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HYPERBOLICITY OF SURFACES MODULO RATIONAL AND ELLIPTIC CURVES

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Let X be a smooth compact complex surface of general type and let D be the union of all rational and elliptic curves in X. If there exist a complex torus T of dimension ≥ 2 and a nontrivial holomorphic map $X \to T$ whose image contains no elliptic curves then X is hyperbolic modulo D. In particular, if X has irregularity $h^0(X, \Omega_X^1) \geq 2$ and its Albanese variety is not isogenous to a product of elliptic curves then X is hyperbolic modulo D.

Introduction. A complex space X is called *hyperbolic* if the Kobayashi pseudo-distance d_X on X is a distance, i.e. if $d_X(x, x') > 0$ for $x \neq x'$ **[K]**. If D is a subset of X and $d_X(x, x') > 0$ unless $x = x'$ or $x, x' \in D$, we say that X is *hyperbolic modulo D*. Let X be a surface of general type. M. Green has made the following conjectures:

Conjecture A. The image of every nonconstant holomorphic map $\mathbf{C} \to X$ lies in a rational or elliptic curve in X, and

Conjecture B. X is hyperbolic modulo the union of all its rational and elliptic curves.

Conjecture A is known to be true for surfaces with irregularity $h^0(X, \Omega_X^1) > 2$ **[GG, OC]** and surfaces with irregularity 2 and simple Albanese variety **[G]**. We use these facts together with Brody's theorem (1.2 below) to prove the following:

THEOREM. *Let X be a smooth compact complex surface of general type and let D be the union of all rational and elliptic curves in X. If there exist a complex torus T of dimension ≥ 2 and a nontrivial holomorphic map $X \to T$ whose image contains no elliptic curves then X is hyperbolic modulo D.*

COROLLARY. *If X is a smooth compact complex surface of general type with irregularity ≥ 2 whose Albanese variety is not isogenous to a product of elliptic curves then X is hyperbolic modulo its rational and elliptic curves.*

We also prove the weaker statement that $X - D$ is hyperbolic in a few additional cases.

I would like to thank Mark Green for suggesting to me this application of Brody's theorem. I am also grateful to R. Treger and F. Catanese for helpful conversations concerning curves in surfaces and the Albanese map.

1. Preliminaries. First we recall some properties of the Kobayashi pseudo-distance and the Albanese map.

(1.1) *Distance-decreasing property.* A holomorphic map $\tau: U \to X$ between complex spaces U and X is always distance-decreasing with respect to d_U and d_X, i.e.

$$d_U(u, u') \geq d_X(\tau(u), \tau(u'))$$

for all $u, u' \in U$.

Let D be a closed analytic subspace of X. Applying (1.1) to the inclusion map $X - D \hookrightarrow X$ shows that $X - D$ is hyperbolic whenever X is hyperbolic modulo D. It also follows from (1.1) that there are no nonconstant holomorphic maps of $\mathbf{C}$ to a hyperbolic space. For compact complex spaces the converse is true:

(1.2) BRODY'S THEOREM ([**BR**], [**L**, III.2.1]). *A compact complex space X is hyperbolic if and only if every holomorphic map $\mathbf{C} \to X$ is constant.*

Nevertheless, a noncompact complex space may fail to be hyperbolic even if every holomorphic map of $\mathbf{C}$ into the space is constant. Green has constructed an example of a Zariski-open subset of $\mathbf{P}^2$ with this property [**L**, p. 79].

Recall also that for every smooth projective variety X there exist an abelian variety $A = \mathrm{Alb}(X)$, the Albanese variety of X, and a holomorphic map $\alpha: X \to A$, the Albanese map, having the universal property that any other holomorphic map of X to a complex torus factors uniquely through α. The dimension of A is equal to the irregularity $q = h^0(X, \Omega^1_X)$ of X. By Poincaré's Complete Reducibility Theorem [**SD**, pp. 56–59], A is isogenous to a product of simple abelian varieties, i.e. there is a finite surjective holomorphic map $A \to A_1 \times \cdots \times A_m$, where $A_1, \ldots, A_m$ are abelian varieties, each containing no nontrivial complex subtori. The factors are unique up to isogeny. Let X be a smooth surface of general type. (Such a surface is always projective [**BPV**, p. 189].) Suppose, as in the hypothesis of the corollary, that $q \geq 2$ and $A = \mathrm{Alb}(X)$ is not isogenous to a product of elliptic curves. Then $\dim(A_i) \geq 2$ for at least one factor A_i. Projecting to A_i gives a map $X \to A_i$ satisfying the hypothesis of the theorem.

