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COMMUTATIVE SUBALGEBRAS OF THE RING OF DIFFERENTIAL OPERATORS ON A CURVE

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Let X denote an irreducible affine algebraic curve over the complex numbers. Let $\mathcal{O}(X)$ be the ring of regular functions on X. Denote by $\mathcal{D}(X)$ the ring of differential operators on X. We wish to characterize $\mathcal{O}(X)$ as a ring theoretic invariant of $\mathcal{D}(X)$. It is proved that $\mathcal{O}(X)$ equals the set of all locally ad-nilpotent elements of $\mathcal{D}(X)$ if and only if X is not simply connected. However, for most simply connected curves, we show there exists a maximal commutative subalgebra of $\mathcal{D}(X)$, consisting of locally ad-nilpotent elements, which is not isomorphic to $\mathcal{O}(X)$.

0. Introduction. Let X be a curve, that is, an irreducible affine algebraic curve over $\mathbb{C}$. Write $\mathcal{O}(X)$ for the ring of regular functions on X and $\mathcal{D}(X)$ for the ring of differential operators on X. See **[8]** for the basic definitions and facts about rings of differential operators on curves. This paper is motivated by the following question. If X and Y are curves with $\mathcal{D}(X) \cong \mathcal{D}(Y)$, is $X \cong Y$? Write $\tilde{X}$ for the normalization of X. Stafford **[9]** considers this question for X with $\tilde{X} = \mathbb{A}^1$, the affine line. He shows that $\mathcal{D}(X) \cong \mathcal{D}(\tilde{X})$ if and only if $X = \tilde{X}$. He also shows that if X is the cubic cusp $y^2 = x^3$ and $\tilde{Y} = \mathbb{A}^1$, then $X \cong Y$ if and only if $\mathcal{D}(X) \cong \mathcal{D}(Y)$. Higher dimensional non-isomorphic varieties can have isomorphic rings of differential operators, see Levasseur, Smith and Stafford **[2]**.

If $u \in \mathcal{D}(X)$, define $\operatorname{ad}(u) \in \operatorname{End}_{\mathbb{C}}(\mathcal{D}(X))$ by $\operatorname{ad}(u)(v) = [u, v] = uv - vu$. We say u is *locally ad-nilpotent* if for every $v \in \mathcal{D}(X)$ there exists $n \in \mathbb{N}$ with $\operatorname{ad}(u)^n(v) = 0$. Write

$$\mathcal{N}(X) = \{u \in \mathcal{D}(X) | u \text{ is locally ad-nilpotent}\}.$$

Note that if $\vartheta: \mathcal{D}(Y) \to \mathcal{D}(X)$ is an isomorphism then $\vartheta(\mathcal{N}(Y)) = \mathcal{N}(X)$. It follows from the definition of $\mathcal{D}(X)$ that $\mathcal{O}(X)$ is a maximal commutative subalgebra of $\mathcal{D}(X)$ and that $\mathcal{O}(X)$ is contained in $\mathcal{N}(X)$. If genus $(X) > 0$ then Makar-Limanov **[3]** shows that $\mathcal{O}(X) = \mathcal{N}(X)$. Hence if $\mathcal{D}(X) \cong \mathcal{D}(Y)$ with genus$(X) > 0$ then $X \cong Y$.

This paper expands on Makar-Limanov's result to prove the following theorem. Let $\pi: \tilde{X} \to X$ denote the canonical surjection.

THEOREM. *Let X be a curve. Then $\mathscr{N}(X) = \mathscr{O}(X)$ if and only if either*

(i) $\tilde{X} \neq \mathbb{A}^1$ *or*

(ii) $\tilde{X} = \mathbb{A}^1$ *and* $\pi\colon \tilde{X} \to X$ *is not injective.*

COROLLARY 1. *Let X and Y be curves with $\mathscr{D}(X) \cong \mathscr{D}(Y)$. If either* (i) $\tilde{X} \neq \mathbb{A}^1$ *or* (ii) $\tilde{X} = \mathbb{A}^1$ *and* $\pi\colon \tilde{X} \to X$ *is not injective, then* $X \cong Y$.

The condition that $\tilde{X} = \mathbb{A}^1$ and $\pi\colon \tilde{X} \to X$ is injective is equivalent to the condition that X is simply connected. This observation is due to S. P. Smith and will be proved in §5. We obtain the following connection between algebra and topology.

COROLLARY 2. *Let X be a curve. Then $\mathscr{O}(X) = \mathscr{N}(X)$ if and only if X is not simply connected. Thus if $\mathscr{D}(X) \cong \mathscr{D}(Y)$ and X is not simply connected then $X \cong Y$.*

If $\mathscr{O}(X) \neq \mathscr{N}(X)$ then $\tilde{X} = \mathbb{A}^1$ and $\pi\colon \tilde{X} \to X$ is injective. In **[8]** it is shown, for such X, that $\mathscr{D}(X)$ is Morita equivalent to $\mathscr{D}(\mathbb{A}^1)$. Thus if X and Y are curves with $\mathscr{O}(X) \neq \mathscr{N}(X)$ and $\mathscr{O}(Y) \neq \mathscr{N}(Y)$ then $\mathscr{D}(X)$ is Morita equivalent to $\mathscr{D}(Y)$. The results of **[8]** also show that $\mathscr{D}(X)$ and $\mathscr{D}(Y)$ are simple, hereditary $\mathbb{C}$-algebras with the same Krull and GK dimensions. Hence it will be difficult to distinguish between them. Nevertheless, distinguishing between such "small" non-commutative algebras is an important problem in ring theory.

We will always write $\mathscr{D}(\mathbb{A}^1) = \mathbb{C}[t, \partial]$ where $\partial = d/dt$. Dixmier **[1]** shows that all maximal commutative subalgebras of $\mathscr{D}(\mathbb{A}^1)$, contained in $\mathscr{N}(\mathbb{A}^1)$, are conjugate under $\operatorname{Aut}\mathscr{D}(\mathbb{A}^1)$. Thus if $\varphi\colon \mathscr{D}(X) \to \mathscr{D}(\mathbb{A}^1)$ is an isomorphism, there exists $\mu \in \operatorname{Aut}\mathscr{D}(\mathbb{A}^1)$ such that $\mu\varphi$ restricts to an isomorphism between $\mathscr{O}(X)$ and $\mathbb{C}[t] = \mathscr{O}(\mathbb{A}^1)$. This leads one to ask, for an arbitrary curve X, if every maximal commutative subalgebra of $\mathscr{D}(X)$ which is contained in $\mathscr{N}(X)$ is isomorphic to $\mathscr{O}(X)$.

Let X be a curve with $\tilde{X} = \mathbb{A}^1$ and $\pi\colon \tilde{X} \to X$ injective. In §§3 and 4, we consider such curves with the additional hypothesis that X has unique singularity at $\pi(0)$. It is shown that for all but one such curve, there are maximal commutative subalgebras of $\mathscr{D}(X)$ contained in $\mathscr{N}(X)$ but not isomorphic to $\mathscr{O}(X)$. The one "well behaved" curve is the cubic cusp $y^2 = x^3$. Thus maximal commutative subalgebras of $\mathscr{D}(X)$ consisting of locally ad-nilpotent elements are not alone sufficient to distinguish these rings from one another.

These results will appear in the author's Ph.D. thesis. I would like to thank my advisor, S. Paul Smith, for interesting me in these questions and for his many helpful comments and suggestions.

1. Curves with $\mathscr{O}(X) = \mathscr{N}(X)$. In this section it is proved that if $\mathscr{O}(X) \neq \mathscr{N}(X)$ then $\tilde{X} = \mathbb{A}^1$ and the canonical surjection $\pi\colon \tilde{X} \to X$ is injective. The rest of the paper considers the case with $\tilde{X} = \mathbb{A}^1$ and π injective; these are the only curves whose differential operator rings are hard to distinguish from one another. Theorem 1.1 and Corollary 1.2 are results of L. Makar-Limanov. We thank him for allowing us to present them here. The proofs will appear in [3]. Most of this section is based on his methods.

THEOREM 1.1 (*Makar-Limanov*). *If X is a curve and $\mathscr{O}(X) \neq \mathscr{N}(X)$ then $\mathscr{O}(X) \subset \mathbb{C}[b]$ for some b transcendental over $\mathbb{C}$.*

COROLLARY 1.2 (*Makar-Limanov*). *If X is a curve with genus greater than zero then $\mathscr{O}(X) = \mathscr{N}(X)$.*

Using Theorem 1.1, we now show that if $\mathscr{O}(X) \neq \mathscr{N}(X)$ then $\mathscr{O}(\tilde{X}) = \mathbb{C}[t]$ for some t transcendental over $\mathbb{C}$. Proposition 1.3 is probably well known but we could find no reference for it.

PROPOSITION 1.3. *Let X be a curve with $\mathscr{O}(X) \subset \mathbb{C}[b]$ for some b transcendental over $\mathbb{C}$. Then there exists $t \in \operatorname{Fract}\mathscr{O}(X)$ with $\mathscr{O}(\tilde{X}) = \mathbb{C}[t]$.*

Proof. Recall that $\tilde{X}$ is an affine nonsingular curve. By Luroth's theorem, $\tilde{X}$ is rational, since $\operatorname{Fract}\mathscr{O}(\tilde{X}) = \operatorname{Fract}\mathscr{O}(X) \subset \mathbb{C}(b)$. Thus $\mathscr{O}(\tilde{X})$ is a UFD and every nonzero prime ideal of $\mathscr{O}(\tilde{X})$ is principal. Since $\mathscr{O}(X) \subset \mathbb{C}[b]$ and $\mathbb{C}[b]$ is integrally closed in $\mathbb{C}(b)$, $\mathscr{O}(\tilde{X}) \subset \mathbb{C}[b]$. Choose $t \in \mathscr{O}(\tilde{X}) \setminus \mathbb{C}$ of minimal degree in b. Then for all $\alpha \in \mathbb{C}$, $t - \alpha$ generates a maximal ideal of $\mathscr{O}(\tilde{X})$. Indeed, if $(t-\alpha)$ were not maximal then $t - \alpha = xy$ with $x, y \in \mathscr{O}(\tilde{X})$ nonunits, since $\mathscr{O}(\tilde{X})$ is a UFD. But then $\deg_b(x) < \deg_b(t)$, a contradiction.

