathscr{H})$ is a closed subset of $\mathscr{C}(\mathscr{H})$ and, consequently, $(W_0(\mathscr{H}), S)$ is a complete metric space. If $A \in 2^X$ and $\varepsilon > 0$, let $N(\varepsilon, A) = \{x \in X: d_X(x,a) < \varepsilon$ for some $a \in A\}$, where $d_X$ is the metric of $X$. Let $\mathbf{N}$ be the set of positive integers.

1.1. PROPOSITION. *$(W(\mathscr{H}), S)$ is topologically complete.*

*Proof*. By [**4**, Theorem 24.12] it is enough to prove that $W(\mathscr{H})$ is a $G_\delta$ subset of $W_0(\mathscr{H})$. For each $n \in \mathbf{N}$, let $F_n = \{f \in W_0(\mathscr{H})$: there exist $A, B \in \mathscr{H}$ such that $f(A) = f(B)$, $A \subset B$ and $B \not\subset N(1/n, A)\}$. We will show that $F_n$ is closed in $W_0(\mathscr{H})$. Take a sequence $(f_m)_m$ in $F_n$ which converges to $f \in W_0(\mathscr{H})$. For each $m \in \mathbf{N}$, let $A_m, B_m \in \mathscr{H}$ be such that $A_m \subset B_m$, $f_m(A_m) = f_m(B_m)$ and $B_m \not\subset N(1/n, A_m)$. Choose points $p_m \in B_m - N(1/n, A_m)$. Then there exist $A, B \in \mathscr{H}$, $p \in X$ and a strictly increasing sequence $(m_k)_k$ in $\mathbf{N}$ such that $A_{m_k} \to A$, $B_{m_k} \to B$ and $p_{m_k} \to p$. Then $f(A) = f(B)$, $A \subset B$, $p \in B$ and $p \notin N(1/n, A)$. So $f \in F_n$. Hence $F_n$ is closed.

For each $n \in N$, let $U_n = W_0(\mathscr{H}) - F_n$. Clearly, $\bigcap\{U_n: n \in \mathbf{N}\} = \{f \in W_0(\mathscr{H}): A \subset B \neq A$ implies that $f(A) < f(B)\}$. Since $\mathscr{H}$ is compact and $\mathscr{H} - (X \cup \{X\})$ is open in $\mathscr{H}$, then there exists a sequence $(\mathscr{H}_n)_n$ of compact subsets of $\mathscr{H}$ such that $\bigcup\{\mathscr{H}_n: n \in \mathbf{N}\} = \mathscr{H} - (X \cup \{X\})$. Define $V_n = \{f \in W_0(\mathscr{H}): 0 < f(A) < 1$ for every $A \in \mathscr{H}_n\}$. Then $V_n$ is open in $W_0(\mathscr{H})$ and, since $(\bigcap\{U_n: n \in \mathbf{N}\}) \cap (\bigcap\{V_n: n \in \mathbf{N}\}) = W(\mathscr{H})$, we have that $W(\mathscr{H})$ is a $G_\delta$ subspace of $W_0(\mathscr{H})$. Hence $W(\mathscr{H})$ is topologically complete.

**2. Embedding** $2^X$ **in** $W(\mathscr{H})$**.** Throughout this section we suppose that $\mathscr{H}$ is a closed subset of $2^X$ such that every point in $X$ is a limit of nondegenerate elements of $\mathscr{H}$, that is, $X$ is contained in the closure of $\mathscr{H} - X$ in$2^X$. In this case, we show that $2^X$ can be naturally embedded in $W(\mathscr{H})$.

Throughout this section, $u$ will denote a fixed Whitney map for $2^X$. For $A \in 2^X$, we define $u_A: 2^X \to [0,1]$ by: $u_A(B) = u(A \cup B)u(B)$.

2.1. THEOREM. *$u_A$ has the following properties:*

(a) *$u_A$ is a Whitney map for $2^X$ for each $A \in 2^X$.*

(b) *If $A, B \in 2^X$ and $A \neq B$, then $u_A|\mathscr{H} \neq u_B|\mathscr{H}$.*

(c) *The function $\phi: 2^X \to W(\mathscr{H})$ given by $\phi(A) = u_A|\mathscr{H}$ is an embedding.*

*Proof.* (b) Take $A, B \in 2^X$ such that $A \neq B$. Suppose that $u_A|\mathscr{H} = u_B|\mathscr{H}$. Take $x \in X$. Let $(C_n)_n$ be a sequence of elements of $\mathscr{H} - X$ such that $C_n \to \{x\}$. Then $A \cup C_n \to A \cup \{x\}$ and $B \cup C_n \to B \cup \{x\}$. Since $u_A(C_n) \equiv u_B(C_n)$, we have that $u(A \cup C_n) = u(B \cup C_n)$. Hence $u(A \cup \{x\}) = u(B \cup \{x\})$ for all $x \in X$.