2. Generically finite maps to complex tori. Throughout this paper a *curve* is a compact complex space of dimension 1 and a *surface* is a reduced irreducible compact complex space of dimension 2. By the *genus* of a curve we always mean its geometric genus, i.e. the genus of its desingularization.

(2.1) PROPOSITION. *Let X be a smooth surface and D the union of reduced irreducible curves $C_1, \dots, C_n$ in X. Suppose that the intersection matrix (C_iC_j) is negative definite and the image of every nonconstant holomorphic map $\mathbf{C} \to X$ lies in D. Then X is hyperbolic modulo D.*

Proof. By Grauert's criterion [**BPV**, III.2.1] there is a holomorphic bimeromorphic map $\psi: X \to Y$ to a surface Y such that D is exceptional for ψ; more precisely, for each connected component C of D there are an open neighbourhood U of C in X, a point y in Y, and a neighbourhood V of y in Y, such that ψ is a biholomorphism of $U - C$ to $V - \{y\}$ and $\psi(C) = y$. Every holomorphic map $\mathbf{C} \to Y$ must be constant; otherwise there would be a nonconstant lifting $\mathbf{C} \to X$ which did not lie in D. By Brody's theorem, Y is hyperbolic. This means that $d_Y(y, y') > 0$ whenever $y, y' \in Y$ and $y \neq y'$. But $d_X(x, x') \geq d_Y(\psi(x), \psi(x'))$, by the distance-decreasing property (1.1), so $d_X(x, x') > 0$ unless $x = x'$ or $x, x' \in D$. □

(2.2) PROPOSITION. *Let X be a smooth surface of general type, T a complex torus of dimension ≥ 2, and $\varphi: X \to T$ a holomorphic map which is generically finite-to-one and such that $\varphi(X)$ contains no elliptic curves. Let D be the union of all rational and elliptic curves in X. Then X is hyperbolic modulo D.*

Proof. A torus contains no rational curves and by assumption $\varphi(X)$ contains no elliptic curves so every rational and elliptic curve in X must be contracted to a point by φ. The map φ has a Stein factorization

$$X \xrightarrow{\rho} X' \xrightarrow{\mu} T$$

where X' is a reduced normal surface, ρ is a holomorphic map with connected fibres, μ is a holomorphic finite-to-one map, and $\mu \circ \rho = \varphi$.

Then ρ is bimeromorphic and the exceptional locus E of ρ is a finite collection of curves which contains D. By Grauert's criterion, the intersection matrix of E is negative definite. The submatrix corresponding to D must also be negative definite.

Next we check that the image of every nonconstant holomorphic map $f: \mathbf{C} \to X$ lies in D. Since rational and elliptic curves are the only irreducible curves admitting nonconstant holomorphic maps of $\mathbf{C}$, it is enough to show that f is algebraically degenerate, i.e. that $f(\mathbf{C})$ is contained in a proper algebraic subvariety of X. Let $\alpha: X \to A = \mathrm{Alb}(X)$ be the Albanese map. By the universal property of α there is a holomorphic map $\gamma: A \to T$ such that $\varphi = \gamma \circ \alpha$. Then $q = \dim(A) \geq 2$ since φ is generically finite. If $q > 2$ then f is algebraically degenerate **[GG]**. Suppose that $q = 2$. Then $\alpha(X) = A$ and $\varphi(X) = \gamma(A)$ is a 2-dimensional complex subtorus of T which contains no elliptic curves. It follows that A itself contains no elliptic curves, so A is simple. By **[G]** f is algebraically degenerate. Now apply Proposition (2.1). □

Remark. The union of all rational and elliptic curves in a surface of general type does not always have negative definite intersection matrix. See Example (5.4).

3. Fibrations over curves of genus ≥ 2.

(3.1) Proposition. *Let X be a smooth surface of general type, C a smooth curve of genus ≥ 2, and $\pi: X \to C$ a surjective holomorphic map with connected fibres. Let D be the union of all rational and elliptic curves in X. Then X is hyperbolic modulo D.*

Before proving Proposition (3.1) we state a result of Zariski which will allow us to apply Grauert's criterion to some of the rational and elliptic curves in X:

(3.2) Lemma **[BPV, III.8.2]**. *Let X be a smooth surface, C a smooth curve, and $\pi: X \to C$ a surjective holomorphic map with connected fibres. Let $F = \sum n_i F_i$ be a fibre of π, where $n_i > 0$ and the curves F_i are the distinct, reduced, irreducible components of F. Then the intersection matrix $(F_i F_j)$ is negative semi-definite and $F^2 = 0$, but the intersection matrix of any proper subset of the collection $\{F_i\}$ is negative definite.*