Let $\mathfrak{m}$ be the ideal generated by t. If $\mathfrak{m}'$ is another maximal ideal and $\varphi\colon \mathscr{O}(\tilde{X}) \to \mathscr{O}(\tilde{X})/\mathfrak{m}'$, then $\varphi(t) = \alpha \in \mathbb{C}$. Thus $\mathfrak{m}'$ contains the maximal ideal $(t-\alpha)$ and hence equals it. If w is an irreducible element of $\mathscr{O}(\tilde{X})$ then $0 \neq (w)$ is prime and hence maximal since $\dim\mathscr{O}(\tilde{X}) = 1$. Thus $(w) = (t-\beta)$ for some $\beta \in \mathbb{C}$ and so $uw = t - \beta$ for some unit of $\mathscr{O}(\tilde{X})$. But the only units of $\mathscr{O}(\tilde{X})$ are in $\mathbb{C}$ since $\mathscr{O}(\tilde{X}) \subset \mathbb{C}[b]$. Hence, by unique factorization, any element of $\mathscr{O}(\tilde{X})$ is of the form $\lambda(t-\alpha_1)\cdots(t-\alpha_k)$ for $\lambda, \alpha_1, \ldots, \alpha_k \in \mathbb{C}$. Thus $\mathcal{O}(\tilde{X}) \subset \mathbb{C}[t]$ and hence there is equality. □

COROLLARY 1.4. *If X is a curve with $\mathcal{O}(X) \neq \mathcal{N}(X)$ then $\tilde{X} = \mathbb{A}^1$. Thus if $\tilde{X} \neq \mathbb{A}^1$ then $\mathcal{O}(X) = \mathcal{N}(X)$.*

Proof. By Theorem 1.1, $\mathcal{O}(X) \subset \mathbb{C}[b]$. Proposition 1.3 implies $\mathcal{O}(\tilde{X}) = \mathbb{C}[t]$ and this yields the result. □

COROLLARY 1.5. *Let X and Y be curves such that $\tilde{X} \neq \mathbb{A}^1$. If $\mathcal{D}(X) \cong \mathcal{D}(Y)$ then $X \cong Y$.* □

If $\tilde{X} = \mathbb{A}^1$ it is still possible to have $\mathcal{O}(X) = \mathcal{N}(X)$. Proposition 1.8 clarifies the situation. First are some technical lemmas. If $\mathcal{O}(\tilde{X}) = \mathbb{C}[t]$ then $\mathcal{D}(X) \subset \mathbb{C}(t)[\partial]$ and hence inherits the filtration by order of differential operator. We form the associated graded ring, denoted $\operatorname{gr}_\partial \mathcal{D}(X)$ to avoid confusion with another graded ring considered later in this paper. Write $\operatorname{gr}_\partial \mathcal{D}(\mathbb{A}^1) = \mathbb{C}[t, \xi]$. If $\tilde{X} = \mathbb{A}^1$ then $\operatorname{gr}_\partial \mathcal{D}(X) \subset \mathbb{C}[t, \xi]$ by [**8**, 3.11]. Throughout the paper we use $\{f, g\}$ to denote the Poisson bracket

$$(\partial f/\partial \xi)(\partial g/\partial t) - (\partial f/\partial t)(\partial g/\partial \xi)$$

with $f, g \in \mathbb{C}(t)[\xi]$. For $u, v \in \mathbb{C}(t)[\partial]$, it is straightforward to check that $\operatorname{gr}_\partial[u, v] = \{\operatorname{gr}_\partial(u), \operatorname{gr}_\partial(v)\}$ if the latter is nonzero. Thus $\{\operatorname{gr}_\partial(u), \cdot\} \in \operatorname{End}_{\mathbb{C}}(\operatorname{gr}_\partial \mathcal{D}(X))$ for all $u \in \mathcal{D}(X)$.

LEMMA 1.6. *Let $p\xi^n \in \mathbb{C}[t, \xi]$ where $p \in \mathbb{C}[t]$ is of degree r and $n > 0$. Let $q \in \mathbb{C}[t]$ be of degree $s > 0$. If $\delta = \{p\xi^n, \cdot\}$ is nilpotent on $q\xi^m$ then, for some $i > 0$, $rm = ns + i(r-n)$.*

Proof. Assume $rm \neq ns + i(r-n)$ for all $i > 0$ and let t^r be the leading term of p. We show by induction on i that $\delta^i(q\xi^m) = q_i\xi^{m+i(n-1)}$ where q_i has leading term $\alpha_i t^{s+i(r-1)}$ with $0 \neq \alpha_i \in \mathbb{C}$. This contradicts the hypothesis that δ is nilpotent on $q\xi^m$ and hence proves the lemma. Of course the result is true for $i = 0$. Note that, if $h \in \mathbb{C}[t]$, $\delta(h\xi^l) = (nph' - lhp')\xi^{l+n-1}$ where f' denotes df/dt. By induction, suppose $\delta^i(q\xi^m) = q_i\xi^{m+i(n-1)}$ where the leading term of q_i is $\alpha_i t^{s+i(r-1)}$ with $0 \neq \alpha_i \in \mathbb{C}$. Then

$$\delta^{i+1}(q\xi^m) = q_{i+1}\xi^{m+(i+1)(n-1)}$$

and the leading term of q_{i+1} is

$$\alpha_i(n(s + i(r-1)) - (m + i(n-1))r)t^{s+(i+1)(r-1)}.$$

Since $(n(s+i(r-1))-(m+i(n-1))r)=ns+i(r-n)-rm\neq 0$ by hypothesis, this completes the proof. □

LEMMA 1.7. *If X is a curve with $\tilde{X}=\mathbb{A}^1$ and $u\in\mathscr{N}(X)\setminus\mathscr{O}(X)$ then $\operatorname{gr}_\partial(u)=\lambda\xi^n$ for some positive integer n and $\lambda\in\mathbb{C}$.*

Proof. By way of contradiction, assume $\operatorname{gr}_\partial(u)=p\xi^n$ with $\deg(p)=r>0$ and let $\operatorname{ann}_{\mathscr{O}(X)}(\mathbb{C}[t]/\mathscr{O}(X))=q\mathbb{C}[t]$. Then $q\mathbb{C}[t,\partial]\subset\mathscr{D}(X)$ and q has order zero, whence $q\mathbb{C}[t,\xi]\subset\operatorname{gr}_\partial\mathscr{D}(X)$. Let $\deg(q)=s>0$ and assume $r\geq n$. Since $\{\operatorname{gr}_\partial(u),\cdot\}$ is nilpotent on q, Lemma 1.6 implies $0=r\cdot 0=ns+i(r-n)\geq s$, a contradiction. Thus $r<n$. Hence if $\{\operatorname{gr}_\partial(u),\cdot\}$ is nilpotent on $q\xi^m$, $rm=ns+i(r-n)\leq ns$. Choose m such that $rm>ns$. Then $q\xi^m\in\operatorname{gr}_\partial\mathscr{D}(X)$ but $\{\operatorname{gr}_\partial(u),\cdot\}$ is not nilpotent on it. This contradiction implies $r=0$. □

PROPOSITION 1.8. *If X is a curve with $\tilde{X}=\mathbb{A}^1$ and $\mathscr{O}(X)\neq\mathscr{N}(X)$ then* $\dim(\mathbb{C}[t,\xi]/\operatorname{gr}_\partial\mathscr{D}(X))<\infty$. *In particular, $\pi\colon\tilde{X}\to X$ is injective.*

Proof. The second assertion follows from the first by [**8**, 3.12]. Choose $u\in\mathscr{N}(X)\setminus\mathscr{O}(X)$. By Lemma 1.7 we may assume that $\operatorname{gr}_\partial(u)=\xi^n$ with $n>0$. As in the proof of Lemma 1.7, pick $q\in\mathbb{C}[t]$ such that $q\mathbb{C}[t,\xi]\subset\operatorname{gr}_\partial\mathscr{D}(X)$ and let t^s be the leading term of q. We claim $\xi^{s(n-1)}\mathbb{C}[t,\xi]\subset\operatorname{gr}_\partial\mathscr{D}(X)$. Assuming the claim, let $v\in\mathbb{C}[t,\xi]\setminus\operatorname{gr}_\partial\mathscr{D}(X)$. Since $q\mathbb{C}[t,\xi]\subset\operatorname{gr}_\partial\mathscr{D}(X)$, we may assume $t-\deg(v)\leq s-1$. Since $\xi^{s(n-1)}\mathbb{C}[t,\xi]\subset\operatorname{gr}_\partial\mathscr{D}(X)$, we may assume $\xi-\deg(v)\leq s(n-1)-1$. Thus v is in the finite dimensional vector space spanned by $\{t^i\xi^j|0\leq i\leq s-1, 0\leq j\leq s(n-1)-1\}$ and this proves the result.

To prove the claim, we show $t^i\xi^j\in\operatorname{gr}_\partial\mathscr{D}(X)$ for all $j\geq s(n-1)$ by induction on i. By the remarks before Lemma 1.6, $\delta=\{\xi^n,-\}\in\operatorname{End}(\operatorname{gr}_\partial\mathscr{D}(X))$. Note that $\delta(f\xi^m)=nf'\xi^{m+(n-1)}$ for $f\in\mathbb{C}[t]$. For all $m\geq 0$, $q\xi^m\in\operatorname{gr}_\partial\mathscr{D}(X)$. Hence so is $\delta^s(q\xi^m)=\alpha\xi^{m+s(n-1)}$ for some $0\neq\alpha\in\mathbb{C}$. It follows that for all $j\geq s(n-1)$, $\xi^j\in\operatorname{gr}_\partial\mathscr{D}(X)$ and we can start the induction. Assume that $i>0$ and for $k<i$, $j\geq s(n-1)$ implies that $t^k\xi^j\in\operatorname{gr}_\partial\mathscr{D}(X)$. If $i<s$ then

$$\delta^{s-i}(q\xi^l)=\beta q^{(s-i)}\xi^{l+(s-i)(n-1)}\in\operatorname{gr}_\partial\mathscr{D}(X)$$

for all $l\geq 0$, where $q^{(s-i)}$ denotes the $s-i$th derivative of q and $0\neq\beta\in\mathbb{C}$. The leading term of $q^{(s-i)}$ is a nonzero constant times t^i so our induction hypothesis implies that $t^i\xi^{l+(s-i)(n-1)}\in\operatorname{gr}_\partial\mathscr{D}(X)$ for all $l\geq i(n-1)$. If $i\geq s$ then $t^{i-s}\xi^l q\in\operatorname{gr}_\partial\mathscr{D}(X)$ for all $l\geq 0$. Since the leading term of $t^{i-s}\xi^l q$ is $t^i\xi^l$ and the induction hypothesis again applies, this completes the induction and the proof. □

COROLLARY 1.9. *If X is a curve such that either*
(i) $\tilde{X} \neq \mathbb{A}^1$ *or*
(ii) $\tilde{X} = \mathbb{A}^1$ *but* $\pi\colon \tilde{X} \to X$ *is not injective*
then $\mathscr{O}(X) = \mathscr{N}(X)$.

Proof. Combine Corollary 1.4 and Proposition 1.8. □

COROLLARY 1.10. *Let X and Y be curves with $\mathscr{D}(X) \cong \mathscr{D}(Y)$. If either $\tilde{X} \neq \mathbb{A}^1$ or $\pi\colon \tilde{X} \to X$ is not injective then $X \cong Y$.* □

2. Locally Ad-**nilpotent elements not in** $\mathscr{O}(X)$. In §1 we saw that if either $\tilde{X} \neq \mathbb{A}^1$ or $\pi\colon \tilde{X} \to X$ is not injective then $\mathscr{O}(X) = \mathscr{N}(X)$. Thus in either of these two cases, $\mathscr{O}(X)$ is the unique maximal commutative subalgebra of $\mathscr{D}(X)$ contained in $\mathscr{N}(X)$. In this section we show, conversely, that if $\tilde{X} = \mathbb{A}^1$ and $\pi\colon \tilde{X} \to X$ is injective then $\mathscr{N}(X)$ properly contains $\mathscr{O}(X)$. Indeed, $\mathscr{D}(X)$ contains a maximal commutative subalgebra $\mathscr{D}_0^t(X)$, consisting of locally ad-nilpotent elements, such that $\mathscr{D}_0^t(X) \cap \mathscr{O}(X) = \mathbb{C}$. Several of the results will be proved in a generality that will prove useful later.

When $X = \mathbb{A}^1$, $\mathscr{D}_0^t(X)$ is just $\mathbb{C}[\partial]$ and in the general case $\mathscr{D}_0^t(X)$ should be thought of as analogous to $\mathbb{C}[\partial]$. However, in general there is no derivation $\delta \in \mathbb{D}(X)$ with $\mathbb{C}[\delta]$ a maximal commutative subalgebra of $\mathscr{D}(X)$ contained in $\mathscr{N}(X)$. Thus finding the correct analogy of $\mathbb{C}[\partial]$ is complicated.

Throughout §2, let X be a curve with $\tilde{X} = \mathbb{A}^1$.

DEFINITION. Let R be a subalgebra of $\mathbb{C}(t)[\partial]$. A commutative subalgebra S of R is *ad-nilpotent* if, for every $u \in R$, there exists $n \in \mathbb{N}$ with $[s_0, [s_1, \dots [s_n, u]]] = 0$ for all $s_0, \dots, s_n \in S$. An ad-nilpotent subalgebra of $\mathscr{D}(X)$ is always contained in $\mathscr{N}(X)$. By the definition of $\mathscr{D}(X)$, $\mathscr{O}(X)$ is an ad-nilpotent subalgebra of $\mathscr{D}(X)$.