If $A \cap B \neq \emptyset$, taking $x \in A \cap B$, we have that $u(A) = u(B)$. Since $A \neq B$, we can choose, for example $y \in B - A$, then $u(A \cup \{y\}) = u(B \cup \{y\}) = u(B) = u(A)$. So $u(A \cup \{y\}) = u(A)$. This is a contradiction, so $A \cap B = \emptyset$. Choose points $a \in A$ and $b \in B$; then $u(A) = u(A \cup \{a\}) = u(B \cup \{a\}) > u(B) = u(B \cup \{b\}) = u(A \cup \{b\}) > u(A)$. This contradiction completes the proof of (b).

(c) It follows from the fact that $H(A, B) < \delta$ implies that $H(A \cup C, B \cup C) < \delta$ for every $C \in 2^X$.

**3.** $W(\mathscr{H})$ **determines** $X$. Throughout this section we will suppose that $\mathscr{H} = C(X)$ and $\mathscr{G} = C(Y)$ or $\mathscr{H} = 2^X$ and $\mathscr{G} = 2^Y$. We say that $W(\mathscr{H})$ is *equivalent* to $W(\mathscr{G})$ if there exists a homeomorphism $\phi: W(\mathscr{H}) \to W(\mathscr{G})$ which is a semigroup isomorphism and preserves strict order (that is, $u \lhd w$ if and only if $\phi(u) \lhd \phi(w)$). In [**1**, Question 14.71.1], S. B. Nadler, Jr., asked the following question: If $W(C(X))$ (resp., $W(2^X)$) is homeomorphic, or both homeomorphic and algebraically isomorphic, to $W(C(Y))$ (resp., $W(2^Y)$), then must $X$ and $Y$ be homeomorphic? The aim of this section is to prove the following partial answer to Nadler's question:

3.1. THEOREM. *The following assertions are equivalent:*

(a) *$W(\mathscr{H})$ is equivalent to $W(\mathscr{G})$.*

(b) *There exists a homeomorphism $F: \mathscr{H} \to \mathscr{G}$ which preserves inclusion (i.e. $A \subset B$ if and only if $F(A) \subset F(B)$).*

(c) *$X$ is homeomorphic to $Y$.*

*Proof.* Proofs of (b) $\Rightarrow$ (a) and (b) $\Leftrightarrow$ (c) are immediate (see page 473 in [**2**]). To prove (a) $\Rightarrow$ (c) it is necessary to introduce some terminology.

We make $\mathscr{H}_1 = \mathscr{H} - (X \cup \{X\})$ and $\mathscr{G}_1 = \mathscr{G} - (Y \cup \{Y\})$. If $A \in \mathscr{H}_1$, $u, w \in W(\mathscr{H})$ and $u \leq w$, we say that $A$ is the *contact between* $u$ and $w$ if $A$ is the only element of $\mathscr{H}_1$ in which $u$ and $w$ agree. We denote by $(\mathscr{C}[0,1], D)$ the metric space of continuous functions of $[0,1]$ in $[0,1]$ with the "sup metric" $D$.

For each $t \in [0,1]$, we define $h_t \colon [0,1] \to [0,1]$ by:

$$h_t(s) \begin{cases} (5/4)s & \text{if } s \in [0, t/2], \\ (3/4)s + (t/4) & \text{if } s \in [t/2, t], \\ (5/4)s - (t/4) & \text{if } s \in [t, (t+1)/2], \\ (3/4)s + (1/4) & \text{if } s \in [(t+1)/2, 1]. \end{cases}$$

Then $h_t$ has the following immediate properties:

(1) $h_t \in \mathscr{C}[0,1]$.

(2) $h_t(s) \geq s$ for each $s \in [0,1]$ and $h_t(s) = s$ if and only if $s \in \{0, t, 1\}$.

(3) $D(h_t, h_r) \leq |t - r|$ for every $t, r \in [0,1]$. Then the function $t \to h_t$ from $[0,1]$ in $(\mathscr{C}[0,1], D)$ is continuous.

(4) $h_t$ is a strictly increasing homeomorphism from $[0,1]$ onto $[0,1]$.

We shall prove a sequence of results which will lead us to the proof of (a) $\Rightarrow$ (b). We suppose that $\phi \colon W(\mathscr{H}) \to W(\mathscr{G})$ is an equivalence between $W(\mathscr{H})$ and $W(\mathscr{G})$.

(A) If $A \in \mathscr{H}_1$ and $u \in W(\mathscr{H})$, then there exists $w \in W(\mathscr{H})$ such that $u \leq w$ and $A$ is the contact between $u$ and $w$.