Proof of Proposition (3.1). The generic fibre of π is a smooth curve of genus ≥ 2 because X is of general type. All rational and elliptic curves in X lie in fibres of π since there are no nonconstant holomorphic maps of rational or elliptic curves to C. Let S be the set of all points z of C such that $\pi^{-1}(z) = X_z$ consists entirely of rational and elliptic curves and let R be the set of all remaining points z of C whose fibres contain rational or elliptic curves. Both S and R are finite. Let $\Gamma = D \cap \pi^{-1}(R)$. The intersection matrix of Γ is negative definite by (3.2) so by Grauert's criterion there exist a surface Y and a holomorphic bimeromorphic map $\psi\colon X \to Y$ which is a biholomorphism off Γ and contracts each connected component of Γ to a point of Y. The union Δ of all rational and elliptic curves in Y is contained in $\psi(D)$. Let $\tau\colon Y \to C$ be the fibration induced by $\pi\colon X \to C$. Then $\Delta = \tau^{-1}(S)$. By the distance-decreasing property (1.1) we have $d_X(x, x') \geq d_Y(\psi(x), \psi(x'))$, so to show that X is hyperbolic modulo D it is enough to show that Y is hyperbolic modulo Δ. We use an argument similar to that of [**L**, I.2.7]. Notice that C and all fibres of τ except those in Δ are hyperbolic since curves of genus ≥ 2 are hyperbolic. If y, y' are points of Y such that $\tau(y) \neq \tau(y')$ then $d_Y(y, y') \geq d_C(\tau(y), \tau(y')) > 0$. Now assume that $y \neq y'$ and $\tau(y) = \tau(y') = z$ and $z \notin S$. Then $\tau^{-1}(z)$ is hyperbolic, so by [**L**, III.3.1] there is a neighbourhood U of z in C such that $\tau^{-1}(U)$ is hyperbolic. Choose $\varepsilon > 0$ small enough that U contains the ball $B(z, 2\varepsilon)$ with centre z and radius 2ε in the metric d_C. The set $V = \tau^{-1}(B(z, 2\varepsilon))$ is hyperbolic since it is contained in $\tau^{-1}(U)$. By [**L**, I.2.5] there is a constant $k > 0$ such that $d_Y(y, y') \geq \min\{\varepsilon, k d_V(y, y')\}$. But $d_V(y, y') > 0$ since V is hyperbolic. □

4. Proof of theorem. Let X be a smooth surface of general type and let D be the union of all rational and elliptic curves in X. Assume that there exist a complex torus T of dimension ≥ 2 and a nontrivial holomorphic map $\varphi\colon X \to T$ such that $\varphi(X)$ contains no elliptic curves. If φ is generically finite-to-one then X is hyperbolic modulo D by Proposition (2.2). Otherwise $\varphi(X)$ is a curve whose normalization C is a smooth curve of genus ≥ 2. If the fibres of the lifting $X \to C$ of φ are not connected we may use Stein factorization to obtain a fibration of X over a smooth curve of genus ≥ 2 with connected fibres. Now by Proposition (3.1) X is hyperbolic modulo D. □

5. Additional cases and examples. We show here how the methods of the previous sections and a theorem of Green can be used to study the Kobayashi pseudo-distance on a surface of general type in a few of the remaining cases. Propositions (5.1) and (5.3) are concerned with fibrations of surfaces over elliptic curves, particularly those which occur when the Albanese variety of a surface is isogenous to a product of elliptic curves. Proposition (5.5) is an application of [**GR**, Theorem 2].

(5.1) PROPOSITION. *Let X be a smooth surface of general type and D the union of all rational and elliptic curves in X. Assume that*

(i) *there is a surjective holomorphic map $\pi: X \to E$ where E is a nonsingular elliptic curve and π has connected fibres,*
(ii) *no fibre of π consists entirely of rational and elliptic curves,*
(iii) *every elliptic curve in X lies in a fibre of π, and*
(iv) *the image of every nonconstant holomorphic map $\mathbf{C} \to X$ lies in D.*

Then X is hyperbolic modulo D.

Proof. Since X is of general type, the generic fibre of π is a smooth curve of genus ≥ 2. Every elliptic and rational curve in X must lie in a fibre of π by (iii) and because there are no nonconstant holomorphic maps of rational curves to E. Then D consists of a finite number of curves and the intersection matrix of D is negative definite by (ii) and Lemma (3.2). By Proposition (2.1) X is hyperbolic modulo D. □

(5.2) REMARK. Conditions (i) and (iv) are satisfied whenever X has irregularity $q \geq 3$ and the Albanese variety of X is isogenous to a product of elliptic curves.