Let $K = \mathbb{C}(t)$ and consider $\mathscr{D}(K) = \mathbb{C}(t)[\partial]$. An element $u \in \mathscr{D}(K)$ can be uniquely expressed $u = \sum f_k \partial^k$ where k ranges over a finite subset of $\mathbb{N}$ and $f_k \in \mathbb{C}(t)$. Since $\tilde{X} = \mathbb{A}^1$, we may identify $\mathscr{D}(X)$ with a subalgebra of $\mathscr{D}(K)$. Set $\deg(f/g) = \deg(f) - \deg(g)$ for $f, g \in \mathbb{C}[t]$.

DEFINITION. If $u = \sum f_k \partial^k \in \mathscr{D}(K)$ define the *t-degree* of u to be $t - \deg(u) = \max\{\deg(f_k)\}$. Set $\mathscr{D}_k^t(K) = \{u \in \mathscr{D}(K) | t - \deg(u) \leq k\}$ for $k \in \mathbb{Z}$. It is clear that $\mathscr{D}_i^t(K) \cdot \mathscr{D}_j^t(K) \subset \mathscr{D}_{i+j}^t(K)$

so we obtain the filtration $\mathscr{D}(K) = \bigcup_{k\in\mathbb{Z}} \mathscr{D}^t_k(K)$. Form the graded ring $\bigoplus_{k\in\mathbb{Z}}(\mathscr{D}^t_k(K)/\mathscr{D}^t_{k-1}(K))$ in the usual way and denote this ring $\mathrm{gr}_t\mathscr{D}(K)$. Define the *t-filtration* on $\mathscr{D}(X)$ by $\mathscr{D}^t_k(X) = \mathscr{D}(X)\cap\mathscr{D}^t_k(K)$. Then $\mathrm{gr}_t\mathscr{D}(X)$ embeds in $\mathrm{gr}_t\mathscr{D}(K)$.

LEMMA 2.1. *For all* $n, m \in \mathbb{Z}$, $[\mathscr{D}^t_n(K), \mathscr{D}^t_m(K)] \subset \mathscr{D}^t_{n+m-1}(K)$ *and hence* $\mathrm{gr}_t\mathscr{D}(K)$ *is commutative.*

Proof. Let $h, f, p, q \in \mathbb{C}[t]$, with $\deg(h/f) = n$ and $\deg(p/q) = m$. If $k, l \in \mathbb{N}$, it suffices to prove $[(h/f)\partial^k, (p/q)\partial^l] \in \mathscr{D}^t_{n+m-1}(K)$. Now

$$[(h/f)\partial^k, (p/q)\partial^l] = (h/f)[\partial^k, p/q]\partial^l + (p/q)[h/f, \partial^l]\partial^k.$$

But

$$\begin{aligned}[h/f, \partial^l] &= -\sum_{0\le i\le l-1} \partial^i[\partial, h/f]\partial^{l-i-1}\\ &= -\sum_{0\le i\le l-1} \partial^i(fh' - hf')f^{-2}\partial^{l-i-1}\end{aligned}$$

so $t-\deg[h/f, \partial^l] \le \deg(fh' - hf')f^{-2} \le n-1$. Thus $(h/f)[\partial^k, p/q]\partial^l$ and $(p/q)[h/f, \partial^l]\partial^k$ are in $\mathscr{D}^t_{n+m-1}(K)$ and the result follows. □

LEMMA 2.2. $\mathrm{gr}_t\mathscr{D}(K) \cong \mathbb{C}[s, s^{-1}][\xi]$.

Proof. Since $\mathrm{gr}_t\mathscr{D}(K)$ is commutative and $\mathrm{gr}_t(t)\,\mathrm{gr}_t(t^{-1}) = 1$, there exists an algebra homomorphism

$$\varphi\colon \mathbb{C}[s, s^{-1}][\xi] \to \mathrm{gr}_t\mathscr{D}(K)$$

given by $\varphi(s) = \mathrm{gr}_t(t)$, $\varphi(s^{-1}) = \mathrm{gr}_t(t^{-1})$ and $\varphi(\xi) = \mathrm{gr}_t(\partial)$. If $\varphi(\sum f_i(\xi)s^i) = 0$ then $\sum \mathrm{gr}_t(f_i(\partial)t^i) = 0$, whence $f_i(\partial) = 0$ for all i. Thus φ is injective. If $f \in K$ then $f = g/h$ with $g, h \in \mathbb{C}[t]$. Write $g = \alpha t^n + p$ and $h = \beta t^m + q$ where $\alpha, \beta \neq 0$, $\deg(p) < n$ and $\deg(q) < m$. Now $f, \alpha\beta^{-1}t^{n-m} \in \mathscr{D}^t_{n-m}(K)$ and

$$\begin{aligned}f - \alpha\beta^{-1}t^{n-m} &= (g - \alpha\beta^{-1}t^{n-m}h)/h\\ &= (p - \alpha\beta^{-1}t^{n-m}q)/h \in \mathscr{D}^t_{n-m-1}(K).\end{aligned}$$

Thus $\mathrm{gr}_t(f) = \mathrm{gr}_t(\alpha\beta^{-1}t^{n-m}) \in \mathrm{Im}(\varphi)$. If $u \in \mathscr{D}(K)$ write $u = \sum f_i\partial^i$ with $f_i \in K$ and set $t - \deg(u) = k$. If $a, b \in \mathscr{D}(K)$ such that $t - \deg(a+b) = t - \deg(a) = t - \deg(b)$ then $\mathrm{gr}_t(a+b) = \mathrm{gr}_t(a) + \mathrm{gr}_t(b)$. It follows that

$$\mathrm{gr}_t(u) = \mathrm{gr}_t\left(\sum\{f_i\partial^i \mid \deg(f_i) = k\}\right) = \sum\{\mathrm{gr}_t(f_i\partial^i) \mid \deg(f_i) = k\}$$

and each $\mathrm{gr}_t(f_i\partial^i) = \mathrm{gr}_t(f_i)\,\mathrm{gr}_t(\partial)^i \in \mathrm{Im}(\varphi)$. Thus φ is surjective. □

If R is any subalgebra of $\mathscr{D}(K)$, define the t-filtration on R any $R_n^t = R \cap \mathscr{D}_n^t(K)$. We now prove the basic results about R_0^t.

LEMMA 2.3. *Let R be a subalgebra of $\mathscr{D}(K)$ such that $\operatorname{gr}_t(R) \subset \mathbb{C}[s, \xi]$ and let $v \in R_0^t$. Let $k > 0$ and $u \in \mathscr{D}(K)$. If $v \notin \mathbb{C}$ then $[v, u] \in \mathscr{D}_{k-1}^t(K)$ implies $u \in \mathscr{D}_k^t(K)$.*

Proof. Assume there exists $n > k$ with $u \in \mathscr{D}_n^t(K) \setminus \mathscr{D}_{n-1}^t(K)$. Write $u = t^n f(\partial) + x$ with $x \in \mathscr{D}_{n-1}^t(K)$. Write $v = g(\partial) + y \in \mathscr{D}_0^t(K)$ with $y \in \mathscr{D}_{-1}^t(K)$. Since $v \notin \mathbb{C}, \deg(g) > 0$ because $\operatorname{gr}_t(R) \subset \mathbb{C}[s, \xi]$. Hence

$$\begin{aligned} [v, u] &= [g(\partial), t^n f(\partial)] + [y, u] + [g(\partial), x] \\ &= nt^{n-1} f(\partial) g'(\partial) + (\text{something in } \mathscr{D}_{n-2}^t(K)). \end{aligned}$$

Thus $[v, u] \notin \mathscr{D}_{k-1}^t(K)$. □

By [**8**, 3.11], $\operatorname{gr}_\partial \mathscr{D}(X) \subset \operatorname{gr}_\partial \mathscr{D}(\tilde{X}) = \mathbb{C}[t, \xi]$ so we may take $R = \mathscr{D}(X)$ in the following result.

PROPOSITION 2.4. *Let R be a subalgebra of $\mathscr{D}(K)$ such that $\operatorname{gr}_\partial(R) \subset \mathbb{C}[t, \xi]$. Then*

(a) $\operatorname{gr}_t(R) \subset \mathbb{C}[s, \xi]$.

(b) *R_0^t is a commutative ad-nilpotent subalgebra of R.*

(c) *$R_0^t \cong \operatorname{gr}_t(R) \cap \mathbb{C}[\xi]$ as $\mathbb{C}$-algebras.*

(d) *If $R_0^t \neq \mathbb{C}$ then it is a maximal commutative subalgebra of R.*

Proof. (a) If $u \in R$ with $t - \deg(u) < 0$ then $u = \sum f_k \partial^k$ with $\deg(f_k) < 0$ for all k, whence $\operatorname{gr}_\partial(u) \notin \mathbb{C}[t, \xi]$. Hence, if $u \in R$ then $t - \deg(u) = n \geq 0$. Write $u = t^n h(\partial) + v$ with $v \in \mathscr{D}_{n-1}^t(K)$ and h a polynomial. Then $\operatorname{gr}_t(u) = s^n h(\xi) \in \mathbb{C}[s, \xi]$.

(b) Since elements of R have non-negative t-degrees, $R_{-1}^t = (0)$. By Lemma 2.1, $[R_0^t, R_n^t] \subset R_{n-1}^t$. Thus R_0^t is ad-nilpotent. It is commutative because $[R_0^t, R_0^t] \subset R_{-1}^t = (0)$.

(c) Note that $\operatorname{gr}_t R \cap \mathbb{C}[\xi]$ is the image of R_0^t under the map $R \to \operatorname{gr}_t(R)$. But this image is isomorphic to $\operatorname{gr}_t(R_0^t)$ under the induced filtration. Since $R_{-1}^t = (0)$,

$$\operatorname{gr}_t(R) \cap \mathbb{C}[\xi] \cong \operatorname{gr}_t(R_0^t) \cong R_0^t / R_{-1}^t \cong R_0^t$$

as $\mathbb{C}$-algebras.

(d) Let $v \in R_0^t \setminus \{\mathbb{C}\}$ and $u \in R$. By (a), $\operatorname{gr}_t(R) \subset \mathbb{C}[s, \xi]$. Thus if $[v, u] = 0 \in R_{-1}^t$ then $u \in R_0^t$ by Lemma 2.3. □

By Proposition 2.4 and the remark before it, $\mathrm{gr}_t \mathscr{D}(X) \subset \mathbb{C}[s,\xi]$. Thus if $\mathscr{D}_0^t(X) \neq \mathbb{C}$, Lemma 2.3 implies that the t-filtration on $\mathscr{D}(X)$ can be defined by the degree of ad-nilpotence of $\mathscr{D}_0^t(X)$ on $\mathscr{D}(X)$. That is

$$\mathscr{D}_n^t(X) = \{u \in \mathscr{D}(X) \mid [v_0,[v_1,\ldots[v_n,u]\ldots]] = 0 \text{ for all } v_0,\ldots,v_n \in \mathscr{D}_0^t(X)\}.$$

This is analogous to the way the order of operator filtration is defined, with $\mathscr{D}_0^t(X)$ playing the role of $\mathscr{O}(X)$.

Our next goal is to prove $\mathscr{D}_0^t(X) \neq \mathbb{C}$, whence it is a maximal commutative ad-nilpotent subalgebra. This will be achieved in Theorem 2.7. Note that $\mathrm{gr}_t \mathscr{D}(\mathbb{A}^1) = \mathbb{C}[s,\xi]$ and let $M \subset N$ be distinct nonzero $\mathrm{gr}_t \mathscr{D}(\mathbb{A}^1)$-submodules of $\mathrm{gr}_t \mathscr{D}(K)$. The *1-length* of N/M is defined, as in [**8**, 3.10], to be the largest integer n such that there exists a chain of $\mathrm{gr}_t \mathscr{D}(\mathbb{A}^1)$-modules $M = M_0 \subset M_1 \subset \cdots \subset M_n = N$ with $\dim_{\mathbb{C}}(M_i/M_{i-1}) = \infty$ for each i.