To prove (A), take a Whitney map $U \colon 2^X \to [0,1]$ which extends $u$ (see §4 in this paper). Consider $u_1 \colon \mathscr{H} \to [0,1]$ defined by $u_1(E) = U(A \cup E)U(E)$. Then (see 2.1 (a)), $u_1 \in W(\mathscr{H})$. Let $t = u(A)$; then $0 < t < 1$. Define $w = h_t \circ \sqrt{u_1}$. For $E \in \mathscr{H}$,

$$w(E) = h_t(\sqrt{U(E \cup A)U(E)}) \geq \sqrt{U(E \cup A)U(E)} \geq u(E).$$

Thus $w \geq u$. Suppose that $E \in \mathscr{H}_1$ is such that $w(E) = u(E)$. Then $U(E \cup A)U(E) = (U(E))^2$ and $\sqrt{U(E \cup A)U(E)} = t$. It follows that $A \subset E$ and $\sqrt{U(E)U(E)} = t = U(A)$. Hence $A = E$.

(B) If $u, w \in W(\mathscr{H})$ and $u \leq w$, then $\phi(u) \leq \phi(w)$.

Choose a sequence of increasing homeomorphisms $(g_n)_n$ of $[0,1]$ on itself such that $g_n \geq \mathrm{Id}$ ($\mathrm{Id}$ = identity of $[0,1]$), $D(g_n, \mathrm{Id}) \to 0$ and $g_n(t) = t$ if and only if $t = 0$ or $t = 1$. Define $w_n = g_n \circ w$. Clearly, $w_n \in W_0(\mathscr{H})$, $u \leq w \lhd w_n$ and $w_n \to w$. Then $\phi(u) \lhd \phi(w_n)$ and $\phi(w_n) \to \phi(w)$. Hence $\phi(u) \leq \phi(w)$.

(C) Let $A \in \mathscr{H}_1$ and $u, w \in W(\mathscr{H})$ be such that $u \leq w$ and $A$ is the contact between $u$ and $w$. Then there exists a unique $B \in \mathscr{G}_1$ such that $B$ is the contact between $\phi(u)$ and $\phi(w)$.

By (B), $\phi(u) \leq \phi(w)$. Since $u \ntrianglelefteq w$, we have that $\phi(u) \ntrianglelefteq \phi(w)$. Then there exists $B \in \mathscr{G}_1$ such that $\phi(u)(B) = \phi(w)(B)$. Take $E \in \mathscr{G}_1$ such that $\phi(u)(E) = \phi(w)(E)$. By (A), there exists $v \in W(\mathscr{G})$ such that $\phi(w) \leq v$ and $E$ is the contact between $\phi(w)$ and $v$. Thus $\phi(u) \leq v$ and $\phi(u)(E) = v(E)$. So $u \leq w \leq \phi^{-1}(v)$ and $u \ntrianglelefteq \phi^{-1}(v)$. This implies that there exists $D \in \mathscr{H}_1$ such that $u(D) = \phi^{-1}(v)(D)$. Then $u(D) = w(D)$. So $D = A$. Hence $u(A) = \phi^{-1}(v)(A)$.

Using (A) again, we have that there exists $v_1 \in W(\mathscr{G})$ such that $\phi(w) \leq v_1$ and $B$ is the contact between $\phi(w)$ and $v_1$. Proceeding as before, $u(A) = \phi^{-1}(v_1)(A)$. Then $u^2 \leq \phi^{-1}(v\ v_1)$ and $u^2(A) = \phi^{-1}(v\ v_1)(A)$. So $u^2 \ntrianglelefteq \phi^{-1}(v\ v_1)$. Thus there exists $C \in \mathscr{G}_1$ such that $\phi(u^2)(C) = v(C)v_1(C)$. If $C \neq E$, then $\phi(u)(C) \leq \phi(w)(C) < v(C)$ and $0 < \phi(u)(C) \leq v_1(C)$. This implies that $\phi(u^2)(C) < v(C)v_1(C)$. This contradiction proves that $C = E$. Similarly, $C = B$. Hence $B$ is the contact between $\phi(u)$ and $\phi(w)$.

(D) Let $u, v, w, z, \in W(\mathscr{H})$ and $A \in \mathscr{H}_1$ be such that $u \leq v$, $w \leq z$ and $A$ is the contact between $u$ and $v$ and $w$ and $z$. Then the contact between $\phi(u)$ and $\phi(v)$ is the contact between $\phi(w)$ and $\phi(z)$.