(5.3) PROPOSITION. *Let X be a smooth surface of general type and D the union of all rational and elliptic curves in X. Assume that condition* (i) *of* (5.1) *holds and*

(ii′) *at least one fibre of π consists entirely of rational and elliptic curves.*

Then $X - D$ is hyperbolic.

Proof. As in (5.1), only a finite number of fibres of π contain curves of D. In addition there may be elliptic curves in X mapping surjectively to E. Let S be the set of all points z in E such that $\pi^{-1}(z)$ consists entirely of curves of D and let R be the set of all remaining points of E whose fibres contain curves of D. Let Γ be the union of all curves of D in $\pi^{-1}(R)$. By (3.2) and Grauert's criterion, there exist a surface Y and a holomorphic bimeromorphic map $\psi: X \to Y$ contracting Γ. Let $\tau: Y \to E$ be the fibration induced by $\pi: X \to E$. By construction, $\tau^{-1}(S)$ consists entirely of rational and elliptic curves, while for every $z \in E-S$ the fibre $\tau^{-1}(z)$ is hyperbolic. By assumption (ii′) S is not empty, so $E-S$ is hyperbolic. Therefore $Y-\tau^{-1}(S)$ is hyperbolic [**L**, III.3.1] and hence so are $X-\pi^{-1}(S)-\Gamma$ and the subset $X-D$. □

(5.4) Example. Let $\pi\colon X \to E$ be a pencil of curves of genus 2, i.e. X is a smooth minimal algebraic surface, E is a smooth curve, and π is a surjective holomorphic map whose generic fibre is a smooth curve of genus 2. Ogg [**OG**] has shown that the singular fibres of such a pencil consist entirely of rational and elliptic curves. If X is of general type and E is an elliptic curve then at least one fibre of π must be singular [**BPV**, V.14 and V.6]. Construction of such a surface is described in [**XI**, pp. 24–28 and 72–73]. This also provides an example in which the union of all rational and elliptic curves does not have negative definite intersection matrix, by Lemma (3.2).

(5.5) Proposition. *Let X be a smooth minimal surface of general type which contains no singular elliptic curves. Let D be the union of all rational and elliptic curves in X. Assume that*

(i) *$D = D_1 + D_2$ where D_1 and D_2 are disjoint effective divisors,*
(ii) *D_1 has negative definite intersection matrix,*
(iii) *every rational curve in D_2 intersects the other curves in D_2 in at least 3 distinct points, and*
(iv) *the image of every nonconstant holomorphic map $\mathbf{C} \to X$ lies in D.*

Then $X-D$ is hyperbolic.

Proof. First note that if E is a nonsingular elliptic curve in X then $E^2 < 0$ since $K_X \cdot E + E^2 = \deg K_E = 0$ (adjunction formula) and $K_X \cdot E > 0$ [**BPV**, VII.2.3]. If E does not intersect any other rational or elliptic curve in X then we may assume that E is in D_1. By Grauert's criterion, there exist a surface Y and a holomorphic bimeromorphic map $\psi\colon X \to Y$ contracting D_1. The image Δ of D_2 in Y consists of rational and elliptic curves $C_1, \ldots, C_n$ with the property that C_i intersects the other curves in Δ in at least 3 distinct points if C_i is rational and in at least one point if C_i is elliptic. Then there is no nonconstant holomorphic map

$$\mathbf{C} \to C_i - \left(\bigcup_{j\neq i} C_j\right)$$

for any i. By Green's theorem [**GR**, Theorem 2], which is also true for singular spaces [**L**, III.3.6], $Y - \Delta$ is hyperbolic. Hence so is $X - D$. □

(5.6) COROLLARY. *Let X be a smooth surface of general type which is embedded in an abelian variety. Let D be the union of all elliptic curves in X. Then $X - D$ is hyperbolic.*

Proof. There are no rational or singular elliptic curves in X since X is contained in an abelian variety. As in (5.5), every elliptic curve in X has negative self-intersection. By a result of Bogomolov [**DE**, 3.4.6], there are only finitely many elliptic curves in X. Let D_1 be the union of all isolated elliptic curves in X and let D_2 be the union of all remaining elliptic curves in X. Since X is embedded in an abelian variety, the irregularity of X is at least 3, so every nonconstant holomorphic map $\mathbf{C} \to X$ lies in D by [**GG**]. Now use (5.5). □

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