LEMMA 2.5. *Let $q \in \mathbb{C}[t]$. (i) If P is a left $\mathscr{D}(\mathbb{A}^1)$-module and $\mathscr{D}(\mathbb{A}^1)q^{-1} \supset P \supset \mathscr{D}(\mathbb{A}^1)$ then*

$$1\text{-}length(\mathrm{gr}_t P/\mathrm{gr}_t \mathscr{D}(\mathbb{A}^1)) = \mathrm{length}_{\mathscr{D}(\mathbb{A}^1)}(P/\mathscr{D}(\mathbb{A}^1)).$$

(ii) *If P is a right $\mathscr{D}(\mathbb{A}^1)$-module and $\mathscr{D}(\mathbb{A}^1) \supset P \supset q\mathscr{D}(\mathbb{A}^1)$ then*

$$1\text{-}length(\mathrm{gr}_t \mathscr{D}(\mathbb{A}^1)/\mathrm{gr}_t P) = \mathrm{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/P).$$

Proof. Let the leading term of q be t^k. We claim

$$1\text{-length}(\mathrm{gr}_t \mathscr{D}(\mathbb{A}^1)/\mathrm{gr}_t(q\mathscr{D}(\mathbb{A}^1))) = \mathrm{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/q\mathscr{D}(\mathbb{A}^1))$$

and

$$1\text{-length}(\mathrm{gr}_t(\mathscr{D}(\mathbb{A}^1)q^{-1})/\mathrm{gr}_t \mathscr{D}(\mathbb{A}^1)) = \mathrm{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)q^{-1}/\mathscr{D}(\mathbb{A}^1)).$$

To prove the claim, note that

$$\mathrm{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/q\mathscr{D}(\mathbb{A}^1)) = \mathrm{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)q^{-1}/\mathscr{D}(\mathbb{A}^1)) = k.$$

If $u = qv \in q\mathscr{D}(\mathbb{A}^1)$ then

$$\mathrm{gr}_t(u) = (\mathrm{gr}_t(q))(\mathrm{gr}_t(v)) = s^k(\mathrm{gr}_t(v)) \in s^k\mathbb{C}[s,\xi].$$

Thus to see $s^k\mathbb{C}[s,\xi] = \operatorname{gr}_t(q\mathscr{D}(\mathbb{A}^1))$, it suffices to note

$$s^{k+i}\xi^j = \operatorname{gr}_t(qt^i\partial^j) \in \operatorname{gr}_t(q\mathscr{D}(\mathbb{A}^1))$$

for all $i, j \in \mathbb{N}$. The first part of the claim follows from

$$k = 1\text{-length}(\mathbb{C}[s,\xi]/s^k\mathbb{C}[s,\xi]).$$

A similar proof, using $\operatorname{gr}_t(q^{-1}) = s^{-k}$, shows

$$\operatorname{gr}_t(\mathscr{D}(\mathbb{A}^1)q^{-1}) = s^{-k}\mathbb{C}[s,\xi].$$

Since $k = \deg(q) = 1\text{-length}(s^{-k}\mathbb{C}[s,\xi]/\mathbb{C}[s,\xi])$, the second part also follows.

Note that if $L \supset N$ are right $\mathscr{D}(\mathbb{A}^1)$-submodules of $\mathscr{D}(K)$ then

$$(*) \qquad \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(L/N) \le 1\text{-length}(\operatorname{gr}_t L/\operatorname{gr}_t N).$$

The result now follows just as in [**8**, 3.10]. By the above paragraph and the inequality $(*)$ we have

$$\begin{aligned}
&\operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/q\mathscr{D}(\mathbb{A}^1)) \\
&\quad = 1\text{-length}(\operatorname{gr}_t \mathscr{D}(\mathbb{A}^1)/\operatorname{gr}_t(q\mathscr{D}(\mathbb{A}^1))) \\
&\quad = 1\text{-length}(\operatorname{gr}_t \mathscr{D}(\mathbb{A}^1)/\operatorname{gr}_t P) + 1\text{-length}(\operatorname{gr}_t P/\operatorname{gr}_t(q\mathscr{D}(\mathbb{A}^1))) \\
&\quad \ge \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/P) + \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(P/q\mathscr{D}(\mathbb{A}^1)) \\
&\quad = \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/q\mathscr{D}(\mathbb{A}^1)).
\end{aligned}$$

Thus

$$\begin{aligned}
&1\text{-length}(\operatorname{gr}_t \mathscr{D}(\mathbb{A}^1)/\operatorname{gr}_t P) + 1\text{-length}(\operatorname{gr}_t P/\operatorname{gr}_t(q\mathscr{D}(\mathbb{A}^1))) \\
&\quad = \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/P) + \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(P/q\mathscr{D}(\mathbb{A}^1)).
\end{aligned}$$

By $(*)$,

$$1\text{-length}(\operatorname{gr}_t \mathscr{D}(\mathbb{A}^1)/\operatorname{gr}_t P) \ge \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/P)$$

and

$$1\text{-length}(\operatorname{gr}_t P/\operatorname{gr}_t(q\mathscr{D}(\mathbb{A}^1))) \ge \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(P/q\mathscr{D}(\mathbb{A}^1))$$

so $1\text{-length}(\operatorname{gr}_t \mathscr{D}(\mathbb{A}^1)/\operatorname{gr}_t P) = \operatorname{length}_{\mathscr{D}(\mathbb{A}^1)}(\mathscr{D}(\mathbb{A}^1)/P)$, proving (ii). The proof of (i) is similar. □

LEMMA 2.6. *Let R be a subalgebra of $\mathscr{D}(K)$ with $\operatorname{gr}_\partial R \subset \mathbb{C}[t,\xi]$. Assume there exist a left $\mathscr{D}(\mathbb{A}^1)$-module P, a right $\mathscr{D}(\mathbb{A}^1)$-module Q, and $q \in \mathbb{C}[t]$ with the following properties.*

(i) $\mathscr{D}(\mathbb{A}^1)q^{-1} \supset P \supset \mathscr{D}(\mathbb{A}^1) \supset Q \supset q\mathscr{D}(\mathbb{A}^1)$,

(ii) $\operatorname{length}(P/\mathscr{D}(\mathbb{A}^1)) = \operatorname{length}(\mathscr{D}(\mathbb{A}^1)/Q)$,
(iii) $QP \subset R$.
Then $\operatorname{gr}_t R \subset \mathbb{C}[s, \xi]$ *and* $\dim_{\mathbb{C}}(\mathbb{C}[s,\xi]/\operatorname{gr}_t R) < \infty$.

Proof. Proposition 2.4 implies $\operatorname{gr}_t R \subset \mathbb{C}[s,\xi]$. The rest of the proof is very similar to the proof of [**8**, 3.12]. We have

$$\mathbb{C}[s,\xi] \supset \operatorname{gr}_t R \supset \operatorname{gr}_t(QP) \supset \operatorname{gr}_t(Q)\operatorname{gr}_t(P).$$

By Lemma 2.5,

$$(**) \qquad \begin{aligned} \text{1-length}(\operatorname{gr}_t P/\operatorname{gr}_t \mathscr{D}(\mathbb{A}^1)) &= \operatorname{length}(P/\mathscr{D}(\mathbb{A}^1)) \\ &= \operatorname{length}(\mathscr{D}(\mathbb{A}^1)/Q) = \text{1-length}(\operatorname{gr}_t \mathscr{D}(\mathbb{A}^1)/\operatorname{gr}_t Q). \end{aligned}$$

Set this common length equal to m. Note that if $I_1 \subset I_2$ and J are nonzero ideals of $\mathbb{C}[s,\xi]$ with $\dim_{\mathbb{C}}(I_2/I_1) = \infty$, then $\dim_{\mathbb{C}}(JI_2/JI_1) = \infty$ also. Thus, if $\mathbb{C}[s,\xi] = I_0 \subset I_1 \subset \cdots \subset I_m = \operatorname{gr}_t P$ is a maximal chain for which $\dim_{\mathbb{C}}(I_j/I_{j-1}) = \infty$ for all j, then

$$\operatorname{gr}_t Q = (\operatorname{gr}_t Q)I_0 \subset \cdots \subset (\operatorname{gr}_t Q)I_m = (\operatorname{gr}_t Q)(\operatorname{gr}_t P)$$

is also a chain for which each factor is infinite dimensional. Hence

$$\text{1-length}((\operatorname{gr}_t Q)(\operatorname{gr}_t P)/\operatorname{gr}_t Q) \geq \text{1-length}(\operatorname{gr}_t P/\mathbb{C}[s,\xi])$$

But

$$\begin{aligned} \text{1-length}(\operatorname{gr}_t P/\mathbb{C}[s,\xi]) &= \text{1-length}(\mathbb{C}[s,\xi]/\operatorname{gr}_t Q) \\ &\geq \text{1-length}((\operatorname{gr}_t Q)(\operatorname{gr}_t P)/\operatorname{gr}_t Q) \end{aligned}$$

by $(**)$, whence

$$\text{1-length}(\mathbb{C}[s,\xi]/\operatorname{gr}_t Q) = \text{1-length}((\operatorname{gr}_t Q)(\operatorname{gr}_t P)/\operatorname{gr}_t Q).$$

Thus $\dim_{\mathbb{C}}(\mathbb{C}[s,\xi]/(\operatorname{gr}_t Q)(\operatorname{gr}_t P)) < \infty$ and this proves the result. □

If X and Y are any pair of birationally equivalent curves then let $K = \operatorname{Fract}\mathscr{O}(X) = \operatorname{Fract}\mathscr{O}(Y)$ and define

$$\mathscr{D}(X,Y) = \{D \in \mathscr{D}(K) | D * f \in \mathscr{O}(Y) \text{ for all } f \in \mathscr{O}(X)\},$$

where $D * f$ denotes the action of D on f. If X and Y are curves such that $\mathscr{O}(X) \subset \mathscr{O}(Y) \subset (\tilde{X})$ then $\mathscr{D}(Y,X)$ is a left ideal of $\mathscr{D}(X)$ and a right ideal of $\mathscr{D}(Y)$. Note that $1 \in \mathscr{D}(X,Y)$ and $\mathscr{D}(Y,X) \cap \mathscr{O}(Y)$ equals the conductor of $\mathscr{O}(Y)$ in $\mathscr{O}(X)$, whence both $\mathscr{D}(X,Y)$ and $\mathscr{D}(Y,X)$ are nonzero. If $\pi\colon \tilde{X} \to X$ is injective then $\mathscr{D}(\tilde{X},X)\mathscr{D}(X,\tilde{X}) = \mathscr{D}(X)$ since it is a nonzero 2-sided ideal of a simple ring, by [**8**, 3.4]. Similarly $\mathscr{D}(X,\tilde{X})\mathscr{D}(\tilde{X},X) = \mathscr{D}(\tilde{X})$.

THEOREM 2.7. *Let $\tilde{X} = \mathbb{A}^1$ and $\pi: \tilde{X} \to X$ be injective. Then*

(a) *$\mathscr{D}_0^t(X)$ is a maximal commutative ad-nilpotent subalgebra of $\mathscr{D}(X)$.*

(b) $\mathscr{D}_0^t(X) \cap \mathscr{O}(X) = \mathbb{C}$.

(c) *$\mathscr{D}_0^t(X)$ is a finitely generated $\mathbb{C}$-domain of dimension 1 with integral closure isomorphic to $\mathbb{C}[\xi]$.*

Proof. As remarked earlier, $\operatorname{gr}_\partial \mathscr{D}(X) \subset \mathbb{C}[t, \xi]$ by [**8**, 3.11]. Thus Proposition 2.4 applies and $\mathscr{D}_0^t(X)$ is a commutative ad-nilpotent subalgebra of $\mathscr{D}(X)$ intersecting $\mathscr{O}(X)$ in $\mathbb{C}$. Furthermore, to see that it is a maximal commutative subalgebra of $\mathscr{D}(X)$, it is enough to show $\mathscr{D}_0^t(X) \neq \mathbb{C}$. Thus it suffices to prove (c).