Since $uw(A) = vz(A)$, we have that $\phi(uw) \leq \phi(vz)$ and $\phi(uw) \ntrianglelefteq \phi(vz)$. Then there exists $B \in \mathscr{G}_1$ such that $\phi(uw)(B) = \phi(vz)(B)$. This implies that $\phi(u)(B) = \phi(v)(B)$ and $\phi(w)(B) = \phi(z)(B)$. Hence $B$ is the contact between $\phi(u)$ and $\phi(v)$ and $\phi(w)$ and $\phi(z)$.

We define $F: \mathscr{H}_1 \to \mathscr{G}_1$ in the following way: For $A \in \mathscr{H}_1$, we take $u, w \in W(\mathscr{H})$ such that $u \leq w$ and $A$ is the contact between $u$ and $w$. We define $F(A)$ as the element of $\mathscr{G}_1$ which is the contact between $\phi(u)$ and $\phi(w)$. Then, by (D), $F$ is well defined. Clearly, we can define $F^{-1}: \mathscr{G}_1 \to \mathscr{H}_1$ in a similar way. Hence $F$ is bijective.

(E) $F$ is continuous.

Take a sequence $(A_n)_n$ of $\mathscr{H}_1$ which converges to an element $A$ from $\mathscr{H}_1$. Let $B_n = F(A_n)$ and $B = F(A)$. Since $\mathscr{G}$ is compact, $(B_n)_n$ has a subsequence which converges to a $D$ in $\mathscr{G}$. Without loss of generality we may suppose that $B_n \to D$. We will prove that $D = B$.

Fix $u \in W(2^X)$. By 2.1(c), $u_{A_n} \to u_A$. So $\sqrt{u_{A_n}} \to \sqrt{u_A}$. Let $t_n = u(A_n)$ and $t = u(A)$. Then $h_{t_n} \to h_t$. So $h_{t_n} \circ \sqrt{u_{A_n}} \to h_t \circ \sqrt{u_A}$. Define $w_n = (h_{t_n} \circ \sqrt{u_{A_n}})|\mathscr{H}$, $w = (h_t \circ \sqrt{u_A})|\mathscr{H}$ and $u_1 = u|\mathscr{H}$. As we showed in (A), $u_1 \leq w_n$, $u_1 \leq w$, $A_n$ is the contact between $u_1$ and $w_n$ and $A$ is the contact between $u_1$ and $w$. So $B_n$ is the contact between

$\phi(u_1)$ and $\phi(w_n)$ and $B$ is the contact between $\phi(u_1)$ and $\phi(w)$. Since $B_n \to D$, $\phi(u_1)(B_n) = \phi(w_n)(B_n)$ and $\phi(w_n) \to \phi(w)$, we have that $\phi(u_1)(D) = \phi(w)(D)$. If $D \in \mathscr{G}_1$, since $B$ is the contact between $\phi(u_1)$ and $\phi(w)$, we have that $D = B$. Then we must show that $D \in \mathscr{G}_1$.

Suppose that $D \in X \cup \{X\}$. Let $s_n = \phi(u_1)(B_n) = \phi(w_n)(B_n)$ and $v_n = h_{s_n} \circ \phi(w_n) \in W(\mathscr{G})$. Since $B_n \to D$, then $s_n \to 0$ or $s_n \to 1$. So $h_{s_n} \to h_0$ $(h_0 = h_1)$. Thus $h_{s_n} \circ \phi(w_n) \to h_0 \circ \phi(w)$. Since $\phi(u_1)(B_n) = s_n = h_{s_n}(s_n) = h_{s_n}(\phi(w_n)(B_n))$, we have that $\phi(u_1) \not\triangleleft h_{s_n} \circ \phi(w_n)$. Then $u_1 \not\triangleleft \phi^{-1}(h_{s_n} \circ \phi(w_n))$, so there exists $C_n \in \mathscr{H}_1$ such that

$$u_1(C_n) \geq \phi^{-1}(h_{s_n} \circ \phi(w_n))(C_n) \geq w_n(C_n) \geq u_1(C_n).$$

Thus $C_n = A_n$ and $\phi^{-1}(h_{s_n} \circ \phi(w_n))(A_n) = u_1(A_n)$. And since $\phi^{-1}(h_{s_n} \circ \phi(w_n)) \to \phi^{-1}(h_0 \circ \phi(w))$, we have that

$$\phi^{-1}(h_0 \circ \phi(w))(A) = u_1(A).$$

But $\phi(u_1) \leq \phi(w) \triangleleft h_0 \circ \phi(w)$ implies that $u_1 \triangleleft \phi^{-1}(h_0 \circ \phi(w))$. This contradiction proves that $D \notin X \cup \{X\}$ and completes the proof of (E).

(F) If $u, w \in W(\mathscr{H})$, then $v = \max\{u, w\} \in W(\mathscr{H})$ and $\phi(v) = \max\{\phi(u), \phi(w)\}$.

(G) Let $u, w \in W(\mathscr{H})$ and $A \in \mathscr{H}_1$ be such that $u(A) = w(A)$. Then $\phi(u)(F(A)) = \phi(w)(F(A))$.

By (A), there exists $v \in W(\mathscr{H})$ such that $\max\{u, w\} \leq v$ and $A$ is the contact between $\max\{u, w\}$ and $v$. Then $u, w \leq v$ and $A$ is the contact between $u$ and $v$ and $w$ and $v$. So $F(A)$ is the contact between $\phi(u)$ and $\phi(v)$ and $\phi(w)$ and $\phi(v)$ where in particular, $\phi(u)(F(A)) = \phi(v)(F(A)) = \phi(w)(F(A))$.

(H) Let $A, B \in \mathscr{H}_1$ be such that $B \not\supset A$. Then there exists $u \in W(\mathscr{H})$ such that $u(A) < u(B)$.

Fix $w \in W(2^X)$. Since $B \not\subset A$, there exists $n \in \mathbf{N}$ such that $(w(A))^n w(A) < (w(A \cup B))^n w(B)$. Define $u_1 \colon 2^X \to [0, 1]$ by $u_1(E) = (w(A \cup E))^n w(E)$ and let $u = u_1|\mathscr{H}$.

(I) If $A, B \in \mathscr{H}_1$ and $B \subset A$, then $F(B) \subset F(A)$.

Suppose that $F(B) \not\subset F(A)$. Let $w \in W(\mathscr{G})$ such that $w(F(A)) < w(F(B))$. Put $u = \phi^{-1}(w)$. Choose $n \geq 2$ such that $\sqrt[n]{u(B)} > u(A)$. Let $r = w(F(A))$, $s = w(F(B))$ and $q = \sqrt[n]{w(F(B))}$. Then $0 < s < q < 1$. So there exists an increasing homeomorphism $\sigma \colon [0, 1] \to [0, 1]$ such that $\sigma(t) = t$ if $t \in [0, r]$; $\sigma(t) > t$ if $t \in (r, 1)$; $\sigma(1) = 1$ and $\sigma(s) = q$. Then $\sigma \circ w \in W(\mathscr{G})$ and $(\sigma \circ w)(F(A)) = r = w(F(A))$. Let $v = \phi^{-1}(\sigma \circ w)$. By analogy to property (G) for $\phi^{-1}$ and $F^{-1}$, we have that $v(A) = \phi^{-1}(w)(A) = u(A)$.

Notice that $(\sigma \circ w)^n \in W(\mathscr{G})$ and $(\sigma \circ w)^n(F(B)) = (\sigma(s))^n = q^n = w(F(B))$. Then $\phi^{-1}((\sigma \circ w)^n(B)) = \phi^{-1}(w)(B)$ (by the analogy to property (G) for $\phi^{-1}$ and $F^{-1}$). Thus $v(B) = \sqrt[n]{u(B)} > u(A) = v(A)$. Hence $v(B) > v(A)$. This contradiction proves (I).

We define $f\colon X \to Y$ in the following way: Given $x \in X$, take a sequence $(A_n)_n$ of elements of $\mathscr{H}_1$ such that $A_1 \supset A_2 \supset \cdots$ and $\{x\} = \bigcap\{A_n\colon n \in \mathbf{N}\}$. Then $F(A_1) \supset F(A_2) \supset \cdots$. Thus $B = \bigcap\{F(A_n)\colon n \in \mathbf{N}\}$ is closed and nonempty. Since $B \subset F(A_1) \in \mathscr{G}_1$, we have that $B \neq Y$. If $B \notin Y$, then $B \in \mathscr{G}_1$. Let $A = F^{-1}(B) \in \mathscr{H}_1$. By (I) applied to $F^{-1}$, $A_n \to A$. So $A = \{x\} \in \mathscr{H}_1$. This contradiction proves that $B$ is as a set of a single point $y$. We define $f(x) = y$.

(J) $f$ is well defined.

Let $(A_n)_n$ and $(B_n)_n$ be sequences in $\mathscr{H}_1$ such that $A_1 \supset A_2 \supset \cdots$; $B_1 \supset B_2 \supset \cdots$ and $\{x\} = \bigcap\{A_n\colon n \in \mathbf{N}\} = \bigcap\{B_n\colon n \in \mathbf{N}\}$. Put $C_n = A_n \cup B_n \in \mathscr{H} - X$. Then $\{x\} = \bigcap\{C_n\colon n \in \mathbf{N}\}$. Thus $\varnothing \neq \bigcap\{F(A_n)\colon n \in \mathbf{N}\}, \bigcap\{F(B_n)\colon n \in \mathbf{N}\} \subset \bigcap\{F(C_n)\colon n \in \mathbf{N}\}$ by the paragraph above, the last set is a singleton. Hence $\bigcap\{F(A_n)\colon n \in \mathbf{N}\} = \bigcap\{F(B_n)\colon n \in \mathbf{N}\}$. This proves (J).