We do this by applying Lemma 2.6 to $P = \mathscr{D}(\tilde{X}, X)$ and $P^* = \mathscr{D}(X, \tilde{X})$. Note that $\operatorname{gr}_t \mathscr{D}(X) \supset (\operatorname{gr}_t(P))(\operatorname{gr}_t(P^*))$. Let $q \in \mathbb{C}[t]$ generate the conductor of $\mathscr{O}(\tilde{X}) = \mathbb{C}[t]$ in $\mathscr{O}(X)$. Then $q\mathscr{D}(\mathbb{A}^1) \subset P$ and $P^* \subset \mathscr{D}(\mathbb{A}^1)q^{-1}$. Since $\mathscr{D}(X)$ is a Noetherian domain, we can identify $\operatorname{Hom}_{\mathscr{D}(\mathbb{A}^1)}(P, \mathscr{D}(\mathbb{A}^1))$ with $P^* = \{u \in \mathscr{D}(K) | uP \subset \mathscr{D}(\mathbb{A}^1)\}$. Thus, since $\mathscr{D}(\mathbb{A}^1)$ is a hereditary domain,

$$\operatorname{length}(P^*/\mathscr{D}(\mathbb{A}^1)) = \operatorname{length}(\mathscr{D}(\mathbb{A}^1)/P).$$

By Lemma 2.6, $\dim_{\mathbb{C}}(\mathbb{C}[s, \xi]/\operatorname{gr}_t \mathscr{D}(X)) < \infty$.

By Proposition 2.4, $\mathscr{D}_0^t(X) \cong \operatorname{gr}_t \mathscr{D}(X) \cap \mathbb{C}[\xi]$. Thus

$$\dim(\mathbb{C}[\xi]/\mathscr{D}_0^t(X)) = \dim(\mathbb{C}[\xi] + (\operatorname{gr}_t \mathscr{D}(X))/\operatorname{gr}_t \mathscr{D}(X)) < \infty$$

and $\mathscr{D}_0^t(X)$ is a domain of dimension 1 with integral closure isomorphic to $\mathbb{C}[\xi]$. That $\mathscr{D}_0^t(X)$ is a finitely generated $\mathbb{C}$-algebra follows from Eakin's Theorem [**5**, §3.5]. This proves (c) and the theorem. □

Combining Theorem 2.7 with Corollary 1.9 yields the theorem stated in the introduction.

THEOREM 2.8. *Let X be a curve. Then $\mathscr{N}(X) = \mathscr{O}(X)$ if and only if either* (i) $\tilde{X} \neq \mathbb{A}^1$ *or* (ii) $\tilde{X} = \mathbb{A}^1$ *and $\pi: \tilde{X} \to X$ is not injective.* □

Let X be a curve with $\tilde{X} = \mathbb{A}^1$ and $\pi: \tilde{X} \to X$ injective. By Theorem 2.7 there exists a curve Y with $\tilde{Y} = \mathbb{A}^1$ such that $\mathscr{D}_0^t(X) \cong \mathscr{O}(Y)$. Moreover $\mathscr{O}(Y) \cong \mathscr{D}_0^t(X)$ is a maximal commutative ad-nilpotent subalgebra of $\mathscr{D}(X)$. It would be interesting to know if there exists a $\mathbb{C}$-algebra isomorphism $\varphi: \mathscr{D}(X) \cong \mathscr{D}(Y)$ such that $\varphi(\mathscr{D}_0^t(X)) = \mathscr{O}(Y)$. If this were true then $\varphi(\mathscr{O}(X)) \subset \mathscr{N}(Y) \setminus \mathscr{O}(Y)$, whence $\pi: \tilde{Y} \to Y$ is injective by Theorem 2.8. In [**7**] we compute an example where $\mathscr{O}(Y) \ncong \mathscr{O}(X)$ and $\pi: \tilde{Y} \to Y$ is injective. However, it is unknown if $\mathscr{D}(X) \cong \mathscr{D}(Y)$ for this example.

3. Non-monomial curves. The remainder of this paper is concerned with those curves X for which $\mathscr{N}(X) \neq \mathscr{O}(X)$. Theorem 2.8 tells us precisely which curves these are. Thus, for the rest of §3 fix a curve X such that $\tilde{X} = \mathbb{A}^1$ and $\pi: \tilde{X} \to X$ is injective. We make the further restriction that X has a unique singularity, which for convenience we take to be $\pi(0)$. Hence if $\mathscr{O}(\mathbb{A}^1) = \mathbb{C}[t]$, it follows that $\mathbb{C}[t] \supset \mathscr{O}(X) \supset t^r\mathbb{C}[t]$ for some r. Our goal is Theorem 3.8 which says that for "most" such X, $\mathscr{D}_0^t(X) \supsetneq \mathscr{O}(X)$.

DEFINITION. A subalgebra R of $\mathbb{C}[t]$ is called a *monomial algebra* if it has a basis consisting of monomials in t and $\dim_{\mathbb{C}}(\mathbb{C}[t]/R) < \infty$. A curve X is a *monomial curve* if $\mathscr{O}(X)$ is a monomial algebra. If $R \subset \mathbb{C}[t]$ is a monomial algebra, then $\Lambda = \{k \in \mathbb{N} | t^k \in R\}$ is a submonoid of the natural numbers and $R = \sum_{\lambda\in\Lambda} \mathbb{C}t^\lambda$. If $R = \mathscr{O}(X)$, call Λ the *monoid associated to* X.

The ring of differential operators on a monomial curve has been studied in **[6]**. Of course $\mathbb{C}[t]$ is a monomial algebra with basis $\{1, t, t^2, \dots\}$ and thus $\mathbb{A}^1$ is a monomial curve. A more interesting example is the planar curve Z given by the equation $y^m = x^n$ where $(m, n) = 1$. Then $\mathscr{O}(Z) = \mathbb{C}[t^m, t^n]$. Note that a monomial curve X has $\tilde{X} = \mathbb{A}^1$ and, if $X \neq \mathbb{A}^1$, unique singularity at $\pi(0)$.

With the above restrictions on X, we will show $\mathscr{O}(X) \supsetneq \mathscr{D}_0^t(X)$ if X is not a monomial curve. Let (t^r) be the conductor $\mathscr{O}(\tilde{X}) = \mathbb{C}[t]$ in $\mathscr{O}(X)$. Thus $\mathscr{D}(X)t^r \subset \mathscr{D}(\tilde{X}) = \mathbb{C}[t, \partial]$ so $\mathscr{D}(X) \subset \mathbb{C}[t, t^{-1}, \partial]$. The element $t\partial \in \mathbb{C}[t, t^{-1}, \partial]$ is ad-semisimple so $\mathbb{C}[t, t^{-1}, \partial]$ decomposes into a direct sum of ad$(t\partial)$-eigenspaces, $\mathbb{C}[t, t^{-1}, \partial] = \bigoplus_{k\in\mathbb{Z}} t^k\mathbb{C}[t\partial]$. Thus $u \in \mathscr{D}(X)$ has a unique expression

$$u = \sum_{m\leq k\leq l} t^k f_k \quad \text{with } k \in \mathbb{Z}, f_k \in \mathbb{C}[t\partial], \text{ and } f_l, f_m \neq 0.$$

Call this the *standard form* for u. Give $\mathscr{D}(X)$ the t-filtration $\{\mathscr{D}_n^t(X)\}_{n\in\mathbb{N}}$ and consider the associated graded ring $\mathrm{gr}_t\,\mathscr{D}(X)$. We will usually apply the following lemma in the case $n = 0$.

LEMMA 3.1. *If $u \in \mathscr{D}(X)$, write $u = \sum_{m\leq k\leq l} t^k f_k$ in standard form. Then $u \in \mathscr{D}_n^t(X)$ if and only if $l \leq n$ and $\deg_{t\partial}(f_k) \leq n - k$ for all $m \leq k \leq l$.*

Proof. If $\deg_{t\partial}(f_k) \leq n - k$ for all $m \leq k \leq l$ then t-$\deg(t^k f_k) \leq n$ and $u \in \mathscr{D}_n^t(X)$. Let $u \in \mathscr{D}_n^t(X)$ and assume, by way of contradiction, that $\{k|k + \deg(f_k) > n\} \neq \varnothing$. Choose j with $s = j + \deg(f_j) > n$ maximal. Note u has a unique expression

$$u = \sum t^i g_i \quad \text{with } g_i \in \mathbb{C}[\partial] \text{ and } i \in \mathbb{Z},$$

since $u \in \mathbb{C}[t, t^{-1}, \partial]$, and $g_i = 0$ for $i > n$ because t-$\deg(u) \leq n$. Set $n(k) = \deg(f_k)$ and note

$$t^k f_k = \alpha_k t^{k+n(k)} \partial^{n(k)} + (\text{terms of lower degree in both } t \text{ and } \partial)$$

where $0 \neq \alpha_k \in \mathbb{C}$. Since $k + n(k) \leq s$ for all k,

$$0 = t^s g_s = \sum \{\alpha_k t^{k+n(k)} \partial^{n(k)} | k + n(k) = s\}.$$

Therefore, since $j + n(j) = s$, there exists $i \neq j$ with $i + n(i) = j + n(j)$ and $n(i) = n(j)$. But then $i = j$, a contradiction. Thus $\deg(f_k) \leq n - k$ for $m \leq k \leq l$. Since $\deg(f_l) \geq 0$, $n - l \geq 0$. □

Definition. If $p \in \mathbb{C}[t]$ write $p = \alpha_0 t^i + \alpha_1 t^{i+1} + \cdots + \alpha_j t^{i+j}$ with $\alpha_0, \alpha_j \neq 0$. Define $\lambda(p) = \alpha_j t^{i+j}$ and $\mu(p) = \alpha_0 t^i$. For any curve Z with $\tilde{Z} = \mathbb{A}^1$, write $\mathscr{O}(\tilde{Z}) = \mathbb{C}[t]$ and define

$$\Gamma(Z) = \{i \in \mathbb{N} | \exists q \in \mathscr{O}(Z) \text{ with } \lambda(q) = t^i\},$$
$$\Omega(Z) = \{i \in \mathbb{N} | \exists q \in \mathscr{O}(Z) \text{ with } \mu(q) = t^i\}.$$

These are both submonoids of $\mathbb{N}$. We write Γ for $\Gamma(Z)$ and Ω for $\Omega(Z)$ if it is clear what curve we mean. If Z is a monomial curve then $\Gamma(Z)$ and $\Omega(Z)$ both equal the monoid associated to Z. We have the following converse.

Lemma 3.2. *If X is not a monomial curve then $\Gamma(X) \setminus \Omega(X) \neq \varnothing$.*

Proof. Since X is not a monomial curve and $t^r\mathbb{C}[t] \subset \mathscr{O}(X)$, we may choose $j \in \Gamma(X)$ maximal such that $t^j \notin \mathscr{O}(X)$. We show $j \in \Gamma(X) \setminus \Omega(X)$. If $j \in \Omega(X)$ choose $q \in \mathscr{O}(X)$ of minimal degree such that $\mu(q) = t^j$. Then $\lambda(q) = \alpha t^{j+i}$ with $i > 0$ and $0 \neq \alpha \in \mathbb{C}$. But $i + j \in \Gamma(X)$ so $t^{i+j} \in \mathscr{O}(X)$ by hypothesis. Hence $p = q - \alpha t^{i+j} \in \mathscr{O}(X)$ with $\mu(p) = t^j$, contradicting the minimality of $\deg(q)$. □

There is an algebra isomorphism $\varphi\colon \mathscr{D}_0^t(X) \cong \mathbb{C}[\xi] \cap \operatorname{gr}_t \mathscr{D}(X)$ by Proposition 2.4. By Theorem 2.7, there exists a curve Y with $\tilde{Y} = \mathbb{A}^1$ such that $\varphi(\mathscr{D}_0^t(X)) = \mathscr{O}(Y) \subset \mathbb{C}[\xi] = \mathscr{O}(\tilde{Y})$. The purpose of the next few results is to compare $\Gamma(Y)$ with $\Omega(X)$ and $\Omega(Y)$ with $\Gamma(X)$. This is done in Proposition 3.7.