Similarly, we can define $f^{-1}\colon Y \to X$, then $f^{-1}$ is well defined and $f$ is bijective.

(K) $f$ is continuous.

Notice that if $A \in \mathscr{H}_1$, then $x \in A$ if and only if $f(x) \in F(A)$. Suppose that $f$ is not continuous in a point $x \in X$. Let $\varepsilon > 0$ and let $(x_n)_n$ be a sequence in $X$ such that $x_n \to x$; $d_Y(f(x), f(x_n)) \geq \varepsilon$ for every $n \in \mathbf{N}$ and $\varepsilon <$ diameter of $X$. Since $Y$ is compact, we may suppose that $f(x_n) \to y$ with $y \in Y$. For $n \in \mathbf{N}$, make $y_n = f(x_n)$ and choose $B_n \in C(Y)$ such that $y_n \in B_n$ and $\varepsilon/4 \leq$ diameter of $B_n \leq \varepsilon/2$. We may suppose also that $B_n \to B$ with $B \in C(Y)$. Then $B \in C(Y) \cap \mathscr{G}_1$ and $y \in B$. Let $A_n = F^{-1}(B_n)$ and $A = F^{-1}(B)$. Then $A_n \to A$ and $x_n = f^{-1}(y_n) \in F^{-1}(B_n) = A_n$. So $x \in A$ and $f(x) \in B$. Thus $f(x)$, $y \in B$, diameter of $B \leq \varepsilon/2$ and $f(x_n) \to y$. This is a contradiction with the choice of $(x_n)_n$.

This completes the proof of Theorem 3.1.

3.2. Corollary. *The following assertions are equivalent*:

(a) *$W(C(X))$ is equivalent to $W(C(Y))$.*

(b) *$W(2^X)$ is equivalent to $W(2^Y)$.*

**4. Embedding $W(\mathscr{H})$ in $W(2^X)$.** In this section we suppose that $\mathscr{H}$ is an arbitrary closed nonempty subset of $2^X$. In [3]. L. E. Ward, Jr., proved that any Whitney map for $\mathscr{H}$ can be extended to a Whitney map for $2^X$. His proof is based in Nachbin's order-theoretic version of Tietze's Theorem for "normally ordered" spaces [**1**]. Here we prove, with explicit formulas, this version of Tietze's Theorem for the particular case of hyperspaces (it states that every map in $W_0(\mathscr{H})$ can be extended to a map in $W_0(2^X)$). Then, using the constructions of L. E. Ward, Jr., we prove that there exists an embedding $\phi: W(\mathscr{H}) \to W(2^X)$ such that $\phi(w)$ is an extension of $w$ for every $w \in W(\mathscr{H})$.

Notice that every map $w \in W(\mathscr{H})$ can be extended, in a natural way, to a map $w_0 \in W(\mathscr{H} \cup X \cup \{X\})$ and the correspondence $w \to w_0$ is continuous. So, we also suppose that $X \cup \{X\} \subset \mathscr{H}$.

Given $T > 0$, we denote by $A([0,T])$ the metric space of bounded real-valued functions defined in $[0,T]$ with the "sup metric".

4.1. LEMMA. *Let $T > 0$ and let $f: [0,T] \to \mathbf{R}$ be a strictly increasing function such that $f(0) = 0$ and $f$ is continuous at 0. Then we can choose a strictly increasing continuous function $g: [0,T] \to \mathbf{R}$ such that $g(0) = 0$, $g(t) \geq f(t)$ for each $t \in [0,T]$ and the correspondence $f \to g$ from $A([0,T])$ in $A([0,T])$ is continuous.*

*Proof*. For $n \geq 2$, let $y_n = f(T/(n-1))$. Let $y_1 = 2f(T)$. Then, for every $n \in \mathbf{N}$ and $t \in [0, T/n]$, $f(t) \leq y_{n+1} < y_n$. Define $g: [0,T] \to \mathbf{R}$ by:

$$g(t) = \begin{cases} (n(t/T) - 1)(n+1)(y_n - y_{n+1}) + y_n & \text{if } t \in [T/(n+1), T/n], \\ 0 & \text{if } t = 0. \end{cases}$$

4.2. THEOREM. *There exists an embedding $\phi: W(\mathscr{H}) \to W(2^X)$ such that $\phi(w)$ is an extension of $w$ for each $w \in W(\mathscr{H})$.*

*Proof*. For each $w \in W_0(\mathscr{H})$ ($W_0(\mathscr{H})$ is defined in §1), we define $f: [0,\infty) \to \mathbf{R}$ by $f(0) = 0$ and, for $t > 0$,

$$f(t) = \sup\{w(A) - w(B): A, B \in \mathscr{H} \text{ and } A \subset N(t, B)\}.$$

(A) $f$ is an increasing function and $f$ is continuous at 0.