LEMMA 3.3. *Let $l \leq 0$ and $u = \sum_{m \leq k \leq l} t^k f_k \in \mathscr{D}(X)$ in standard form. Then $\prod_{i \in \Gamma \setminus \Gamma - l}(t\partial - i)$ divides f_l and $\prod_{i \in \Omega \setminus \Omega - m}(t\partial - i)$ divides f_m in $\mathbb{C}[t\partial]$.*

Proof. If $i \in \Gamma$ and $(t\partial - i)$ does not divide f_l, choose $q \in \mathscr{O}(X)$ with $\lambda(q) = t^i$. Then $i \in \Gamma - l$ because $u * q \in \mathscr{O}(X)$ and $\lambda(u * q) = f_l(i)t^{i+l}$. Thus $\prod_{i \in \Gamma \setminus \Gamma - l}(t\partial - i)$ divides f_l.

Similarly, if $i \in \Omega$ and $(t\partial - i)$ does not divide f_m, choose $q \in \mathscr{O}(X)$ with $\mu(q) = t^i$. Then $i \in \Omega - m$ since $u * q \in \mathscr{O}(X)$ and $\mu(u * q) = f_m(i)t^{i+m}$. Thus $\prod_{i \in \Omega \setminus \Omega - m}(t\partial - i)$ divides f_m. □

LEMMA 3.4. *Let Λ be a submonoid of $\mathbb{N}$ containing $r + \mathbb{N}$ for some $r \in \mathbb{N}$. If $k \in \mathbb{Z}$, set $n(k)$ equal to the cardinality of the finite set $\Lambda \setminus (\Lambda - k)$. Then $n(-k) = n(k) + k$.*

Proof. By symmetry it is enough to prove the result for $k \in \mathbb{N}$. Suppose $k > 0$. Partition $\mathbb{N}$ into the disjoint union $\mathbb{N} = \coprod_{i \in \mathbb{N}} \mathbb{N}_i$, where $\mathbb{N}_i = \{j \in \mathbb{N} | ik \leq j < (i+1)k\}$, and set $\Lambda_i = \Lambda \cap \mathbb{N}_i$. Note $\Lambda \setminus (\Lambda - k) = ((\Lambda + k) \setminus \Lambda) - k$. We have

$$\Lambda \setminus (\Lambda + k) = \coprod_{i \in \mathbb{N}} \Lambda_i \setminus (\Lambda_{i-1} + k)$$

and

$$(\Lambda + k) \setminus \Lambda = \coprod_{i \in \mathbb{N}} (\Lambda_{i-1} + k) \setminus \Lambda_i$$

where $\Lambda_{-1} + k = \varnothing$. Because $r + \mathbb{N} \subset \Lambda$, there exists $N \in \mathbb{N}$ such that $i \geq N$ implies $\Lambda_{i-1} + k = \Lambda_i = \mathbb{N}_i$. Now

$$\begin{aligned}
n(-k) - n(k) &= |\Lambda \setminus (\Lambda + k)| - |\Lambda \setminus (\Lambda - k)| \\
&= |\Lambda \setminus (\Lambda + k)| - |(\Lambda + k) \setminus \Lambda| \\
&= \sum_{0 \leq i \leq N} |\Lambda_i \setminus (\Lambda_{i-1} + k)| - \sum_{1 \leq i \leq N} |(\Lambda_{i-1} + k) \setminus \Lambda_i| \\
&= |\Lambda_0| + \sum_{1 \leq i \leq N} \{|\Lambda_i \setminus (\Lambda_{i-1} + k)| - |(\Lambda_{i-1} + k) \setminus \Lambda_i|\} \\
&= |\Lambda_0| + \sum_{1 \leq i \leq N} \{|\Lambda_i| - |(\Lambda_{i-1} + k)|\} \\
&= |\Lambda_0| + \sum_{1 \leq i \leq N} \{|\Lambda_i| - |\Lambda_{i-1}|\} = |\Lambda_N| = |\mathbb{N}_N| = k.
\end{aligned}$$

Note that we have used the fact that if A and B are finite sets then $|A \setminus B| - |B \setminus A| = |A| - |B|$. □

LEMMA 3.5. *Let $m, l \leq 0$ and set $g_l = \prod_{i \in \Gamma \setminus \Gamma - l}(t\partial - i)$ and $h_m = \prod_{i \in \Omega \setminus \Omega - m}(t\partial - i)$. Then*
(a) $\deg_{t\partial}(g_l) \geq -l$ *with equality if and only if* $-l \in \Gamma$
(b) $\deg_{t\partial}(h_m) \geq -m$ *with equality if and only if* $-m \in \Omega$.

Proof. Recall that Γ and Ω are both submonoids of $\mathbb{N}$. Thus, by Lemma 3.4, $\deg(g_l) = |\Gamma \setminus (\Gamma - l)| = -l + |\Gamma \setminus (\Gamma + l)| \geq -l$ with equality if and only if $|\Gamma \setminus (\Gamma + l)| = 0$. If $-l \in \Gamma$ then $|\Gamma \setminus (\Gamma + l)| = |(\Gamma - l) \setminus \Gamma| = 0$ since $\Gamma - l \subset \Gamma$. Conversely, $|(\Gamma - l) \setminus \Gamma| = 0$ implies $-l \in \Gamma$ because $0 \in \Gamma$. This proves the assertion for g_l. The proof for h_m is identical. □

PROPOSITION 3.6. *If $u \in \mathscr{D}_0^t(X)$ write $u = \sum_{m \leq k \leq l} t^k f_k$ in standard form. Then $f_m \in \mathbb{C}h_m$ and $f_l \in \mathbb{C}g_l$. Moreover, $-l \in \Gamma(X)$ and $-m \in \Omega(X)$.*

Proof. By Lemma 3.1, $l \leq 0$ and $\deg(f_k) \leq -k$ for all k. By Lemma 3.3, g_l divides f_l. By Lemma 3.5, $\deg(g_l) \geq -l$. Thus $-l \geq \deg(f_l) \geq \deg(g_l) \geq -l$, whence $f_l \in \mathbb{C}g_l \setminus \{0\}$ and $-l \in \Gamma$ since $\deg(g_l) = -l$. A similar argument works for f_m. □

PROPOSITION 3.7. *Write $\varphi(\mathscr{D}_0^t(X)) = \mathscr{O}(Y) \subset \mathbb{C}[\xi]$. Then $\Gamma(Y) \subset \Omega(X)$ and $\Omega(Y) \subset \Gamma(X)$.*

Proof. Choose $u \in \mathscr{D}_0^t(X)$. Write $u = \sum_{m \leq k \leq l} t^k f_k$ in standard form with $l \leq 0$ and $\deg(f_k) \leq -k$. Then

$$\varphi(u) = \sum \{\mathrm{gr}_t(t^k f_k) | \deg(f_k) = -k\} \in \mathscr{O}(Y) \subset \mathbb{C}[\xi].$$

Thus $\mu(u) = \mathrm{gr}_t(t^l f_l) = \alpha_l \xi^{-l}$, for some $0 \neq \alpha_l \in \mathbb{C}$. By Proposition 3.6, $-l \in \Gamma(X)$. Thus $\Omega(Y) \subset \Gamma(X)$. A similar argument shows $\Gamma(Y) \subset \Omega(X)$. □

THEOREM 3.8. *Let X be a curve with $\tilde{X} = \mathbb{A}^1$, $\pi\colon \tilde{X} \to X$ injective and unique singularity at $\pi(0)$. If X is not a monomial curve then $\mathscr{D}_0^t(X)$ is a maximal commutative ad-nilpotent subalgebra of $\mathscr{D}(X)$ which is not isomorphic to $\mathscr{O}(X)$.*

Proof. Assume X is not a monomial curve. By Theorem 2.7, $\mathscr{D}_0^t(X)$ is a maximal commutative ad-nilpotent subalgebra of $\mathscr{D}(X)$ and is isomorphic to $\mathscr{O}(Y)$ for some curve Y with $\tilde{Y} = \mathbb{A}^1$. If there exists $\vartheta\colon \mathscr{O}(X) \cong \mathscr{O}(Y)$ then, since $\mathbb{C}[t]$ is the integral closure of both

$\mathcal{O}(X)$ and $\mathcal{O}(Y)$, ϑ extends to an automorphism of $\mathbb{C}[t]$. Thus $\vartheta(t) = \alpha t + \beta$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. By Lemma 3.2, there exists $i \in \Gamma(X) \setminus \Omega(X)$. Choose $q \in \mathcal{O}(X)$ with $\lambda(q) = t^i$. Then $\lambda\vartheta(q) = \alpha^i t^i$, whence $i \in \Gamma(Y)$. But $\Gamma(Y) \subset \Omega(X)$ by Proposition 3.7, a contradiction. □

The results in this section were motivated by the explicit computation, given in [7], of $\mathcal{D}(X)$ when $\mathcal{O}(X) = \mathbb{C} \oplus \mathbb{C}(t^2+t^3) \oplus t^4\mathbb{C}[t]$. In this case $\mathcal{D}_0^t(X)$ is isomorphic to the ring of regular functions on a curve with two singularities, while X has only one.

4. Monomial curves. Recall that $\mathcal{D}_0^t(\mathbb{A}^1) = \mathbb{C}[\partial]$ and that there exists an automorphism of $\mathcal{D}(\mathbb{A}^1)$ which interchanges t and $-\partial$. For a monomial curve X we will see, in this section, that $\mathcal{D}_0^t(X) \cong \mathcal{O}(X)$. In general, however, there is no automorphism of $\mathcal{D}(X)$ interchanging $\mathcal{O}(X)$ and $\mathcal{D}_0^t(X)$ when X is a monomial curve. This is proved in [7]. Also in this section it is proved that for all but two monomial curves X there exists a maximal commutative ad-nilpotent subalgebra of $\mathcal{D}(X)$ which is not isomorphic to $\mathcal{O}(X)$. Indeed, this subalgebra is not isomorphic to any monomial algebra.

Throughout §4, let X be a monomial curve. Hence $\mathcal{D}(X) \subset \mathbb{C}[t, t^{-1}, \partial]$ and, since $\mathcal{O}(X)$ is spanned by monomials, $t\partial \in \mathcal{D}(X)$. Thus $\mathcal{D}(X)$ is an ad$(t\partial)$-stable subspace of $\mathbb{C}[t, t^{-1}, \partial]$, whence $\mathcal{D}(X) = \bigoplus_{k\in\mathbb{Z}}\{t^k\mathbb{C}[t\partial] \cap \mathcal{D}(X)\}$. If $u \in \mathcal{D}(X)$ and $f \in \mathcal{O}(X)$, recall that $u * f$ denotes the action of u on f. The following lemma is implicit in [6].