Clearly, $f$ is an increasing function. Suppose that $f$ is not continuous at 0. Then $\varepsilon = \inf\{f(t): t > 0\} > 0$. So $f(1/n) > \varepsilon/2$ for every $n \in \mathbf{N}$. Thus there exist sequences $(A_n)_n$ and $(B_n)_n$ in $\mathscr{H}$ such that $w(A_n) - w(B_n) \geq \varepsilon/2$ and $A_n \subset N(1/n, B_n)$. Let $A, B \in \mathscr{H}$ and let $(n_k)_k$ be a strictly increasing sequence in $\mathbf{N}$ such that $A_{n_k} \to A$ and

$B_{n_k} \to B$. Then $\varepsilon/2 \leq w(A) - w(B)$, so $w(B) < w(A)$. But $A_{n_k} \subset N(1/(n_k), B_{n_k})$ for all $k$, implies that $A \subset B$. Thus $w(A) \leq w(B)$. This contradiction proves (A).

Define $T = 2$(diameter of $X$) and $f_1 : [0, T] \to \mathbf{R}$ by: $f_1(t) = f(t)+t$.

(B) The correspondence $w \to f_1$ from $W_0(\mathscr{H})$ in $A([0, T])$ is continuous, $f_1$ is strictly increasing and continuous in 0.

By Lemma 4.1, we can associated a continuous function $g$ to $f_1$ such that $g(t) \geq f_1(t)$ for each $t \in [0, T]$, $g(0) = 0$ and $g$ is strictly increasing. Then the correspondence $w \to g$ from $W_0(\mathscr{H})$ in $A([0, T])$ is continuous. We define $v : 2^X \to \mathbf{R}$ by: $v(B) = \inf\{w(A) + g(t) : A \in \mathscr{H}, 0 < t \leq T \text{ and } B \subset N(t, A)\}$.

(C) $v$ is continuous, $v$ extends $w, v \in W_0(2^X)$ and the correspondence $w \to v$ is continuous.

Take $\varepsilon > 0$. Let $\delta > 0$ be such that $B_1, B_2 \in \mathscr{H}$; $H(B_1, B_2) < \delta$ and $s, t \in [0, T]$; $|s - t| < \delta$ implies that $|w(B_1) - w(B_2)| < \varepsilon/2$ and $|g(s) - g(t)| < \varepsilon/2$. Let $B_1, B_2 \in \mathscr{H}$ be such that $H(B_1, B_2) < \delta/2$. Take $A_1 \in \mathscr{H}$ and $t \in (0, T]$ such that $B_1 \subset N(t, A_1)$ and $v(B_1) \leq w(A_1) + g(t) < v(B_1) + \varepsilon/2$. Since $B_2 \subset N(\delta/2, B_1)$ and $B_1 \subset N(t, A_1)$, we have that $B_2 \subset N(t + \delta/2, A_1)$. Let $s = \min\{T, t + \delta/2\}$. Then $B_2 \subset N(s, A_1)$ and $|s - t| < \delta$. Thus $|g(s) - g(t)| < \varepsilon/2$ and

$$v(B_2) \leq w(A_1) + g(s) \leq w(A_1) + g(t) + \varepsilon/2 < v(B_1) + \varepsilon.$$

Hence $v(B_2) < v(B_1) + \varepsilon$. Similarly it can be proved that $v(B_1) < v(B_2) + \varepsilon$. This proves the continuity of $v$.

If $B \in \mathscr{H}$, since $B \subset N(B, t)$ for all $t \in (0, T]$, then $v(B) \leq w(B)$. From definition of $f$ it follows that $v(B) \geq w(B)$. Hence $v(B) = w(B)$. It is easy to prove that $v \in W_0(2^X)$ and that the correspondence $w \to v$ is continuous.

From now on, we copy Ward's proof giving explicit constructions for some steps in order to be able to check that $\phi$ is continuous.

Let $\beta$ be a denumerable basis for the topology of $X$. For each finite sequence $L = (T_i, T_2, \ldots, T_k)$ of elements of $\beta$, define $G(L) = \{A \in 2^X : A \subset \mathrm{Cl}_X(T_1 \cup \cdots \cup T_k)\}$ and

$$K(L) = \{A \in 2^X : A \cap \mathrm{Cl}_X(T_1) \neq \varnothing; \cdots ; A \cap \mathrm{Cl}_X(T_k) \neq \varnothing\}.$$

Then $G(L)$, $K(L) \in 2^{2^X}$ and $X \in K(L)$. Let $((L_n, M_n))_n$ be an enumeration of $\{(L, M) : L \text{ and } M \text{ are finite sequences of elements of } \beta$ such that $G(L) \cap K(M) = \varnothing\}$.

Given $w \in W(\mathscr{H})$, take $v \in W_0(2^X)$ as in (C). For each $n \in \mathbf{N}$, let $r_n = \max w|(G(L_n) \cap \mathscr{H})$ and $R_n = \min w|(K(M_n) \cap \mathscr{H})$.

Define $v_n \colon G(L_n) \cup \mathscr{H} \cup K(M_n) \to \mathbf{R}$ by:

$$v_n(A) = \begin{cases} \min\{r_n, v(A)\} & \text{if } A \in G(L_n), \\ \max\{R_n, v(A)\} & \text{if } A \in K(M_n), \\ w(A) & \text{if } A \in \mathscr{H}. \end{cases}$$

It is easy to prove that:

(D) $v_n$ is well defined, $v_n \in W_0(G(L_n) \cup \mathscr{H} \cup K(M_n))$ and the correspondence $w \to v_n$ is continuous.

Take $u_n \in W_0(2^X)$ an extension of $v_n$ as in (C) (then the correspondence $v_n \to u_n$ is continuous).

Define $u \colon 2^X \to [0, 1]$ by $u = \sum(u_n)/(2^n)$. Then:

(E) $u \in W(2^X)$, $u$ extends $w$ and the correspondence $w \to u$ is continuous.

Let $A, B \in 2^X$ be such that $A \subset B \neq A$. We will prove that $u(A) < u(B)$. For this, it is enough to show that there exists $n \in \mathbf{N}$ such that $A \in G(L_n)$, $B \in K(M_n)$ and $r_n < R_n$. Assume the last assertion is false. Let $K \in \mathbf{N}$ be such that $B \subset N(2/K, A)$. For $k \geq K$, choose finite sequences $L = (T_1, \dots, T_a)$ and $M = (S_1, \dots, S_b)$ of $\beta$ such that $A \subseteq T_1 \cup \cdots \cup T_a \subset N(1/k, A)$; $B \subset S_1 \cup \cdots \cup S_b$; diameter of $T_j < 1/k$; diameter of $S_i < 1/k$ and $B \cap S_i \neq \varnothing$ for all $i \in \{1, \dots, b\}$. Then $A \in G(L)$, $B \in K(M)$ and $G(L) \cap K(M) = \varnothing$. So there exists $s_k \in \mathbf{N}$ such that $(L, M) = (L_{s_k}, M_{s_k})$.

Because we are assuming, $r_{s_k} \geq R_{s_k}$. So there exist $A_k \in G(L_{s_k}) \cap \mathscr{H}$ such that $w(A_k) \geq w(B_k)$. Let $A_0, B_0 \in \mathscr{H}$ and $(A_{k_c})_c, (B_{k_c})_c$ be subsequences of $(A_k)_k$, $(B_k)_k$, respectively, such that $A_{k_c} \to A_0$ and $B_{k_c} \to B_0$. Then $A_0 \subset A$, $B_0 \subset B$ and $w(A_0) \geq w(B_0)$. This is a contradiction since $w$ is a Whitney map. This completes the proof of $u(A) < u(B)$.

Define $\phi \colon W(\mathscr{H}) \to W(2^X)$ by $\phi(w) = u$. Then $\phi$ is continuous. Define $\psi \colon W(2^X) \to W(\mathscr{H})$ by $\psi(u) = u|\mathscr{H}$, then $\psi \circ \phi =$ Identity of $W(\mathscr{H})$ and $\phi \circ (\psi|\phi(W(\mathscr{H}))) =$ Identity of $\operatorname{Im}\phi$. Hence $\phi$ is an embedding.

4.3. COROLLARY. *$W(\mathscr{H})$ is homeomorphic to a subspace of $W(2^X)$ which is a strong deformation retract of $W(2^X)$.*

*Proof.* Let $\phi$ and $\psi$ be as above; then $\operatorname{Im}\phi$ is homeomorphic with $W(\mathscr{H})$ and $\operatorname{Im}\phi$ is a retract of $W(2^X)$. Thus, since $W(2^X)$ is convex ([**2**, 14.71.3]), $\operatorname{Im}\phi$ is a strong deformation retract of $W(2^X)$.

*Added in proof*. The author has proved that $W(C(X))$ is homeomorphic to $l_2$ for every continuum $X$. ("The space of Whitney levels" preprint).

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Received October 25, 1987 and in revised form May 24, 1988.

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