LEMMA 4.1. *If X is a monomial curve with associated monoid Λ, then $\mathcal{D}(X) = \bigoplus_{k\in\mathbb{Z}} t^k f_k \mathbb{C}[t\partial]$ with $f_k = \Pi\{t\partial - j \mid j \in \Lambda \setminus (\Lambda - k)\} \in \mathbb{C}[t\partial]$.*

Proof. By the above paragraph, $\mathcal{D}(X) = \bigoplus_{k\in\mathbb{Z}}\{t^k\mathbb{C}[t\partial] \cap \mathcal{D}(X)\}$. But $\mathcal{D}(X) \cap t^k\mathbb{C}[t\partial]$ is a $\mathbb{C}[t\partial]$-submodule of $t^k\mathbb{C}[t\partial]$. Since $t^k\mathbb{C}[t\partial] \cong \mathbb{C}[t\partial]$ as $\mathbb{C}[t\partial]$-module and $\mathbb{C}[t\partial]$ is a PID, $t^k\mathbb{C}[t\partial] \cap \mathcal{D}(X) = t^k f_k \mathbb{C}[t\partial]$ for some $f_k \in \mathbb{C}[t\partial]$. Set $h_k = \Pi\{t\partial - j \mid j \in \Lambda \setminus (\Lambda - k)\}$. If $g \in \mathbb{C}[t\partial]$ then $t^k g * t^n = g(n)t^{n+k}$. Using this it is easy to check that $t^k h_k * \mathcal{O}(X) \subset \mathcal{O}(X)$, whence $t^k h_k \in \mathcal{D}(X)$. Let $t^k g \in \mathcal{D}(X)$ and $j \in \Lambda$. If $g(j) \neq 0$ then $g(j)t^{k+j} = t^k g * t^j \in \mathcal{O}(X)$ so $k + j \in \Lambda$. This implies h_k divides g in $\mathbb{C}[t\partial]$ and thus $f_k = h_k$. □

THEOREM 4.2. *If X is a monomial curve with associated monoid Λ and $\mathcal{D}(X) = \bigoplus_{k\in\mathbb{Z}} t^k f_k \mathbb{C}[t\partial]$ then $\mathcal{D}_0^t(X) = \sum_{i\in\Lambda} \mathbb{C}t^{-i} f_{-i}$. In particular, $\mathcal{D}_0^t(X) \cong \mathcal{O}(X)$.*

Proof. Let $u = \sum t^k p_k \in \mathscr{D}_0^t(X)$, with $p_k \in \mathbb{C}[t\partial]$. Note $\Lambda = \Gamma(X) = \Omega(X)$ so, in the notation of Proposition 3.5, $f_k = h_k = g_k$. Since X is a monomial curve, $t^k p_k \in \mathscr{D}(X)$ for all k. Thus t-deg$(t^k p_k) \geq 0$. But, by Lemma 3.1, t-deg$(t^k p_k) \leq 0$. Thus t-deg$(t^k p_k) = 0$ so, if $p_k \neq 0$, $-k \in \Lambda$ and $p_k = g_k = f_k$ by Proposition 3.6. Of course, if $-k \in \Lambda$ then $t^{-k} f_{-k} = t^{-k} g_{-k} \in \mathscr{D}_0^t(X)$.

By Proposition 2.4, $\mathscr{D}_0^t(X) \cong \mathrm{gr}_t \mathscr{D}_0^t(X)$ as $\mathbb{C}$-algebras. Combining this with the observation that $\mathrm{gr}_t(t^{-k} f_{-k}) = \xi^k$ for $k \in \Lambda$ gives the second statement from the first. □

Combining Theorems 4.2 and 3.8 yields the following.

COROLLARY 4.3. *Let X be a curve with $\tilde{X} = \mathbb{A}^1$, $\pi\colon \tilde{X} \to X$ injective and unique singularity at $\pi(0)$. Then $\mathscr{D}_0^t(X) \cong \mathscr{O}(X)$ if and only if X is a monomial curve.* □

The next results show that, for all but two monomial curves, there exists a maximal commutative ad-nilpotent subalgebra of $\mathscr{D}(X)$ not isomorphic to $\mathscr{O}(X)$. For $v \in \mathscr{D}(K)$, define $\mathrm{ord}(v)$ to be the order of v as a differential operator, the "∂-degree". If R is any ring and $u \in R$ is locally ad-nilpotent we may consider, as in **[1]**, $\exp(\mathrm{ad}(u)) \in \mathrm{Aut}(R)$.

LEMMA 4.4. *Set $\Phi = \exp(\mathrm{ad}(\lambda t^k)) \in \mathrm{Aut}(\mathbb{C}[t, t^{-1}, \partial])$ where $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$. Let X be a curve with $\tilde{X} = \mathbb{A}^1$, $\pi\colon \tilde{X} \to X$ injective and unique singularity at $\pi(0)$. If $R = \Phi\mathscr{D}(X)$ then $\mathrm{gr}_\partial R = \mathrm{gr}_\partial \mathscr{D}(X)$.*

Proof. If $u \in \mathscr{D}(X)$, let $\mathrm{ord}(u) = n$ and write $u = f\partial^n + v$ with $f \in \mathbb{C}[t] \setminus \{0\}$, $v \in \mathbb{C}[t, t^{-1}, \partial]$ and $\mathrm{ord}(v) < n$. Since $\mathrm{ord}([\lambda t^k, w]) < \mathrm{ord}(w)$ for all $w \in \mathscr{D}(K)$, $\mathrm{ord}\,\Phi(v) < \mathrm{ord}\,\Phi(f\partial^n)$ and

$$\begin{aligned}\Phi(u) &= \Phi(f\partial^n) + \Phi(v) \\ &= f\partial^n + \left(\sum_{i\geq 1} (i!)^{-1}\, \mathrm{ad}(\lambda t^k)^i (f\partial^n)\right) + \Phi(v) = f\partial^n + y\end{aligned}$$

where $y \in \mathbb{C}[t, t^{-1}, \partial]$ and $\mathrm{ord}(y) < n$. Thus $\mathrm{gr}_\partial(\Phi(u)) = f\xi^n = \mathrm{gr}_\partial(u)$. □

PROPOSITION 4.5. *Set $\Phi = \exp(\mathrm{ad}(\lambda t^k)) \in \mathrm{Aut}(\mathbb{C}[t, t^{-1}, \partial])$, where $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$. Let X be a curve with $\tilde{X} = \mathbb{A}^1$, $\pi\colon \tilde{X} \to X$ injective and unique singularity at $\pi(0)$. Set R_0 equal to the* 0th *part of the t-filtration on $R = \Phi\mathscr{D}(X) \subset \mathbb{C}[t, t^{-1}, \partial]$. Then R_0 is a maximal* *commutative ad-nilpotent subalgebra of* R. *Moreover,* R_0 *is a finitely generated domain with integral closure isomorphic to* $\mathbb{C}[\xi]$.

Proof. Lemma 4.4 implies $\operatorname{gr}_\partial R \subset \mathbb{C}[t,\xi]$. Thus, by Proposition 2.4, $\operatorname{gr}_t R \subset \mathbb{C}[s,\xi]$ and R_0 is isomorphic to $(\operatorname{gr}_t R) \cap \mathbb{C}[\xi]$. Moreover, R_0 is a maximal commutative ad-nilpotent subalgebra of R if $R_0 \neq \mathbb{C}$. Set $P = \Phi\mathscr{D}(X,\tilde{X})$, $Q = \Phi\mathscr{D}(\tilde{X},X)$ and let $q \in \mathbb{C}[t]$ generate the conductor of $\mathbb{C}[t]$ in $\mathscr{O}(X)$. Then $\operatorname{length}(P/\mathbb{C}[t,\partial]) = \operatorname{length}(\mathbb{C}[t,\partial]/Q)$ as $\mathbb{C}[t,\partial]$-modules because $\Phi \in \operatorname{Aut}(\mathbb{C}[t,\partial])$. Furthermore

$$QP = \Phi(\mathscr{D}(\tilde{X},X)\mathscr{D}(X,\tilde{X})) = \Phi\mathscr{D}(X) = R.$$

Since Φ fixes $\mathbb{C}(t)$ and

$$\mathbb{C}[t,\partial]q^{-1} \supset \mathscr{D}(X,\tilde{X}) \supset \mathbb{C}[t,\partial] \supset \mathscr{D}(\tilde{X},X) \supset q\mathbb{C}[t,\partial],$$

we have

$$\mathbb{C}[t,\partial]q^{-1} \supset P \supset \mathbb{C}[t,\partial] \supset Q \supset q\mathbb{C}[t,\partial].$$

Thus by Lemma 2.6, $\dim_{\mathbb{C}}(\mathbb{C}[s,\xi]/\operatorname{gr}_t R) < \infty$ and $(\operatorname{gr}_t R) \cap \mathbb{C}[\xi] \cong R_0$ is of finite codimension in $\mathbb{C}[\xi]$. The rest follows as in the proof of Theorem 2.7. □

Let X be a monomial curve with associated monoid Λ. Write $\mathscr{D}(X) = \bigoplus_{k\in\mathbb{Z}} t^k f_k \mathbb{C}[t\partial]$ with $f_k \in \mathbb{C}[t\partial]$. If $\Lambda \cup \{1\} \neq \mathbb{N}$, we show $\mathscr{D}(X)$ contains a maximal commutative ad-nilpotent subalgebra which is not isomorphic to any monomial algebra. Given $m \in \mathbb{N} \setminus (\Lambda \cup \{1\})$, set $\Phi = \exp(\operatorname{ad}(m^{-1}t^m))$ and $R = \Phi(\mathscr{D}(X)) \subset \mathbb{C}[t,t^{-1},\partial]$. Then R_0, the 0th part of the t-filtration on R, is a maximal commutative ad-nilpotent subalgebra of R by Proposition 4.5. Moreover, $R_0 \cong \mathbb{C}[\xi] \cap \operatorname{gr}_t(R)$. Note that $\Phi^{-1} = \exp(\operatorname{ad}(-m^{-1}t^m))$.

LEMMA 4.6. *Let* $\Psi = \exp(\operatorname{ad}(-k^{-1}t^k)) \in \operatorname{Aut}\mathbb{C}[t,t^{-1},\partial]$ *with* $0 \neq k \in \mathbb{N}$. *If* $n \in \mathbb{Z}$ *and* $\alpha_1,\dots,\alpha_r \in \mathbb{C}$ *then there exist* $h_0,\dots,h_r \in \mathbb{C}[t\partial]$ *such that*

$$\Psi\left(t^n \prod_{1\le i\le r} (t\partial - \alpha_i)\right) = t^n \sum_{0\le j\le r} t^{jk} h_j.$$

Moreover, the leading term of h_j *is* $\binom{r}{j}(t\partial)^{r-j}$ *and* $h_0 = \prod_{1\le i\le r}(t\partial - \alpha_i)$.

Proof.

$$\Psi\left(t^n \prod_{1\le i\le r}(t\partial-\alpha_i)\right) = \Psi(t^n)\Psi\left(\prod_{1\le i\le r}(t\partial-\alpha_i)\right) = t^n \prod_{1\le i\le r}((t\partial-\alpha_i)+t^k).$$

The result follows by expanding the product using $(t\partial-\alpha_i)t^l = t^l(t\partial-\alpha_i+l)$. For example,

$$t^k h_1 = (t\partial-\alpha_1)\cdots(t\partial-\alpha_{r-1})t^k + \cdots + t^k(t\partial-\alpha_2)\cdots(t\partial-\alpha_r). \quad \square$$

LEMMA 4.7. *If $0 \neq n \in \mathbb{N}$ and $f \in \mathbb{C}[t\partial]\setminus\{0\}$ then $t^{-n}f \notin R_0$.*

Proof. Assume $t^{-n}f \in R_0$ and note $\deg(f) = n$. Then $\Phi^{-1}(t^{-n}f) = t^{-n}\sum_{0\le j\le n} t^{jm}h_j$ with $h_0 = f$, by Lemma 4.6. Since X is a monomial curve, $t^{-n}f = t^{-n}h_0 \in \mathscr{D}_0^t(X)$ and so $f = f_{-n}$ with $n \in \Lambda$. Thus write

$$f_{-n} = \prod\{t\partial - j \mid j \in \Lambda\setminus(\Lambda+n)\} = \prod_{1\le i\le n}(t\partial-\alpha_i) \quad \text{with } \alpha_n = 0.$$

Now $t^{m-n}h_1 \in \mathscr{D}(X)$ and

$$\begin{aligned} t^m h_1 &= [(t\partial-\alpha_1)\cdots(t\partial-\alpha_{n-1})t^m] \\ &\quad + [(t\partial-\alpha_1)\cdots(t\partial-\alpha_{n-2})t^m(t\partial-\alpha_n)] \\ &\quad + \cdots + [t^m(t\partial-\alpha_2)\cdots(t\partial-\alpha_n)] \\ &= t^m\{[(t\partial-\alpha_1+m)\cdots(t\partial-\alpha_{n-1}+m)] \\ &\qquad + [(t\partial-\alpha_1+m)\cdots(t\partial-\alpha_{n-2}+m)(t\partial-\alpha_n)] + \dots \\ &\qquad + [(t\partial-\alpha_2)\cdots(t\partial-\alpha_n)]\}. \end{aligned}$$

Since $\alpha_n = 0$, $\{\prod_{1\le i\le n-1}(-\alpha_i+m)\}t^{m-n} = t^{m-n}h_1 * 1 \in \mathscr{O}(X)$. Since $m \notin \Lambda$, $\prod_{1\le i\le n-1}(-\alpha_i+m) \neq 0$. Thus $m-n \in \Lambda$ and $n \notin \Lambda$, a contradiction. $\square$

LEMMA 4.8. *Let $u = \sum_{0\le k\le n} t^{-k}g_k \in R_0$ be in standard form. Then $v = \sum_{l\ge 0} t^{lm-n}g_{n-lm} \in R_0$ also.*

Proof. Let $r(k) = \deg(g_k)$. By Lemma 4.6, set

$$\Phi^{-1}(t^{-k}g_k) = t^{-k}\sum_{0\le i\le r(k)} t^{im}h_{k,i}$$

for each k. Then

$$\Phi^{-1}(u) = \sum_{0\le k\le n}\sum_{0\le i\le r(k)} t^{im-k}h_{k,i} = \sum_{s\ge -n} t^s \sum_{im-k=s} h_{k,i}.$$

By Lemma 4.1, $t^s\sum_{im-k=s} h_{k,i} \in \mathscr{D}(X)$ for each s. For $k \le n$, $im-k = jm-n$ if and only if $k = n - lm$ for some $l \in \mathbb{N}$. Thus

$$\Phi^{-1}(v) = \sum_{j\ge 0} t^{jm-n}\sum\{h_{k,i}|im-k = jm-n\} \in \mathscr{D}(X).$$

Thus $v \in \Phi\mathscr{D}(X) = R$ and the result follows. □

LEMMA 4.9. *If R_0 is isomorphic to $\mathscr{D}(X)$ then for every $n \in \Lambda$ there exists $u \in R_0$ such that $u = \sum_{0\le i\le n} t^{-i}g_i$, in standard form, with $g_n \neq 0$.*

Proof. Consider both R_0 and $\mathscr{O}(X)$ as subalgebras of $\mathbb{C}[\xi]$ of finite co-dimension. If $\vartheta\colon \mathscr{O}(X) \cong R_0$ then ϑ extends to an automorphism of $\mathbb{C}[\xi]$. Thus $\vartheta(\xi^n) = (a\xi + b)^n$ and there exists $u \in R_0$ such that $\mathrm{gr}_t(u) = a^n\xi^n +$ (lower degree terms). Write $u = \sum_{k\le i\le l} t^{-i}g_i$, in standard form, with $g_l \neq 0$. Then $l \ge n$ and to prove that $l = n$ it suffices to show that $t-\deg(t^{-l}g_l) = 0$. By Lemma 3.1, $t-\deg(t^{-i}g_i) \le 0$ for all i. In particular $k \ge 0$. As in the proof of Lemma 4.8, if we write

$$\Phi^{-1}(u) = \sum_{s\ge -l} t^s \sum_{im-k=s} h_{k,i}$$

then $t^{-l}g_l = t^{-l}\sum_{im-k=-l} h_{k,i} \in \mathscr{D}(X)$. By Proposition 2.4, $\mathrm{gr}_t\mathscr{D}(X) \subset \mathbb{C}[t,\xi]$, from which it follows that $t-\deg(t^{-l}g_l) \ge 0$. □

THEOREM 4.10. *Let X be a monomial curve with associated monoid Λ such that $\Lambda\cup\{1\} \neq \mathbb{N}$. Then $\mathscr{D}(X)$ contains a maximal commutative ad-nilpotent subalgebra which is not isomorphic to $\mathscr{O}(X)$.*

Proof. Assume there exists an isomorphism $\vartheta\colon \mathscr{O}(X) \to R_0$. Suppose there exists $0 \neq n \in \Lambda$ such that $n < m$. By Lemma 4.9 there exists $\sum_{0\le i\le n} t^{-i}g_i \in R_0$, in standard form, with $g_n \neq 0$. Since $lm - n > 0$ for all $l \in \mathbb{N}$, Lemma 4.8 implies $t^{-n}g_n \in R_0$, contradicting Lemma 4.7. Thus we may assume $\Lambda = \{0\} \cup (\mathbb{N} + r)$ for some $r \ge 3$ and take $m = r - 1$. By Lemma 4.9, there exists $u \in R_0$ such that $u = \sum_{0\le i\le r} t^{-i}g_i$ in standard form with $g_r \neq 0$. By Lemma 4.8, take $u = t^{-r}g_r + t^{-1}g_1$ where $\deg(g_1) \le 1$.

Make the identification $R_0 \cong \mathrm{gr}_t(R_0)$ and consider R_0 and $\mathscr{O}(X)$ as subalgebras of $\mathbb{C}[\xi]$. As in the proof of Lemma 4.9, ϑ extends to an automorphism of $\mathbb{C}[\xi]$. If $\deg(g_1) = 1$ then $\mathrm{gr}_t(u) = c\xi^r + d\xi$ with $c, d \in \mathbb{C} \setminus \{0\}$. If $v = \vartheta^{-1}(u) \in \mathscr{O}(X)$ then $v = e\xi^r + f$ for some $e, f \in \mathbb{C}$, since $\vartheta(\xi^n) = (a\xi + b)^n$. But $r > 2$ so,

$$\vartheta(v) = e(a\xi + b)^r + f \neq c\xi^r + d\xi = \mathrm{gr}_t(u).$$

Thus $\deg(g_1) = 0$ and $u = t^{-r}g_r + \alpha t^{-1}$ for some $\alpha \in \mathbb{C}$.

We have $t^{-r}g_r \in \mathscr{D}(X)$, as in the proof of Lemma 4.9, and hence $g_r = \Pi\{t\partial - j | j \in \Lambda \setminus (\Lambda + r)\}$ by Lemma 4.2. Note that $2m = 2(r-1) \in \Lambda \setminus (\Lambda + r)$. As in the proof of Lemma 4.6,

$$\begin{aligned} \Phi^{-1}(u) &= \Phi^{-1}(t^{-r}g_r) + \Phi^{-1}(\alpha t^{-1}) \\ &= t^{-r}\Phi^{-1}(g_r) + \alpha t^{-1} \\ &= t^{-r}\Pi\{t\partial - j + t^m | j \in \Lambda \setminus (\Lambda + r)\} + \alpha t^{-1} \\ &= t^{-r} \sum_{0 \leq i \leq r} t^{im} h_i + \alpha t^{-1} \end{aligned}$$

for some $h_i \in \mathbb{C}[t\partial]$. Factor $g_r = (t\partial - \alpha_1) \cdots (t\partial - \alpha_r)$ with $\alpha_r = 0$ and $\alpha_1 = 2m$ and note $t^{r-2}h_2 = t^{-r+2m}h_2 \in \mathscr{D}(X)$. Now

$$\begin{aligned} t^{2m}h_2 &= (t\partial - \alpha_1) \cdots (t\partial - \alpha_{r-2})t^{2m} \\ &\quad + (t\partial - \alpha_1) \cdots (t\partial - \alpha_{r-3})t^m(t\partial - \alpha_{r-1})t^m \\ &\quad + \cdots + t^{2m}(t\partial - \alpha_2) \cdots (t\partial - \alpha_r) \\ &= t^{2m}[(t\partial - \alpha_1 + 2m) \cdots (t\partial - \alpha_{r-2} + 2m) \\ &\qquad + (t\partial - \alpha_1 + 2m) \cdots (t\partial - \alpha_{r-3} + 2m)(t\partial - \alpha_{r-1} + m) \\ &\qquad\qquad + \cdots + (t\partial - \alpha_2) \cdots (t\partial - \alpha_r)]. \end{aligned}$$

The only summand of h_2 which does not have a factor of $t\partial$, in the form of $t\partial - \alpha_r$ or $t\partial - \alpha_1 + 2m$, is $(t\partial - \alpha_2 + m) \cdots (t\partial - \alpha_{r-1} + m)$. Thus

$$(-\alpha_2 + m) \cdots (-\alpha_{r-1} + m)t^{r-2} = t^{r-2}h_2 * 1 \in \mathscr{O}(X).$$

Since $m \notin \Lambda$ and $\alpha_i \in \Lambda$ for all i, $(-\alpha_2 + m) \cdots (-\alpha_{r-1} + m) \neq 0$. But $r - 2 \notin \Lambda$, a contradiction. □

The only monomial curves with $\Lambda \cup \{1\} = \mathbb{N}$ and $\mathbb{A}^1$ and $y^2 = x^3$. In [1], it is shown that all maximal commutative subalgebras of $\mathscr{D}(\mathbb{A}^1)$, contained in $\mathscr{N}(\mathbb{A}^1)$, are isomorphic to $\mathscr{O}(\mathbb{A}^1) = \mathbb{C}[t]$. Mimicking this proof, the same is shown for $y^2 = x^3$ in [7]. Hence $y^2 = x^3$ is the only curve with normalization equal to $\mathbb{A}^1$, π injective and unique singularity at $\pi(0)$ such that all maximal commutative ad-nilpotent subalgebras of its ring of differential operators are isomorphic.

5. Simply connected curves. This section is devoted to proving the following proposition, referred to in the introduction.

PROPOSITION 5.1. *Let X be a curve. Then $\pi_1(X) = 0$ if and only if $\tilde{X} = \mathbb{A}^1$ and $\pi\colon \tilde{X} \to X$ is injective.*

Proof. Note that $\pi_1(\mathbb{A}^1) = 0$ and π is continuous in the usual topology. Assume $\tilde{X} = \mathbb{A}^1$ and $\pi\colon \tilde{X} \to X$ is injective, hence bijective. Thus π is a homomorphism and induces an isomorphism between $\pi_1(X)$ and $\pi_1(\mathbb{A}^1)$. Therefore $\pi_1(X) = 0$.

Assume $\pi_1(X) = 0$. We first show π is injective and $\pi_1(\tilde{X}) = 0$. Recall π identifies at most a finite number of points. Factor π into a sequence of maps

$$\tilde{X} \to Y_0 \to Y_1 \to \cdots \to Y_n = X$$

where $\tilde{X} \to Y_0$ is injective and $Y_i \to Y_{i+1}$ identifies just two points. Thus consider the map $A \to B$ where A equals some Y_i and B is A modulo the identification of two points $x, y \in A$. Without affecting the homotopy of A, we may draw out two thin "whiskers" at points x and y and, since A is path connected, assume they both originate at the same point. Thus B is homotopically equivalent to $A \vee S^1$, the one point union of A with a circle. By a standard application of Van Kampen's Theorem [4], it follows that $\pi_1(B) \cong \pi_1(A) * \mathbb{Z}$, the free product. By induction, $\pi_1(X) \cong \pi_1(\tilde{X}) * \mathbb{Z} * \cdots * \mathbb{Z}$, where there are n copies of $\mathbb{Z}$. Thus $\pi_1(\tilde{X}) = 0$ and $n = 0$, whence π is injective.

To see $\tilde{X} = \mathbb{A}^1$, we first show genus$(\tilde{X}) = 0$. Let Z be the nonsingular projective model for $\tilde{X}$. Then Z is a complex nonsingular projective curve and hence is homeomorphic to a compact Riemann surface. It is enough to see that genus$(Z) = 0$. Since $\tilde{X}$ equals Z less a finite number of points, there exists a series of inclusions

$$\tilde{X} = Y_0 \subset Y_1 \subset \cdots \subset Y_n = Z$$

where Y_i is Y_{i+1}, less one point. Another easy application of Van Kampen's theorem implies that $\pi_1(Y_{i+1})$ is a homomorphic image of $\pi_1(Y_i)$. Since $\pi_1(\tilde{X}) = 0$ we have $\pi_1(Z) = 0$, by induction. Hence $Z = \mathbb{P}^1$ and genus$(Z) = 0$. Thus genus$(\tilde{X}) = 0$, whence $\tilde{X}$ equals $\mathbb{A}^1$ less a finite number of points. But $\pi_1(\tilde{X}) = 0$, so $\tilde{X} = \mathbb{A}^1$. □

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